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Derivation of Some Identities and Applications

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Abstract Certain identities based on recurrently published results about explicit solutions of general second-order linear recurrences are proved and are used to derive explicit solutions of four well-known recurrence relations with polynomial coefficients.

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1. Introduction

Let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ with $b_0 = 1$ be two sequences of complex numbers. Let y_0 and y_1 be initial values of the second-order linear recurrence relation

$$y_n = b_{n-1}y_{n-1} + a_{n-2}y_{n-2}, \quad n \geq 2. \tag{1.1}$$

Set $D_0 = D_1 = \{\emptyset\}$ and for $n \geq 2$, define

$$D_n = \{X \in \mathcal{P}([0, n-2]);\ |X| = 0, 1 \text{ or } |u - v| \geq 2 \text{ for all distinct } u, v \in X\},$$

where $[i, j] := \{i, i+1, \ldots, j\} \subseteq \mathbb{N}\ \ (i \leq j)$, the set $\mathcal{P}([i, j])$ refers to the power set of the set $[i, j]$, and $|X|$ denotes the cardinality of the set X.

Recently in [1], the following theorem was proved.

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Theorem 1.1. *The second order linear recurrence* (1.1)*, with initial values* y_0, y_1*, has an explicit solution given, for* $n \geq 2$*, by*

$$y_n = y_0\mathcal{D}_n + y_1\mathcal{E}_n, \tag{1.2}$$

where

$$\mathcal{D}_n = \sum_{\substack{X \in D_n \\ 0 \in X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X + \{1\}))} b_i \right) \left(\prod_{i \in X} a_i \right), \tag{1.3}$$

$$\mathcal{E}_n = \sum_{\substack{X \in D_n \\ 0 \notin X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X + \{1\}))} b_i \right) \left(\prod_{i \in X} a_i \right), \tag{1.4}$$

and, by convention, the empty product is taken to be 1*, the empty sum to be* 0*, and the set* $X + \{1\}$ *is defined as the set of elements resulting from adding* 1 *to every element in the set* X*.*

The primary objective of this work is to derive, using Theorem 1.1, new identities by making explicit the values $\mathcal{D}_n$ and $\mathcal{E}_n$ through specializing the set D_n. Our second objective is to apply the identities so derived to find explicit solutions of second-order homogeneous linear difference equations with polynomial coefficients focusing on those equations possessing well-known solution functions, such as Fibonacci, Lucas, and Chebyshev polynomials. Although there have already appeared results similar to ours, we believe our approach offers a more lucid and simpler exposition.

2. Identities

Our sought after identities are derived from the counting function defined in the next theorem.

Theorem 2.1. *For integers* $n \geq 2$ *and* $k \geq 0$*, let*

$$F(n,k) := \sum_{\substack{X \in D_n \\ |X| = k}} 1;$$

equivalently, $F(n,k)$ *counts the number of sets* X *in* D_n *containing* k *elements. Then*

$$F(m,k) = \begin{cases} 1 & \text{if } k = 0,\ m \geq 2 \\ F(m-1,k) + F(m-2,k-1) & \text{if } 1 \leq k \leq \lfloor m/2 \rfloor,\ m \geq 3 \\ 0 & \text{if } k > \lfloor m/2 \rfloor,\ m \geq 3. \end{cases} \tag{2.1}$$

Proof. Let $X \in D_m$ with $|X| = k$. If $k = 0$ and $m \geq 2$, then the only set X is the empty set, and so $F(m,0) = 1$.

Next consider the case $m \geq 3$. If $k > \lfloor m/2 \rfloor$, since the elements in X are differed by at least 2, there is no such X, and so $F(m,k) = 0$. If $1 \leq k \leq \lfloor m/2 \rfloor$, then there are two possible cases, i.e., the largest element in X is $< m-2$ or the largest element in X is $m-2$. In the former case, the number of such X is $F(m-1,k)$. In the latter case, consider discarding the largest element, we see that the number of such X is $F(m-2,k-1)$, and so $F(m,k) = F(m-1,k) + F(m-2,k-1)$. ∎

Based on Theorems 2.1, we next deduce our anticipated identities.

Corollary 2.2. *For integers $k \geq 0$, $n \geq 2$ and $n \geq k+1$, we have*

I. $F(n,k) = \binom{n-k}{k}$;

II. $$\sum_{\substack{X\in D_n\\ 0\notin X,\ |X|=k}} 1 = \sum_{\substack{X\in D_{n-1}\\ |X|=k}} 1 = \binom{n-1-k}{k}.$$

Proof. I. If $k=0$, then by Theorem 2.1 we have $F(n,0) = 1 = \binom{n-0}{0}$ for all $n \geq 2$.
If $k=1$, $n \geq 2$, there are $n-1$ singleton sets in D_n, namely, $\{0\}, \{1\}, \ldots, \{n-2\}$, and so $F(n,1) = n-1 = \binom{n-1}{1}$.
If $k=2$, $n \geq k+1 = 3$, the number of the sets in D_n with cardinality 2 is equal to the number of sets containing 2 elements from the set $\{0,1,\ldots,n-2\}$ subtracted by the number of sets containing two consecutive elements from the set $\{0,1,\ldots,n-2\}$, i.e., $F(n,2) = \binom{n-1}{2} - (n-2) = \binom{n-2}{2}$.

For $k \geq 2, n \geq k+1$, assume now that $F(n,k) = \binom{n-k}{k}$.
If $k+1 > \lfloor n/2 \rfloor$, then by Theorem 2.1, $F(n,k+1) = 0$ which is also equal to $\binom{n-k-1}{k+1}$ by convention.
If $k+1 \leq \lfloor n/2 \rfloor$, by the recurrence relation in Theorem 2.1, we get

$$\begin{aligned} F(n,k+1) &= F(n-1,k+1) + F(n-2,k)\\ F(n-1,k+1) &= F(n-2,k+1) + F(n-3,k)\\ &\vdots\\ F(2k+3,k+1) &= F(2k+2,k+1) + F(2k+1,k)\\ F(2k+2,k+1) &= F(2k+1,k+1) + F(2k,k). \end{aligned}$$

Summing all the identities, we obtain

$$F(n,k+1) = F(n-2,k) + F(n-3,k) + \cdots + F(2k+1,k) + F(2k,k) + F(2k+1,k+1).$$

Since $F(2k+1,k+1) = 0$ (Theorem 2.1), by induction, we get

$$F(n,k+1) = \binom{n-k-2}{k} + \binom{n-k-3}{k} + \cdots + \binom{k}{k} = \binom{n-k-1}{k+1},$$

which proves the first assertion.

II. Consider any set X belonging to the second summation $\sum_{\substack{X\in D_{n-1}\\ |X|=k}} 1$. If $0 \notin X$, then X also belongs to the first summation. If $0 \in X$, then discard this 0, subtract 1 from each remaining element, and insert the element $n-2$ to the set X to get a new set that clearly belongs to the first summation. This shows the elements in the second summation form a subset of those in the first summation $\sum_{\substack{X\in D_n\\ 0\notin X,\ |X|=k}} 1$. Reversing the preceding arguments, the opposite inclusion is verified yielding their equality. The binomial value is merely an application of the first assertion. ∎

3. Applications

Let $(y_n(t))_{n\geq 0}$ be a sequence of functions that satisfies the second-order homogeneous difference equation

$$y_n(t) = b_{n-1}(t)y_{n-1}(t) + a_{n-2}(t)y_{n-2}(t), \quad n \geq 2, \tag{3.1}$$

where $(a_n(t))_{n\geq 0}$ and $(b_n(t))_{n\geq 1}$ are two sequences of polynomials in t. As applications to our results in Section 1, explicit solutions of four well-known recurrences (3.1) are determined.

3.1. Fibonacci Polynomials

Consider the recurrence (3.1) with initial values of the form

$$y_n(t) = ty_{n-1}(t) + y_{n-2}(t) \quad (n \geq 2), \qquad y_0(t) = 0,\ y_1(t) = 1. \tag{3.2}$$

Comparing with (3.1), we have $a_i(t) = 1$ $(i \geq 0)$, $b_i(t) = t$ $(i \geq 1)$. The expressions (1.3) and (1.4) are

$$\mathcal{D}_n(t) = \sum_{\substack{X\in D_n \\ 0\in X}} \left(\prod_{i\in[1,n-1]\setminus(X\cup(X+\{1\}))} t \right) \left(\prod_{i\in X} 1 \right) = \sum_{\substack{X\in D_n \\ 0\in X}} t^{n-2|X|},$$

$$\mathcal{E}_n(t) = \sum_{\substack{X\in D_n \\ 0\notin X}} \left(\prod_{i\in[1,n-1]\setminus(X\cup(X+\{1\}))} t \right) \left(\prod_{i\in X} 1 \right) = \sum_{\substack{X\in D_n \\ 0\notin X}} t^{n-1-2|X|}.$$

Appealing to Theorem 1.1 and Corollary 2.2, the recurrence relation (3.2) has an explicit solution given by

$$\begin{aligned}
y_n(t) &= y_0(t)\mathcal{D}_n(t) + y_1(t)\mathcal{E}_n(t) \\
&= 0 + \sum_{\substack{X\in D_n \\ 0\notin X}} t^{n-1-2|X|} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\substack{X\in D_n \\ 0\notin X, |X|=k}} t^{n-1-2k} \right) \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} t^{n-1-2k} \left(\sum_{\substack{X\in D_n \\ 0\notin X, |X|=k}} 1 \right) \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} t^{n-1-2k} \left(\sum_{\substack{X\in D_{n-1} \\ |X|=k}} 1 \right) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} t^{n-1-2k} \binom{n-1-k}{k}.
\end{aligned}$$

3.2. Generalized Fibonacci Polynomials (Panwar et al. [2])

Consider the recurrence (3.1) with initial values of the form

$$y_n(t) = ty_{n-1}(t) + y_{n-2}(t) \quad (n \geq 2), \qquad y_0(t) = c,\ y_1(t) = ct \quad (c \in \mathbb{C}). \tag{3.3}$$

Proceeding as before, $a_i(t) = 1$ $(i \geq 0)$, $\quad b_i(t) = t$ $(i \geq 1)$ and

$$\mathcal{D}_n(t) = \sum_{\substack{X \in D_n \\ 0 \in X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X+\{1\}))} t \right) \left(\prod_{i \in X} 1 \right) = \sum_{\substack{X \in D_n \\ 0 \in X}} t^{n-2|X|},$$

$$\mathcal{E}_n(t) = \sum_{\substack{X \in D_n \\ 0 \notin X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X+\{1\}))} t \right) \left(\prod_{i \in X} 1 \right) = \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-1-2|X|}.$$

An explicit solution of (3.3) is given by

$$\begin{aligned} y_n(t) &= y_0(t)\mathcal{D}_n(t) + y_1(t)\mathcal{E}_n(t) \\ &= c \sum_{\substack{X \in D_n \\ 0 \in X}} t^{n-2|X|} + ct \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-1-2|X|} = c \sum_{\substack{X \in D_n \\ 0 \in X}} t^{n-2|X|} + c \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-2|X|} \\ &= c \sum_{X \in D_n} t^{n-2|X|} = c \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} t^{n-2k} \left(\sum_{\substack{X \in D_n \\ |X|=k}} 1 \right) = c \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} t^{n-2k} \binom{n-k}{k}. \end{aligned}$$

3.3. Generalized Lucas Polynomials (Panwar et al. [2])

Consider the recurrence (3.1) with initial values of the form

$$y_n(t) = ty_{n-1}(t) + y_{n-2}(t), \quad (n \geq 2), \qquad y_0(t) = 2c, \; y_1(t) = ct \quad (c \in \mathbb{C}). \tag{3.4}$$

Proceeding as before $a_i(t) = 1$ $(i \geq 0)$, $b_i(t) = t$ $(i \geq 1)$ and

$$\mathcal{D}_n(t) = \sum_{\substack{X \in D_n \\ 0 \in X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X+\{1\}))} t \right) \left(\prod_{i \in X} 1 \right) = \sum_{\substack{X \in D_n \\ 0 \in X}} t^{n-2|X|},$$

$$\mathcal{E}_n(t) = \sum_{\substack{X \in D_n \\ 0 \notin X}} \left(\prod_{i \in [1,n-1] \setminus (X \cup (X+\{1\}))} t \right) \left(\prod_{i \in X} 1 \right) = \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-1-2|X|}.$$

An explicit solution of (3.4) is given by

$$\begin{aligned} y_n(t) &= y_0(t)\mathcal{D}_n(t) + y_1(t)\mathcal{E}_n(t) \\ &= 2c \sum_{\substack{X \in D_n \\ 0 \in X}} t^{n-2|X|} + ct \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-1-2|X|} \\ &= 2c \sum_{X \in D_n} t^{n-2|X|} - c \sum_{\substack{X \in D_n \\ 0 \notin X}} t^{n-2|X|} \end{aligned}$$

$$
\begin{aligned}
&= c\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} t^{n-2k}\left(2\sum_{\substack{X\in D_n\\ |X|=k}} 1-\sum_{\substack{X\in D_n\\ 0\notin X,|X|=k}} 1\right)\\
&= c\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} t^{n-2k}\left(2\binom{n-k}{k}-\binom{n-1-k}{k}\right)\\
&= c\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} t^{n-2k}\binom{n-k}{k}\frac{n}{n-k}.
\end{aligned}
$$

3.4. Chebyshev Polynomials of the Second Kind

Consider the recurrence (3.1) with initial values of the form

$$
y_n(t) = 2ty_{n-1}(t) - y_{n-2}(t) \quad (n \geq 2), \qquad y_0(t) = 1,\ y_1(t) = 2t. \tag{3.5}
$$

Proceeding as before, $a_i(t) = -1\ (i \geq 0),\ b_i(t) = 2t\ (i \geq 1)$ and

$$
\mathcal{D}_n(t) = \sum_{\substack{X\in D_n\\ 0\in X}}\left(\prod_{i\in[1,n-1]\setminus(X\cup(X+\{1\}))} 2t\right)\left(\prod_{i\in X}(-1)\right) = \sum_{\substack{X\in D_n\\ 0\in X}} (2t)^{n-2|X|}(-1)^{|X|},
$$

$$
\mathcal{E}_n(t) = \sum_{\substack{X\in D_n\\ 0\notin X}}\left(\prod_{i\in[1,n-1]\setminus(X\cup(X+\{1\}))} 2t\right)\left(\prod_{i\in X}(-1)\right) = \sum_{\substack{X\in D_n\\ 0\notin X}} (2t)^{n-1-2|X|}(-1)^{|X|}.
$$

An explicit solution of (3.5) is given by

$$
\begin{aligned}
y_n(t) &= y_0(t)\mathcal{D}_n(t) + y_1(t)\mathcal{E}_n(t)\\
&= \sum_{\substack{X\in D_n\\ 0\in X}} (2t)^{n-2|X|}(-1)^{|X|} + 2t\sum_{\substack{X\in D_n\\ 0\notin X}} (2t)^{n-1-2|X|}(-1)^{|X|}\\
&= \sum_{\substack{X\in D_n\\ 0\in X}} (2t)^{n-2|X|}(-1)^{|X|} + \sum_{\substack{X\in D_n\\ 0\notin X}} (2t)^{n-2|X|}(-1)^{|X|} = \sum_{X\in D_n} (2t)^{n-2|X|}(-1)^{|X|}\\
&= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\left(\sum_{\substack{X\in D_n\\ |X|=k}} (2t)^{n-2k}(-1)^k\right) = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(2t)^{n-2k}(-1)^k\left(\sum_{\substack{X\in D_n\\ |X|=k}} 1\right)\\
&= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(2t)^{n-2k}(-1)^k\binom{n-k}{k}.
\end{aligned}
$$

Our demonstration of the explicit solution of Chebyshev polynomials appears to be less complicated compared to the proof presented in Mallik's work [3].

Postscript (December 2023). The results in this paper have now been further extended in [4].

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