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The $(s,t)-$Jacobsthal Hybrid Numbers and $(s,t)-$Jacobsthal-Lucas Hybrid Numbers

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Abstract The $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas hybrid numbers are introduced. Several properties of these numbers are derived, including the Binet formulas, generating functions, exponential generating functions, summation formulas and identities such as those due to Catalan, Cassini and d'Ocagne. In addition, a matrix generator for these numbers is presented. The obtained results extend and generalize well-known theorems.

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1. Introduction

Numerous researchers have dedicated their time and effort in the study of number sequences, owing to their widespread utility in the realms of science, engineering, art, and nature. A particularly fascinating sequence is the Jacobsthal sequence [3], which has been extensively researched. This sequence is named after Ernst Jacobsthal, a renowned mathematician hailing from Germany. It boasts a notable attribute, that is, the enumeration of microcontroller skip instructions [1].

In 2014, Falcon [2] defined $k-$Jacobsthal and $k-$Jacobsthal-Lucas sequences and presented some properties. Subsequently, Uygun [9] introduced $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas sequences, as follows:

Definition 1.1. For any real numbers s,t such that $s>0, t\neq 0$ and $s^2+8t>0$, the $(s,t)-$Jacobsthal sequence $\{j_n(s,t)\}_{n\in\mathbb{N}}$ and the $(s,t)-$Jacobsthal-Lucas sequence $\{c_n(s,t)\}_{n\in\mathbb{N}}$ are defined recurrently by, for $n\geq 2$

$$j_n(s,t)=sj_{n-1}(s,t)+2tj_{n-2}(s,t) \tag{1.1}$$

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and

$$c_n(s,t) = sj_{n-1}(s,t) + 2tj_{n-2}(s,t), \tag{1.2}$$

respectively, where $j_0(s,t) = 0,\ j_1(s,t) = 1,\ c_0(s,t) = 2,\ c_1(s,t) = s.$

Remark 1.2. In Definition 1.1, it can be observed that when $s = 1$ and $2t = k$, the resulted sequences are the $k-$Jacobsthal sequence and the $k-$Jacobsthal-Lucas sequence, respectively. In the event that $s = t = 1$, the resulted sequences are the Jacobsthal sequence and the Jacobsthal-Lucas sequence, respectively.

In 2018, Özdemir [7] presented a new number system within the structure of non-commutative algebra known as hybrid numbers, which is a generalization of complex, hyperbolic and dual numbers. The hybrid number $\mathbf{Z}$ can be written in form

$$\mathbf{Z} = a + \boldsymbol{i}b + \boldsymbol{\varepsilon}c + \boldsymbol{h}d$$

where $a, b, c, d \in \mathbb{R}$ and $\boldsymbol{i}, \boldsymbol{\varepsilon}, \boldsymbol{h}$ are hybrid units such that

$$\boldsymbol{i}^2 = -1, \boldsymbol{h}^2 = 1, \boldsymbol{\varepsilon}^2 = 0$$

and

$$\boldsymbol{ih} = -\boldsymbol{hi} = \boldsymbol{\varepsilon} + \boldsymbol{i}.$$

Let $\mathbf{Z}_1 = a_1 + \boldsymbol{i}b_1 + \boldsymbol{\varepsilon}c_1 + \boldsymbol{h}d_1$ and $\mathbf{Z}_2 = a_2 + \boldsymbol{i}b_2 + \boldsymbol{\varepsilon}c_2 + \boldsymbol{h}d_2$ denote two hybrid numbers. Equality, addition, subtraction, scalar multiplication, and multiplication of two hybrid numbers can be defined as follows:

$$\begin{aligned}
\mathbf{Z}_1 = \mathbf{Z}_2 &\quad \text{only if } a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2 \\
\mathbf{Z}_1 + \mathbf{Z}_2 &= (a_1 + a_2) + \boldsymbol{i}(b_1 + b_2) + \boldsymbol{\varepsilon}(c_1 + c_2) + \boldsymbol{h}(d_1 + d_2) \\
\mathbf{Z}_1 - \mathbf{Z}_2 &= (a_1 - a_2) + \boldsymbol{i}(b_1 - b_2) + \boldsymbol{\varepsilon}(c_1 - c_2) + \boldsymbol{h}(d_1 - d_2) \\
s\mathbf{Z}_1 &= sa_1 + \boldsymbol{i}sb_1 + \boldsymbol{\varepsilon}sc_1 + \boldsymbol{h}sd_1, s \in \mathbb{R} \\
\mathbf{Z}_1\mathbf{Z}_2 &= (a_1 + \boldsymbol{i}b_1 + \boldsymbol{\varepsilon}c_1 + \boldsymbol{h}d_1)(a_2 + \boldsymbol{i}b_2 + \boldsymbol{\varepsilon}c_2 + \boldsymbol{h}d_2) \\
&= (a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2) \\
&\quad + \boldsymbol{i}(a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2) \\
&\quad + \boldsymbol{\varepsilon}(a_1c_2 + b_1d_2 + c_1a_2 - c_1d_2 - d_1b_2 + d_1c_2) \\
&\quad + \boldsymbol{h}(a_1d_2 - b_1c_2 + c_1b_2 + d_1a_2).
\end{aligned}$$

The subsequent table presents the product of any two hybrid units.

$\cdot$	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
$\boldsymbol{i}$	-1	1-$\boldsymbol{h}$	$\boldsymbol{\varepsilon}$+ $\boldsymbol{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$+1	0	-$\boldsymbol{\varepsilon}$
$\boldsymbol{h}$	-$\boldsymbol{\varepsilon}$-$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	1

Table 1. The hybrid numbers multiplication

It is apparent that the multiplication of hybrid numbers has the property of associativity, but it lacks commutativity. The conjugate of a hybrid number $\mathbf{Z}$ is defined by

$$\overline{\mathbf{Z}} = \overline{a + \boldsymbol{i}b + \boldsymbol{\varepsilon}c + \boldsymbol{h}d} = a - \boldsymbol{i}b - \boldsymbol{\varepsilon}c - \boldsymbol{h}d.$$

The character of the hybrid number $\mathbf{Z}$ is

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2.$$

Many special kinds of hybrid numbers have been studied, including Fibonacci and Lucas hybrid numbers [5], Padovan hybrid numbers [6], Jacobsthal and Jacobsthal-Lucas hybrid numbers [8], as well as $k-$Jacobsthal and $k-$Jacobsthal-Lucas hybrid numbers [4]. These research articles have motivated us to define $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas hybrid numbers and explore their properties.

2. Preliminaries

The roots of the characteristic equation $x^2 - sx - 2t = 0$ associated with the recurrence relation in Definition 1.1 are

$$\alpha = \frac{s + \sqrt{s^2 + 8t}}{2} \quad \textit{and} \quad \beta = \frac{s - \sqrt{s^2 + 8t}}{2}. \tag{2.1}$$

Thus, the Binet formulas for $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas sequences can be expressed as follows:

$$j_n(s,t) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.2}$$

and

$$c_n(s,t) = \alpha^n + \beta^n. \tag{2.3}$$

By utilizing (2.1), it is evident that

$$\alpha + \beta = s, \quad \alpha \cdot \beta = -2t, \quad \alpha - \beta = \sqrt{s^2 + 8t}. \tag{2.4}$$

Uygun [9] has demonstrated the summation formulas for $(s,t)-$Jacobsthal sequence and $(s,t)-$Jacobsthal-Lucas sequence as follows:

$$\sum_{k=0}^{n} j_k(s,t) = \frac{1 - j_{n+1}(s,t) - 2tj_n(s,t)}{1 - s - 2t} \tag{2.5}$$

and

$$\sum_{k=0}^{n} c_k(s,t) = \frac{2 - c_{n+1}(s,t) - s - 2tc_n(s,t)}{1 - s - 2t}. \tag{2.6}$$

Additionally, he has established the relationships between $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas sequences, which are listed below:

$$c_n(s,t) = j_{n+1}(s,t) + 2tj_{n-1}(s,t) \tag{2.7}$$

and

$$sj_n(s,t) + c_n(s,t) = 2j_{n+1}(s,t). \tag{2.8}$$

3. Main Results

Definition 3.1. Let $n \geq 1$ be an integer. For any real numbers $s, t \in \mathbb{R}$ such that $s > 0, t \neq 0$ and $s^2 + 8t > 0$, the nth $(s,t)-$Jacobsthal hybrid numbers, $Hj_n(s,t)$, and the nth $(s,t)-$Jacobsthal-Lucas hybrid numbers, $Hc_n(s,t)$, are defined by

$$Hj_n(s,t) = j_n(s,t) + \boldsymbol{i}j_{n+1}(s,t) + \boldsymbol{\varepsilon}j_{n+2}(s,t) + \boldsymbol{h}j_{n+3}(s,t) \tag{3.1}$$

and

$$Hc_n(s,t) = c_n(s,t) + \boldsymbol{i}c_{n+1}(s,t) + \boldsymbol{\varepsilon}c_{n+2}(s,t) + \boldsymbol{h}c_{n+3}(s,t), \tag{3.2}$$

respectively, where $\boldsymbol{i}, \boldsymbol{\varepsilon}, \boldsymbol{h}$ are hybrid units.

Remark 3.2. In Definition 3.1, if $s = 1$ and $2t = k$, then we have the $k-$Jacobsthal and the $k-$Jacobsthal-Lucas hybrid numbers, respectively. If $s = t = 1$, then we have the Jacobsthal and Jacobsthal-Lucas hybrid numbers, respectively.

Lemma 3.3. *Suppose that $s, t \in \mathbb{R}$ such that $s > 0, t \neq 0$ and $s^2 + 8t > 0$. Let $n \geq 1$ be an integer. Then*

$$\text{I. } Hj_{n+1}(s,t) = sHj_n(s,t) + 2tHj_{n-1}(s,t), \tag{3.3}$$

$$\text{II. } Hc_{n+1}(s,t) = sHc_n(s,t) + 2tHc_{n-1}(s,t), \tag{3.4}$$

with $Hj_0(s,t) = \boldsymbol{i} + \boldsymbol{\varepsilon}(s) + \boldsymbol{h}(s^2+2t)$, $Hj_1(s,t) = 1 + \boldsymbol{i}(s) + \boldsymbol{\varepsilon}(s^2+2t) + \boldsymbol{h}(s^3+4st)$, $Hc_0(s,t) = 2 + \boldsymbol{i}(s) + \boldsymbol{\varepsilon}(s^2+4t) + \boldsymbol{h}(s^3+6st)$ and $Hc_1(s,t) = s + \boldsymbol{i}(s^2+4t) + \boldsymbol{\varepsilon}(s^3+6st) + \boldsymbol{h}(s^4+8s^2t+8t^2)$.

Proof. Using (1.1) and (3.1), we obtain

$$\begin{aligned}
Hj_{n+1}(s,t) &= j_{n+1}(s,t) + \boldsymbol{i}j_{n+2}(s,t) + \boldsymbol{\varepsilon}j_{n+3}(s,t) + \boldsymbol{h}j_{n+4}(s,t) \\
&= sj_n(s,t) + 2tj_{n-1}(s,t) + \boldsymbol{i}[sj_{n+1}(s,t) + 2tj_n(s,t)] \\
&\quad + \boldsymbol{\varepsilon}[sj_{n+2}(s,t) + 2tj_{n+1}(s,t)] + \boldsymbol{h}[sj_{n+3}(s,t) + 2tj_{n+2}(s,t)] \\
&= s[j_n(s,t) + \boldsymbol{i}j_{n+1}(s,t) + \boldsymbol{\varepsilon}j_{n+2}(s,t) + \boldsymbol{h}j_{n+3}(s,t)] \\
&\quad + 2t[j_{n-1}(s,t) + \boldsymbol{i}j_n(s,t) + \boldsymbol{\varepsilon}j_{n+1}(s,t) + \boldsymbol{h}j_{n+2}(s,t)] \\
&= sHj_n(s,t) + 2tHj_{n-1}(s,t).
\end{aligned}$$

From (1.2) and (3.2) to obtain

$$\begin{aligned}
Hc_{n+1}(s,t) &= c_{n+1}(s,t) + \boldsymbol{i}c_{n+2}(s,t) + \boldsymbol{\varepsilon}c_{n+3}(s,t) + \boldsymbol{h}c_{n+4}(s,t) \\
&= sc_n(s,t) + 2tc_{n-1}(s,t) + \boldsymbol{i}[sc_{n+1}(s,t) + 2tc_n(s,t)] \\
&\quad + \boldsymbol{\varepsilon}[sc_{n+2}(s,t) + 2tc_{n+1}(s,t)] + \boldsymbol{h}[sc_{n+3}(s,t) + 2tc_{n+2}(s,t)] \\
&= s[c_n(s,t) + \boldsymbol{i}c_{n+1}(s,t) + \boldsymbol{\varepsilon}c_{n+2}(s,t) + \boldsymbol{h}c_{n+3}(s,t)] \\
&\quad + 2t[c_{n-1}(s,t) + \boldsymbol{i}c_n(s,t) + \boldsymbol{\varepsilon}c_{n+1}(s,t) + \boldsymbol{h}c_{n+2}(s,t)] \\
&= sHc_n(s,t) + 2tHc_{n-1}(s,t).
\end{aligned}$$

∎

Theorem 3.4. (Binet formulas for $(s,t)-$Jacobsthal hybrid number and $(s,t)-$Jacobsthal-Lucas hybrid number) *Suppose that $s, t \in \mathbb{R}$ such that $s > 0, t \neq 0$ and $s^2 + 8t > 0$. Let $m \geq 0$ be an integer. Then*

$$\text{I. } Hj_m(s,t) = \frac{\alpha^m \hat{\alpha} - \beta^m \hat{\beta}}{\alpha - \beta}, \tag{3.5}$$

$$\text{II. } Hc_m(s,t) = \alpha^m \hat{\alpha} + \beta^m \hat{\beta}, \tag{3.6}$$

where $\hat{\alpha} = 1 + \boldsymbol{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \boldsymbol{h}\alpha^3$ *and* $\hat{\beta} = 1 + \boldsymbol{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \boldsymbol{h}\beta^3$.

Proof. By using (3.1) and (2.2), we have

$$\begin{aligned}
Hj_m(s,t) &= j_m(s,t) + \boldsymbol{i}j_{m+1}(s,t) + \boldsymbol{\varepsilon}j_{m+2}(s,t) + \boldsymbol{h}j_{m+3}(s,t) \\
&= \frac{\alpha^m - \beta^m}{\alpha - \beta} + \boldsymbol{i}\left(\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta}\right) + \boldsymbol{\varepsilon}\left(\frac{\alpha^{m+2} - \beta^{m+2}}{\alpha - \beta}\right) \\
&\quad + \boldsymbol{h}\left(\frac{\alpha^{m+3} - \beta^{m+3}}{\alpha - \beta}\right) \\
&= \frac{\alpha^m(1 + \boldsymbol{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \boldsymbol{h}\alpha^3) - \beta^m(1 + \boldsymbol{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \boldsymbol{h}\beta^3)}{\alpha - \beta} \\
&= \frac{\alpha^m\hat{\alpha} - \beta^m\hat{\beta}}{\alpha - \beta}.
\end{aligned}$$

Next, we use (3.2) and (2.3) to obtain

$$\begin{aligned}
Hc_m(s,t) &= c_m(s,t) + \boldsymbol{i}c_{m+1}(s,t) + \boldsymbol{\varepsilon}c_{m+2}(s,t) + \boldsymbol{h}c_{m+3}(s,t) \\
&= \alpha^m + \beta^m + \boldsymbol{i}(\alpha^{m+1} + \beta^{m+1}) + \boldsymbol{\varepsilon}(\alpha^{m+2} + \beta^{m+2}) + \boldsymbol{h}(\alpha^{m+3} + \beta^{m+3}) \\
&= \alpha^m(1 + \boldsymbol{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \boldsymbol{h}\alpha^3) + \beta^m(1 + \boldsymbol{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \boldsymbol{h}\beta^3) \\
&= \alpha^m\hat{\alpha} + \beta^m\hat{\beta}.
\end{aligned}$$

∎

Theorem 3.5. *Suppose that* $s, t \in \mathbb{R}$ *such that* $s > 0, t \neq 0$ *and* $s^2 + 8t > 0$. *The generating function for* $(s,t)-$*Jacobsthal hybrid number and* (s,t)*-Jacobsthal hybrid number are*

$$\text{I.}\quad \sum_{m=0}^{\infty} Hj_m(s,t)x^m = \frac{Hj_0(s,t) + x[Hj_1(s,t) - sHj_0(s,t)]}{1 - sx - 2tx^2}, \tag{3.7}$$

$$\text{II.}\quad \sum_{m=0}^{\infty} Hc_m(s,t)x^m = \frac{Hc_0(s,t) + x[Hc_1(s,t) - sHc_0(s,t)]}{1 - sx - 2tx^2}, \tag{3.8}$$

respectively.

Proof. Assume that the generating function of the (s,t)-Jacobsthal hybrid number sequence $Hj_n(s,t)$ has the form $A(x) = \sum_{m=0}^{\infty} Hj_m(s,t)x^m$. Then

$$A(x) = Hj_0(s,t) + xHj_1(s,t) + x^2Hj_2(s,t) + \ldots \tag{3.9}$$

Multiply (3.9) on both sides by $-sx$ and then by $-2tx^2$ we have

$$-sxA(x) = -sxHj_0(s,t) - sx^2Hj_1(s,t) - sx^3Hj_2(s,t) - \ldots \tag{3.10}$$

$$-2tx^2A(x) = -2tx^2Hj_0(s,t) - 2tx^3Hj_1(s,t) - 2tx^4Hj_2(s,t) - \ldots \tag{3.11}$$

By adding (3.9)-(3.11), we have

$$\begin{aligned}(1-sx-2tx^2)A(x) =& Hj_0(s,t)+x[Hj_1(s,t)-sHj_0(s,t)]\\ &+x^2[Hj_2(s,t)-sHj_1(s,t)-2tHj_0(s,t)]\\ &+x^3[Hj_3(s,t)-sHj_2(s,t)-2tHj_1(s,t)]\\ &+x^4[Hj_4(s,t)-sHj_3(s,t)-2tHj_2(s,t)]+...\end{aligned}$$

Since the coefficients of t^m for $m \geq 2$ are equal to zero,

$$A(x)=\sum_{m=0}^{\infty} Hj_m(s,t)t^m = \frac{Hj_0(s,t)+x[Hj_1(s,t)-sHj_0(s,t)]}{1-sx-2tx^2}.$$

Assume that the generating function of the (s,t)-Jacobsthal hybrid number sequence $Hc_n(s,t)$ has the form $B(x)=\sum_{m=0}^{\infty} Hc_m(s,t)x^m$. Then

$$B(x)=Hc_0(s,t)+xHc_1(s,t)+x^2Hc_2(s,t)+... \tag{3.12}$$

Multiply (3.12) on both sides by $-sx$ and then by $-2tx^2$ we have

$$-sxB(x)=-sxHc_0(s,t)-sx^2Hc_1(s,t)-sx^3Hc_2(s,t)-... \tag{3.13}$$

$$-2tx^2B(x)=-2tx^2Hc_0(s,t)-2tx^3Hc_1(s,t)-2tx^4Hc_2(s,t)-... \tag{3.14}$$

By adding (3.12)-(3.14), we have

$$\begin{aligned}(1-sx-2tx^2)B(x) =& Hc_0(s,t)+x[Hc_1(s,t)-sHc_0(s,t)]\\ &+x^2[Hc_2(s,t)-sHc_1(s,t)-2tHc_0(s,t)]\\ &+x^3[Hc_3(s,t)-sHc_2(s,t)-2tHc_1(s,t)]\\ &+x^4[Hc_4(s,t)-sHc_3(s,t)-2tHc_2(s,t)]+...\end{aligned}$$

Since the coefficients of t^m for $m \geq 2$ are equal to zero,

$$B(x)=\sum_{m=0}^{\infty} Hc_m(s,t)t^m = \frac{Hc_0(s,t)+x[Hc_1(s,t)-sHc_0(s,t)]}{1-sx-2tx^2}.$$

■

Theorem 3.6. *Suppose that $s,t \in \mathbb{R}$ such that $s>0, t\neq 0$ and $s^2+8t>0$. The exponential generating function for $(s,t)-$Jacobsthal hybrid numbers and $(s,t)-$Jacobsthal-Lucas hybrid numbers are*

$$\text{I. } \sum_{m=0}^{\infty} Hj_m(s,t)\frac{y^m}{m!} = \frac{\hat{\alpha}e^{\alpha y}-\hat{\beta}e^{\beta y}}{\alpha-\beta}, \tag{3.15}$$

$$\text{II. } \sum_{m=0}^{\infty} Hc_m(s,t)\frac{y^m}{m!} = \hat{\alpha}e^{\alpha y}+\hat{\beta}e^{\beta y}, \tag{3.16}$$

respectively.

Proof. By using (3.5), we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} Hj_m(s,t)\frac{y^m}{m!} &= \sum_{m=0}^{\infty}\left(\frac{\alpha^m\hat{\alpha}-\beta^m\hat{\beta}}{\alpha-\beta}\right)\frac{y^m}{m!}\\ &= \left(\frac{\hat{\alpha}}{\alpha-\beta}\right)\sum_{m=0}^{\infty}\frac{(\alpha y)^m}{m!}-\left(\frac{\hat{\beta}}{\alpha-\beta}\right)\sum_{m=0}^{\infty}\frac{(\beta y)^m}{m!}\\ &= \frac{\hat{\alpha}e^{\alpha y}}{\alpha-\beta}-\frac{\hat{\beta}e^{\beta y}}{\alpha-\beta}\\ &= \frac{\hat{\alpha}e^{\alpha y}-\hat{\beta}e^{\beta y}}{\alpha-\beta}.\end{aligned}$$

By using (3.6), we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} Hc_m(s,t)\frac{y^m}{m!} &= \sum_{m=0}^{\infty}(\alpha^m\hat{\alpha}+\beta^m\hat{\beta})\frac{y^m}{m!}\\ &= \hat{\alpha}\sum_{m=0}^{\infty}\frac{(\alpha y)^m}{m!}+\hat{\beta}\sum_{m=0}^{\infty}\frac{(\beta y)^m}{m!}\\ &= \hat{\alpha}e^{\alpha y}+\hat{\beta}e^{\beta y}.\end{aligned}$$

■

Theorem 3.7. *Suppose that $s,t\in\mathbb{R}$ such that $s>0, t\neq 0$ and $s^2+8t>0$. Let $n\geq 0$ be an integer. Then*

I. $$\sum_{k=0}^{n} Hj_k(s,t) = \frac{1+Hj_0(s,t)-Hj_{n+1}(s,t)-2tHj_n(s,t)+2\varepsilon tj_1(s,t)+2htj_2(s,t)}{1-s-2t} \tag{3.17}$$

II. $$\sum_{k=0}^{n} Hc_k(s,t) = \frac{Hc_0(s,t)-Hc_{n+1}(s,t)-2tHc_n(s,t)-c_1(s,t)+2tic_0(s,t)+2t\varepsilon c_1(s,t)+2thc_2(s,t)}{1-s-2t}. \tag{3.18}$$

Proof. Using (1.1), (2.5) and (3.1), we have

$$\begin{aligned}&\sum_{k=0}^{n} Hj_k(s,t) = Hj_0(s,t)+Hj_1(s,t)+Hj_2(s,t)+...+Hj_n(s,t)\\ &= (j_0(s,t)+\boldsymbol{i}j_1(s,t)+\boldsymbol{\varepsilon}j_2(s,t)+\boldsymbol{h}j_3(s,t))+(j_1(s,t)+\boldsymbol{i}j_2(s,t)+\boldsymbol{\varepsilon}j_3(s,t)\\ &\quad+\boldsymbol{h}j_4(s,t))+...+(j_n(s,t)+\boldsymbol{i}j_{n+1}(s,t)+\boldsymbol{\varepsilon}j_{n+2}(s,t)+\boldsymbol{h}j_{n+3}(s,t))\\ &= (j_0(s,t)+j_1(s,t)+...+j_n(s,t))+\boldsymbol{i}(j_1(s,t)+j_2(s,t)+...+j_{n+1}(s,t)+j_0(s,t)\\ &\quad-j_0(s,t))+\boldsymbol{\varepsilon}(j_2(s,t)+j_3(s,t)+...+j_{n+2}(s,t)+j_0(s,t)-j_0(s,t)+j_1(s,t)\\ &\quad-j_1(s,t))+\boldsymbol{h}(j_3(s,t)+j_4(s,t)+...+j_{n+3}(s,t)+j_0(s,t)-j_0(s,t)+j_1(s,t)\\ &\quad-j_1(s,t)+j_2(s,t)-j_2(s,t))\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1 - j_{n+1}(s,t) - 2tj_n(s,t)}{1-s-2t}\right) + \boldsymbol{i}\left(\frac{1 - j_{n+2}(s,t) - 2tj_{n+1}(s,t)}{1-s-2t} - j_0(s,t)\right) \\
&\quad + \boldsymbol{\varepsilon}\left(\frac{1 - j_{n+3}(s,t) - 2tj_{n+2}(s,t)}{1-s-2t} - j_0(s,t) - j_1(s,t)\right) \\
&\quad + \boldsymbol{h}\left(\frac{1 - j_{n+4}(s,t) - 2tj_{n+3}(s,t)}{1-s-2t} - j_0(s,t) - j_1(s,t) - j_2(s,t)\right) \\
&= \frac{j_1(s,t) + \boldsymbol{i}j_1(s,t) + \boldsymbol{\varepsilon}j_2(s,t) + \boldsymbol{h}j_3(s,t) + 2\boldsymbol{\varepsilon}tj_1(s,t) + 2\boldsymbol{h}tj_2(s,t)}{1-s-2t} \\
&\quad \frac{-Hj_{n+1}(s,t) - 2tHj_n(s,t) + j_0(s,t) - j_0(s,t)}{1-s-2t} \\
&= \frac{1 + Hj_0(s,t) - Hj_{n+1}(s,t) - 2tHj_n(s,t) + 2\boldsymbol{\varepsilon}tj_1(s,t) + 2\boldsymbol{h}tj_2(s,t)}{1-s-2t}.
\end{aligned}
$$

The proof of equation (3.18) is similar and is omitted. ∎

Theorem 3.8. *(Catalan's identity) Suppose that $s,t \in \mathbb{R}$ such that $s > 0, t \neq 0$ and $s^2 + 8t > 0$. Let m and r be integers such that $m \geq r \geq 0$. Then, we obtain*

I. $$Hj_{m-r}(s,t)Hj_{m+r}(s,t) - Hj_m^2(s,t) = \frac{1}{s^2+8t}[\hat{\alpha}\hat{\beta}(-2t)^m(1-\frac{\beta^r}{\alpha^r}) + \hat{\beta}\hat{\alpha}(-2t)^m(1-\frac{\alpha^r}{\beta^r})] \tag{3.19}$$

II. $$Hc_{m-r}(s,t)Hc_{m+r}(s,t) - Hc_m^2(s,t) = \hat{\alpha}\hat{\beta}(-2t)^m(\frac{\beta^r}{\alpha^r} - 1) + \hat{\beta}\hat{\alpha}(-2t)^m(\frac{\alpha^r}{\beta^r} - 1). \tag{3.20}$$

Proof. By using (3.5), we have
$Hj_{m-r}(s,t)Hj_{m+r}(s,t) - Hj_m^2(s,t)$

$$
\begin{aligned}
&= \left(\frac{\alpha^{m-r}\hat{\alpha} - \beta^{m-r}\hat{\beta}}{\alpha-\beta}\right)\left(\frac{\alpha^{m+r}\hat{\alpha} - \beta^{m+r}\hat{\beta}}{\alpha-\beta}\right) - \left(\frac{\alpha^m\hat{\alpha} - \beta^m\hat{\beta}}{\alpha-\beta}\right)^2 \\
&= \frac{-\hat{\alpha}\hat{\beta}\alpha^{m-r}\beta^{m+r} - \hat{\beta}\hat{\alpha}\beta^{m-r}\alpha^{m+r} + \hat{\alpha}\hat{\beta}\alpha^m\beta^m + \hat{\beta}\hat{\alpha}\beta^m\alpha^m}{(\alpha-\beta)^2} \\
&= \frac{\left(\hat{\alpha}\hat{\beta}(\alpha\beta)^m\left(1 - \frac{\beta^r}{\alpha^r}\right)\right) + \left(\hat{\beta}\hat{\alpha}(\beta\alpha)^m\left(1 - \frac{\alpha^r}{\beta^r}\right)\right)}{(\alpha-\beta)^2} \\
&= \frac{1}{s^2+8t}\left[\hat{\alpha}\hat{\beta}(-2t)^m\left(1 - \frac{\beta^r}{\alpha^r}\right) + \hat{\beta}\hat{\alpha}(-2t)^m\left(1 - \frac{\alpha^r}{\beta^r}\right)\right].
\end{aligned}
$$

The proof (3.20) is similar to the proof of (3.19), so it is omitted. ∎

Remark 3.9. For $r = 1$ in Theorem 3.8, we have the Cassini's identity for both $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas hybrid numbers, that is,

I. $$Hj_{m-1}(s,t)Hj_{m+1}(s,t) - Hj_m^2(s,t) = \frac{1}{s^2+8t}\left[\hat{\alpha}\hat{\beta}(-2t)^m\left(1 - \frac{\beta}{\alpha}\right) + \hat{\beta}\hat{\alpha}(-2t)^m\left(1 - \frac{\alpha}{\beta}\right)\right]$$

II. $$Hc_{m-1}(s,t)Hc_{m+1}(s,t) - Hc_m^2(s,t) = \hat{\alpha}\hat{\beta}(-2t)^m\left(\frac{\beta}{\alpha} - 1\right) + \hat{\beta}\hat{\alpha}(-2t)^m\left(\frac{\alpha}{\beta} - 1\right).$$

Theorem 3.10. *Suppose that $s,t\in\mathbb{R}$ such that $s>0, t\neq 0$ and $s^2+8t>0$. Let m and n be integers such that $m\geq n\geq 0$. Then*

$$\text{I. } Hj_m(s,t)Hj_{n+1}(s,t)-Hj_{m+1}(s,t)Hj_n(s,t)=\frac{(-2t)^n(\alpha^{m-n}\hat{\alpha}\hat{\beta}-\beta^{m-n}\hat{\alpha}\hat{\beta})}{\sqrt{s^2+8t}}, \quad (3.21)$$

$$\text{II. } Hc_m(s,t)Hc_{n+1}(s,t)-Hc_{m+1}(s,t)Hc_n(s,t)=\sqrt{s^2+8t}\,(-2t)^n(\beta^{m-n}\hat{\alpha}\hat{\beta}-\alpha^{m-n}\hat{\alpha}\hat{\beta}), \quad (3.22)$$

$$\text{III. } Hj_m(s,t)Hc_n(s,t)-Hc_m(s,t)Hj_n(s,t)=\frac{2(-2t)^n(\alpha^{m-n}\hat{\alpha}\hat{\beta}-\beta^{m-n}\hat{\alpha}\hat{\beta})}{\sqrt{s^2+8t}}. \quad (3.23)$$

Proof. If we consider (3.6) and the (2.4), we obtain
$Hj_m(s,t)Hj_{n+1}(s,t)-Hj_{m+1}(s,t)Hj_n(s,t)$

$$\begin{aligned}
&=\left(\frac{\alpha^m\hat{\alpha}-\beta^m\hat{\beta}}{\alpha-\beta}\right)\left(\frac{\alpha^{n+1}\hat{\alpha}-\beta^{n+1}\hat{\beta}}{\alpha-\beta}\right)-\left(\frac{\alpha^{m+1}\hat{\alpha}-\beta^{m+1}\hat{\beta}}{\alpha-\beta}\right)\left(\frac{\alpha^n\hat{\alpha}-\beta^n\hat{\beta}}{\alpha-\beta}\right)\\
&=\frac{\hat{\alpha}\hat{\beta}(\alpha\beta)^n(\alpha^{m-n}-\beta^{m-n})}{\alpha-\beta}\\
&=\frac{(-2t)^n(\alpha^{m-n}\hat{\alpha}\hat{\beta}-\beta^{m-n}\hat{\alpha}\hat{\beta})}{\sqrt{s^2+8t}}.
\end{aligned}$$

The proofs of equations (3.22) and (3.23) are similar and have been omitted. ∎

Equations (3.21) and (3.22) represent the d'Ocagne's identity for $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal-Lucas hybrid numbers, correspondingly.

Theorem 3.11. *Suppose that $s,t\in\mathbb{R}$ such that $s>0, t\neq 0$ and $s^2+8t>0$. Let $n\geq 0$ and $m\geq 0$ be an integer. Then*

$$Hc_n(s,t)=Hj_{n+1}(s,t)+2tHj_{n-1}(s,t),$$

$$sHj_m(s,t)+Hc_m(s,t)=2Hj_{m+1}(s,t).$$

Proof. Using (3.1) and (2.7), we have

$$\begin{aligned}
Hc_n(s,t)&=c_n(s,t)+\boldsymbol{i}c_{n+1}(s,t)+\boldsymbol{\varepsilon}c_{n+2}(s,t)+\boldsymbol{h}c_{n+3}(s,t)\\
&=j_{n+1}(s,t)+2tj_{n-1}(s,t)+\boldsymbol{i}(j_{n+2}(s,t)+2tj_n(s,t))+\boldsymbol{\varepsilon}(j_{n+3}(s,t)\\
&\quad+2tj_{n+1}(s,t))+\boldsymbol{h}(j_{n+4}(s,t)+2tj_{n+2}(s,t))\\
&=(j_{n+1}(s,t)+\boldsymbol{i}j_{n+2}(s,t)+\boldsymbol{\varepsilon}j_{n+3}(s,t)+\boldsymbol{h}(j_{n+4}(s,t))\\
&\quad+2t(j_{n-1}(s,t)+\boldsymbol{i}j_n(s,t)+\boldsymbol{\varepsilon}j_{n+1}(s,t)+\boldsymbol{h}j_{n+2}(s,t))\\
&=Hj_{n+1}(s,t)+2tHj_{n-1}(s,t).
\end{aligned}$$

By virtue of (3.1), (3.2) and (2.8), we find that

$$\begin{aligned} sHj_m(s,t) + Hc_m(s,t) &= (sj_m(s,t) + s\boldsymbol{i}j_{m+1}(s,t) + s\boldsymbol{\varepsilon}j_{m+2}(s,t) + s\boldsymbol{h}j_{m+3}(s,t)) \\ &\quad + (c_m(s,t) + \boldsymbol{i}c_{m+1}(s,t) + \boldsymbol{\varepsilon}c_{m+2}(s,t) + \boldsymbol{h}c_{m+3}(s,t)) \\ &= sj_m(s,t) + c_m(s,t) + \boldsymbol{i}(sj_{m+1}(s,t) + c_{m+1}(s,t)) \\ &\quad + \boldsymbol{\varepsilon}(sj_{m+2}(s,t) + c_{m+2}(s,t)) + \boldsymbol{h}(sj_{m+3}(s,t) + c_{m+3}(s,t)) \\ &= 2j_{m+1}(s,t) + 2\boldsymbol{i}j_{m+2}(s,t) + 2\boldsymbol{\varepsilon}j_{m+3}(s,t) + 2\boldsymbol{h}j_{m+4}(s,t) \\ &= 2Hj_{m+1}(s,t). \end{aligned}$$

■

Next, we present a matrix generator for the computation of $(s,t)-$Jacobsthal hybrid numbers and $(s,t)-$Jacobsthal-Lucas hybrid numbers, as follows:

Theorem 3.12. *Suppose that $s,t \in \mathbb{R}$ such that $s > 0, t \neq 0$ and $s^2 + 8t > 0$. Let $m \geq 0$ be an integer. Then*

$$\text{I.}\ \begin{bmatrix} Hj_{m+2}(s,t) & Hj_{m+1}(s,t) \\ Hj_{m+1}(s,t) & Hj_m(s,t) \end{bmatrix} = \begin{bmatrix} Hj_2(s,t) & Hj_1(s,t) \\ Hj_1(s,t) & Hj_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix}^m$$

$$\text{II.}\ \begin{bmatrix} Hc_{m+2}(s,t) & Hc_{m+1}(s,t) \\ Hc_{m+1}(s,t) & Hc_m(s,t) \end{bmatrix} = \begin{bmatrix} Hc_2(s,t) & Hc_1(s,t) \\ Hc_1(s,t) & Hc_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix}^m.$$

Proof. For $m = 0$, we let the matrix to the power 0 be the identity matrix. Therefore, the result can be readily obtained. Consider $m = 1$. By (3.3), we have

$$\begin{aligned} \begin{bmatrix} Hj_3(s,t) & Hj_2(s,t) \\ Hj_2(s,t) & Hj_1(s,t) \end{bmatrix} &= \begin{bmatrix} sHj_2(s,t) + 2tHj_1(s,t) & Hj_2(s,t) \\ sHj_1(s,t) + 2tHj_0(s,t) & Hj_1(s,t) \end{bmatrix} \\ &= \begin{bmatrix} Hj_2(s,t) & Hj_1(s,t) \\ Hj_1(s,t) & Hj_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix}^1. \end{aligned}$$

Therefore, the case $m = 1$ is true. Next, assume for some integer $m \geq 1$,

$$\begin{bmatrix} Hj_{m+2}(s,t) & Hj_{m+1}(s,t) \\ Hj_{m+1}(s,t) & Hj_m(s,t) \end{bmatrix} = \begin{bmatrix} Hj_2(s,t) & Hj_1(s,t) \\ Hj_1(s,t) & Hj_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix}^m.$$

By (3.3), we have

$$\begin{aligned} \begin{bmatrix} Hj_{m+3}(s,t) & Hj_{m+2}(s,t) \\ Hj_{m+2}(s,t) & Hj_{m+1}(s,t) \end{bmatrix} &= \begin{bmatrix} sHj_{m+2}(s,t) + 2tHj_{m+1}(s,t) & Hj_{m+2}(s,t) \\ sHj_{m+1}(s,t) + 2tHj_m(s,t) & Hj_{m+1}(s,t) \end{bmatrix} \\ &= \begin{bmatrix} Hj_{m+2}(s,t) & Hj_{m+1}(s,t) \\ Hj_{m+1}(s,t) & Hj_m(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix} \\ &= \begin{bmatrix} Hj_2(s,t) & Hj_1(s,t) \\ Hj_1(s,t) & Hj_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & m \end{bmatrix}^m \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix} \\ &= \begin{bmatrix} Hj_2(s,t) & Hj_1(s,t) \\ Hj_1(s,t) & Hj_0(s,t) \end{bmatrix} \begin{bmatrix} s & 1 \\ 2t & 0 \end{bmatrix}^{m+1}. \end{aligned}$$

Thus, the proof is completed. Using a similar approach, we can construct a matrix generator for (s,t)-Jacobsthal-Lucas hybrid numbers. ■

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References

[1] A. Daşdemir, On the Jacobsthal numbers by matrix method, Fen Derg. 7 (2012) 69–76.

[2] S. Falcon, On the $k-$Jacobsthal numbers, American Review of Mathematics and Statistic 2 (1) (2014) 67–77.

[3] A.F. Horadam, Jacobsthal representation numbers, Fibonacci Quarterly 34 (1996) 40–54.

[4] N. Kilic, On $k-$Jacobsthal and $k-$Jacobsthal-Lucas hybrid numbers, Journal of Discrete Mathematical Sciences and Cryptography 24 (4) (2021) 1063–1074.

[5] C. Kizilates, A new generalization of Fibonacci hybrid and Lucas hybrid numbers, Chaos, Solitons and Fractals 130 (2020) 109449.

[6] M.C.D.S. Mangueira, R.P.M. Vieira, F.R.V. Alves, P.M.M.C. Catarino, In Annales Mathematicae Silesianae 34 (2) (2020) 256.

[7] M. Özdemir, Introduction to hybrid numbers, Adv. Appl. Clifford Algebr 28 (1) (2018) Article no. 11.

[8] A. Szynal-Liana, I. Wloch, On Jacobsthal and Jacobsthal-Lucas hybrid numbers, Annales Mathematicae Silesianae 33 (2019) 276–283.

[9] Ş. Uygun, The $(s, t)-$Jacobsthal and $(s, t)-$Jacobsthal Lucas sequences, Appl. Math. Sci. 70 (2015) 3467–3476.