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On Cubic Exponential Diophantine Equations $\pm 3^x \pm a^y = z^3$

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Abstract We investigate the integer solutions of the cubic exponential Diophantine equations in the form $\pm 3^x \pm a^y = z^3$ using elementary techniques. In particular, we also prove that there are infinitely many cubic exponential Diophantine equations in that form that have no integer solutions.

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1. Introduction

An exponential Diophantine equation is an equation of the form $a^x \pm b^y = z^n$ for fixed nonnegative integers a, b, n. Diophantine equations of this type have been studied for fixed a, b and $n = 2$. One of the important theorems is known as Catalan's conjecture [1] (stated in Theorem 2.1) which was proved by Mihăilescu in 2004. After that, most mathematicians who are interested in solving the exponential Diophantine equations studied those when n is even. However, there are a few studies of the exponential Diophantine equation in the case of odd integer n. For example, Burshtein [2], in 2020, studied the solutions of the Diophantine equation $p^x + q^y = z^3$ when both p and q are distinct primes and both x and y are between 1 and 2. Recently, in 2022, Aquibo and Bacani [3] studied the equation $p^x + q^y = z^3$ where p and q are twin primes. Furthermore, in the same year Mina and Bacani [4] studied the same where p and q are cousin primes; that is, primes that differ by four.

In this article, we are interested in the exponential Diophantine equations when $n = 3$ and either a or b is 3 with integer solutions. More precisely, we study the following

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equations:

$$3^x + a^y = z^3$$
$$3^x - a^y = z^3$$
$$a^x - 3^y = z^3$$

for some conditions on the positive integer a. Most of the techniques in this article are quite simple but some of them are different from the old ones (when $n = 2$), e.g. the cubic residue.

The structure of this article was designed as follows: in the next section, we shall equip some useful lemmas to prove our main results in the next two sections. In Section 3 we study the Diophantine equation $3^x + a^y = z^3$ for some conditions of the integer a. For the next section, we also do the same manner for the other two equations in Section 4. In the last, we provide some examples and note some open problems which are related to our exponential Diophantine equations.

2. Preliminaries

In this section, we state some well-known facts and recall some definitions which are useful in this article. We first start with the well-known result in [1], called *Catalan's conjecture*.

Theorem 2.1 (Catalan's conjecture)**.** *The only integer solution* (a, b, x, y) *of the equation* $a^x - b^y = 1$ *such that* $\min\{a, b, x, y\} \geq 2$ *is* $(3, 2, 2, 3)$.

Now we shall pause to note some applications of Catalan's conjecture (Theorem 2.1).

Lemma 2.2. *Let a be a positive integer. The equation* $1 + a^y = z^3$ *has at most one integer solution for fixed a, namely* $(y, z, a) = \left(1, \sqrt[3]{a+1}, a\right)$.

Proof. It suffices to assume that $a \geq 2$ and $z \geq 2$. Suppose that $y \geq 2$. By Catalan's conjecture (Theorem 2.1), this equation has no integer solution. Hence, y must be 1 and then the integer solution is $(y, z, a) = \left(1, \sqrt[3]{a+1}, a\right)$. ■

Lemma 2.3. *Let a be a positive integer. All nonnegative integer solutions* (x, z, a) *of the equation* $a^x - 1 = z^3$ *are* $(0, 0, a)$, $(x, 0, 1)$, $\left(1, \sqrt[3]{a-1}, a\right)$, $(2, 2, 3)$.

Proof. For $z = 0, 1$, it is easy to see that all integer solutions (x, z, a) are $(0, 0, a)$, $(x, 0, 1)$, $(1, 1, 2)$. Assume that $z \geq 2$. It implies that $a \geq 2$. If $x = 0, 1$, we get that $(x, z, a) = \left(1, \sqrt[3]{a-1}, a\right)$ is the only one solution. If $x \geq 2$, we get the other solutions are $(x, z, a) = (2, 2, 3)$ by Catalan's conjecture (Theorem 2.1). Observe that $(1, 1, 2)$ can be absorbed into the form $\left(1, \sqrt[3]{a-1}, a\right)$ when $a = 2$. ■

Next, we shall discuss the prime which is congruent 1 in modulo 3. (In fact, it is congruent 1 in modulo 6.) It is well-known that the prime which is congruent to 1 in modulo 4 can be expressed in the sum of the square of two integers. In the same way, we can express the prime which is congruent 1 in modulo 3 as follows:

Theorem 2.4. *Let p be a prime number such that* $p \equiv 1 \pmod 3$. *Then there are unique positive integers L and M such that* $4p = L^2 + 27M^2$.

Proof. See [5, Proposition 8.3.2]. ■

This expression is useful to determine the existence of solutions of some equations. Recall that let two integers $m \geq 2$ and a be such that $\gcd(a, m) = 1$. We call a a *quadratic residue* of m if the congruence $x^2 \equiv a \pmod{m}$ has a solution. Similarly, we call a a *cubic residue* of m if the congruence $x^3 \equiv a \pmod{m}$ has a solution. The following determines whether 3 is a cubic residue in modulo p or not.

Theorem 2.5. *Let p be a prime number such that $p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod{3}$. Then 3 is a cubic residue modulo p if and only if $3 \mid M$.*

Proof. See [6, Proposition 7.2]. ∎

Observe that p in the above theorem can be written in that form by Theorem 2.4. Although there is a notation for the cubic residue like the Legendre symbol in quadratic case, we do not use it in this article.

3. Additive Cubic Exponential Diophantine Equations

In this section, our main goal is to find all integer solutions of the equation $3^x + p^y = z^3$ for some prime p. The answer is shown in Corollary 3.8. However, we shall study the equation in a more general form $3^x + a^y = z^3$, where a is not necessarily prime.

First, we shall consider the equation $3^x + a^y = z^3$ where a is not a multiple of 3. We start with the case where a is a cubic integer.

Theorem 3.1. *Let a be a positive integer such that $3 \nmid a$. The equation $3^x + a^{3y} = z^3$ has no nonnegative integer solution.*

Proof. Let (x, y, z, a) be a nonnegative integer solution of the equation $3^x + a^{3y} = z^3$. Then

$$3^x = z^3 - a^{3y} = (z - a^y)\left(z^2 + a^y z + a^{2y}\right).$$

Let $z - a^y = 3^k$ for some nonnegative integer k. Then

$$3^{x-k} = z^2 + a^y z + a^{2y} = 3^{2k} + 3^{k+1}a^y + 3a^{2y}.$$

It is easy to see that x is a positive integer by Lemma 2.2 and then so is k. Since $9 \leq 3^{2k} \leq 3^{x-k}$, we have $x - k \geq 2$. Then

$$3a^{2y} \equiv 3^{2k} + 3^{k+1}a^y + 3a^{2y} = 3^{x-k} \equiv 0 \pmod{9}.$$

This implies that $3 \mid a$ which contradicts the fact that $3 \nmid a$. This proof is completed. ∎

Observe that if we can force the exponent of a to be divisible by 3, we can apply the previous theorem to show that it has no nonnegative integer solution. The following is an application of Theorem 3.1.

Theorem 3.2. *Let a be a positive integer such that $3 \nmid a$ and $a \not\equiv \pm 1 \pmod{9}$. The nonnegative integer solutions (x, y, z, a) of the equation $3^x + a^y = z^3$ are $\left(0, 1, \sqrt[3]{a+1}, a\right)$ or $x = 1$.*

Proof. Assume that $x \neq 1$. For $x = 0$ or $y = 0$, the solution $(x, y, z, a) = \left(0, 1, \sqrt[3]{a+1}, a\right)$ is only one solution (for fixed a) by Lemma 2.2. From now on, we assume that $y \geq 1$ and $x \geq 2$. It is easy to see that $3 \mid y$ by the condition of the integer a. There is a positive integer y_o such that $y = 3y_o$. Applying Theorem 3.1 to the equation $3^x + a^{3y_o} = z^3$, we obtain that there is no nonnegative integer solution in this case. ∎

According to Theorem 3.2, several integer solutions may arise from the case $x = 1$. To deal with this issue, we say that an integer a has **property P** if there is a prime $p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod 6$ such that $p \mid a$ and $3 \nmid M$. This idea follows from the cubic residue (like the quadratic residue).

Theorem 3.3. *Let a be a positive integer such that $3 \nmid a$ and $a \not\equiv \pm 1 \pmod 9$. If the integer a satisfies the property P, then the equation $3^x + a^y = z^3$ has at most one integer solution and that solution is $(x, y, z, a) = \left(0, 1, \sqrt[3]{a+1}, a\right)$.*

Proof. Suppose that $x = 1$. It is easy to see that $y \neq 0$ and then we have

$$z^3 = 3 + a^y \equiv 3 \pmod p.$$

Since $3 \nmid M$, we get that 3 is not a cubic residue modulo p by Theorem 2.5. Hence, $x \neq 1$. By Theorem 3.2, the equation $3^x + a^y = z^3$ has at most one integer solution, namely $(x, y, z, a) = \left(0, 1, \sqrt[3]{a+1}, a\right)$. ∎

Now, we shall consider the case where $3 \mid a$. The main goals in this part are Theorems 3.6 and 3.7. We can find some conditions which guarantee that the equation has no integer solution. Before we go to our goal, we shall consider two equations which are the main tools for our main goals. Both theorems describe the existence of the integer solutions of the equation $1 + 3^x a^y = z^3$ when a is not multiple of 3. Now, we start the case where the integer a is odd.

Lemma 3.4. *Let a be an odd positive integer such that $3 \nmid a$ and $a \equiv 1, 3 \pmod 8$. Then the equation $1 + 3^x a^y = z^3$ has no nonnegative integer solution.*

Proof. Let (x, y, z, a) be a nonnegative integer solution of the equation $1 + 3^x a^y = z^3$. Note that z is even. We consider

$$3^x a^y = z^3 - 1 = (z-1)\left(z^2 + z + 1\right)$$

and

$$\gcd\left(z - 1, z^2 + z + 1\right) = \gcd(z - 1, 3) = 1, 3.$$

Now, we write $a = st$ where $\gcd(s, t) = 1$. If $\gcd\left(z - 1, z^2 + z + 1\right) = 1$, then

- if $z - 1 = 3^x s^y$, then $t^y = 3^{2x} s^{2y} + 3^{x+1} s^y + 3$ which implies that $x = 0$. Then by Lemma 2.2, we get that $y = 1$. This is impossible since

  $$0 \equiv z^3 = 1 + a \equiv 2, 4 \pmod 8.$$

- if $z - 1 = s^y$, then $3^x t^y = s^{2y} + 3s^y + 3$ which also implies that $x = 0$. It is impossible in the same way as the previous one.

Similarly, if $\gcd\left(z - 1, z^2 + z + 1\right) = 3$, then

- if $z - 1 = 3s^y$, then $3^{x-1} t^y = 3^2 s^{2y} + 3^2 s^y + 3$ which implies that $x = 2$ and y is odd. This is impossible since

  $$0 \equiv z^3 = 1 + 9a^y \equiv 2, 4 \pmod 8.$$

- if $z - 1 = 3^{x-1} s^y$, then $3t^y = 3^{2x-2} s^{2y} + 3^x s^y + 3$ which implies that y is odd. This is also impossible. More precisely, if x is odd, then

  $$a = st \equiv s^2 + 4s \equiv 5 \pmod 8.$$

  If x is even, then $a \equiv 7 \pmod 8$ for the same reason.

Consequently, the equation has no integer solution. ∎

Next, we also consider the equation $1 + 3^x a^y = z^3$ for the even integer a.

Lemma 3.5. *Let a be an odd positive integer such that $3 \nmid a$. For any positive integer k such that $a < 4^k$, the equation $1 + 3^x \left(2^k a\right)^y = z^3$ has no nonnegative integer solution.*

Proof. Let (x, y, z, a) be a nonnegative integer solution of the equation $1 + 3^x \left(2^k a\right)^y = z^3$. We consider

$$3^x 2^{ky} a^y = z^3 - 1 = (z-1)\left(z^2 + z + 1\right)$$

and

$$\gcd\left(z-1, z^2+z+1\right) = \gcd(z-1, 3) = 1, 3.$$

Note that z^2+z+1 is odd and write $a = st$ where $\gcd(s,t) = 1$. If $\gcd\left(z-1, z^2+z+1\right) = 1$, then

- if $z - 1 = 3^x 2^{ky} s^y$, then $t^y = 3^{2x} 2^{2ky} s^{2y} + 3^{x+1} 2^{ky} s^y + 3$ which implies that $x = 0$. By Lemma 2.2, we have $y = 1$, so $t = 2^{2k} s^2 + 3 \cdot 2^k s + 3$. It is impossible since

  $$a^3 \geq t^3 = 2^{2k} a^2 + 3 \cdot 2^k a t + 3t^2 \implies a > 4^k.$$

- if $z - 1 = 2^{ky} s^y$, then $3^x t^y = 2^{2ky} s^{2y} + 3 \cdot 2^{ky} s^y + 3$ which also implies that $x = 0$. It is also impossible by the same argument as the previous case.

Similarly, if $\gcd\left(z-1, z^2+z+1\right) = 3$, then

- if $z - 1 = 3 \cdot 2^{ky} s^y$, then $3^{x-1} t^y = 3^2 2^{2ky} s^{2y} + 3^2 2^{ky} s^y + 3$ which implies that $x = 2$. It is also impossible since

  $$a^3 \geq t^3 > 2^{2k} s^2 t^2 = 2^{2k} a^2 \implies a > 4^k.$$

- if $z - 1 = 3^{x-1} 2^{ky} s^y$, then $3t^y = 3^{2x-2} 2^{2ky} s^{2y} + 3^x 2^{ky} s^y + 3$. Similarly to the previous one, it is also impossible.

Consequently, the equation has no nonnegative integer solution. ∎

Now, we are ready to consider the equation when $a = 3^n a_*$ is a multiple of 3 where $3 \nmid a_*$. The first one is considered when the integer a_* is cubic.

Theorem 3.6. *Let n and $a = 2^k a_o$ be positive integers (where a_o is odd) satisfying one of the followings:*

- $a \equiv 1, 19 \pmod{24}$ *(i.e. $a \equiv 1 \pmod 3$ and $a \equiv 1, 3 \pmod 8$), or*
- $a_o < 4^k$ *and* $a \equiv 1 \pmod 3$.

Then the equation $3^x + \left(3^n a^3\right)^y = z^3$ has no nonnegative integer solution.

Proof. If $x < ny$, then we have

$$z^3 = 3^x \left(1 + 3^{ny-x} a^{3y}\right).$$

This implies that $1 + 3^{ny-x} a^{3y}$ is cubic. There is no nonnegative integer solution in this case by Lemma 3.4 for an odd integer a and Lemma 3.5 for an even integer a, respectively. If $x \geq ny$, then we have

$$z^3 = 3^{ny} \left(3^{x-ny} + a^{3y}\right).$$

Since $a \equiv 1 \pmod 3$, this implies that $3 \mid ny$ and $3^{x-ny} + a^{3y}$ is cubic. By Theorem 3.1, there is no nonnegative integer solution in this case. ∎

The following is similar to the previous one but the integer a_* is not cubic where $a = 3^n a_*$ (with $3 \nmid a_*$).

Theorem 3.7. *Let n and $a = 2^k a_o$ be positive integers (where a_o is odd) satisfying one of the following:*

- $a \equiv 25, 43, 49, 67 \pmod{72}$ *(i.e. $a \equiv 4, 7 \pmod 9$ and $a \equiv 1, 3 \pmod 8$), or*
- $a_o < 4^k$, *and* $a \equiv 4, 7 \pmod 9$.

If the integer a satisfies the property P, then the equation $3^x + (3^n a)^y = z^3$ has no non-negative integer solution.

Proof. If $x < ny$, then we have

$$z^3 = 3^x \left(1 + 3^{ny-x} a^y\right).$$

This implies that $1 + 3^{ny-x}a^y$ is cubic. There are no nonnegative integer solutions in this case by Lemma 3.4 for an odd integer a and Lemma 3.5 for an even integer a, respectively. If $x \geq ny$, then we have

$$z^3 = 3^{ny} \left(3^{x-ny} + a^y\right).$$

Since $a \equiv 1 \pmod 3$, this implies that $3 \mid ny$ and $3^{x-ny} + a^y$ is cubic. By Theorem 3.3, it implies that $y = 1$ and $x = n$. Hence, $(x, y, z, n, a) = \left(3\alpha, 1, 3^\alpha \sqrt[3]{a+1}, 3\alpha, a\right)$ for all nonnegative integer α. It is easy to see that if a is odd, then $a \equiv 7 \pmod 8$ since $\sqrt[3]{a+1}$ is an even integer. Hence, a is even. Then $a_o < 4^k$. Let $z_o = \sqrt[3]{a+1}$. Then

$$2^k a_o = a = z_o^3 - 1 = (z_o - 1)\left(z_o^2 + z_o + 1\right).$$

This implies that $z_o \geq 2^k + 1$. Since $z_o^3 = 2^k a_o + 1 \leq 8^k$, we have $z_o \leq 2^k$. Hence, this equation has no nonnegative integer solution. ∎

Finally, we end this section with the equation

$$3^x + p^y = z^3,$$

where p is prime. Note that for any prime p, it is easy to see that the number $\sqrt[3]{p+1}$ is an integer if and only if $p = 7$. We immediately obtain the following consequences:

Corollary 3.8. *Let p be prime.*

(1) *The equation $1 + p^y = z^3$ has a unique nonnegative integer solution $(y, z, p) = (1, 2, 7)$.*
(2) *The equation $3^x + 3^y = z^3$ has no nonnegative integer solution.*
(3) *If $p \neq 3$ and $p \not\equiv \pm 1 \pmod 9$, then the nonnegative integer solutions (x, y, z, p) of the equation $3^x + p^y = z^3$ are $(0, 1, 2, 7)$ or $x = 1$.*
(4) *If $p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod 6$ such that $3 \nmid M$, then the equation $3^x + p^y = z^3$ has a unique nonnegative integer solution, namely, $(x, y, z, p) = (0, 1, 2, 7)$.*

Proof. The first item follows from Lemma 2.2. The second item follows from Theorem 3.6 when $a = 1$. The third item follows from Theorem 3.2 and finally, the last item follows from Theorem 3.3. ∎

4. Negative Cubic Exponential Diophantine Equations

In this section, we shall do the same manner as Section 3. Note that there are bijections

$$\begin{array}{ccc} \{\text{the integer solutions of } 3^x + a^y = z^3\} & \Longleftrightarrow & \{\text{the integer solutions of } -3^x - a^y = z^3\} \\ (x,y,z,a) = (x_o, y_o, z_o, a_o) & \mapsto & (x,y,z,a) = (x_o, y_o, -z_o, a_o) \end{array}$$

and

$$\begin{array}{ccc} \{\text{the integer solutions of } 3^x - a^y = z^3\} & \Longleftrightarrow & \{\text{the integer solutions of } a^x - 3^y = z^3\} \\ (x,y,z,a) = (x_o, y_o, z_o, a_o) & \mapsto & (x,y,z,a) = (y_o, x_o, -z_o, a_o)\,. \end{array}$$

Hence, it suffices to consider only two equations $a^x - 3^y = z^3$ and $3^x - a^y = z^3$ for nonnegative integer z. First, we study the equation $a^x - 3^y = z^3$ when a is cubic.

Theorem 4.1. *Let a be a positive integer such that $3 \nmid a$. The nonnegative integer solutions of the equation $a^{3x} - 3^y = z^3$ are $(x,y,z,a) = (0,0,0,a)\,, (x,0,1,1)$.*

Proof. If $y = 0$, the nonnegative integer solutions in this case are $(0,0,0,a)\,, (x,0,1,1)$ by Lemma 2.3. Assume that $y > 0$. We consider

$$3^y = a^{3x} - z^3 = (a^x - z)\left(a^{2x} + a^x z + z^2\right).$$

Let $a^x - z = 3^k$ for some nonnegative integer k. Then we have

$$3^{y-k} = a^{2x} + a^x z + z^2 = 3a^{2x} - 3^{k+1}a^x + 3^{2k}.$$

It is easy to see that k is a positive integer and $y - k \geq 2$. This is impossible since $3 \nmid a$. ∎

Similarly to Theorem 3.2, we can replace the cubic condition with the condition $a \not\equiv \pm 1 \pmod 9$ as follows:

Theorem 4.2. *Let a be a positive integer such that $3 \nmid a$ and $a \not\equiv \pm 1 \pmod 9$. All nonnegative integer solutions (x,y,z,a) of the equation $a^x - 3^y = z^3$ are $(0,0,0,a), \left(1,0,\sqrt[3]{a-1},a\right)$ or $y = 1$.*

Proof. Assume that $y \neq 1$. If $y = 0$, then all nonnegative integer solutions in this case are

$$(0,0,0,a)\,, \left(1,0,\sqrt[3]{a-1},a\right)$$

by Lemma 2.3. If $y \geq 2$, then by the conditions of a it is easy to see that $3 \mid x$. We write $x = 3x_o$ for some nonnegative integer x_o. By Theorem 4.1, this case has no nonnegative integer solution. ∎

Next, we shall consider the equation $3^x - a^y = z^3$ when an integer a is cubic.

Theorem 4.3. *Let a be a positive integer such that $3 \nmid a$. All nonnegative integer solutions (x,y,z,a) of the equation $3^x - a^{3y} = z^3$ are $(0,0,0,a)$, $(0,y,0,1)$, $(2,0,2,a)$, $(2,y,2,1)$, $(2,1,1,2)$.*

Proof. For $x = 0, 1$, this equation has exactly two nonnegative integer solutions, namely $(0,0,0,a)\,, (0,y,0,1)$. Assume $x \geq 2$. We consider

$$3^x = z^3 + a^{3y} = (z + a^y)\left(z^2 - a^y z + a^{2y}\right).$$

Let $z + a^y = 3^k$ for some nonnegative integer k. This implies that $3^{x-k} = 3^{2k} - 3^{k+1}a^y + 3a^{2y}$. It is easy to see that k is a positive integer and $x - k \leq 1$ since $3 \nmid a$. If $x - k = 0$, then we have $a^{2y} \leq \frac{4}{3}$ which implies that $a^y = 1$. This case does not occur by Lemma 2.3. Now, we have $x - k = 1$. Then $3 = z^2 - a^y z + a^{2y}$. It implies that $a^{2y} \geq 4\left(a^{2y} - 3\right)$, i.e. $a^{2y} \leq 4$. Hence, we obtain three nonnegative integer solutions (x, y, z, a) in this case, namely, $(2, 0, 2, a), (2, y, 2, 1)(2, 1, 1, 2)$. ∎

Similarly, the cubic condition can be changed to the condition $a \not\equiv \pm 1 \pmod 9$ as follows:

Theorem 4.4. *Let a be a positive integer such that $3 \nmid a$ and $a \not\equiv \pm 1 \pmod 9$. All nonnegative integer solutions (x, y, z, a) of the equation $3^x - a^y = z^3$ are $(0, 0, 0, a), (1, 1, 1, 2)$, $(2, 0, 2, a), (2, 3, 1, 2)$.*

Proof. For $x = 0, 1$, all nonnegative integer solutions (x, y, z, a) are $(0, 0, 0, a), (1, 1, 1, 2)$. Assume that $x \geq 2$. By the conditions of the integer a, we obtain that $3 \mid y$. There is a nonnegative integer y_o such that $y = 3y_o$. Applying Theorem 4.3, the other nonnegative integer solutions (x, y, z, a) are $(2, 0, 2, a), (2, 3, 1, 2)$. ∎

In this section, we cannot use the same technique when $3 \mid a$ as in Section 3 because the conditions are too strong to find an integer a satisfying that conditions. However, we would like to fulfill when $3 \mid a$. We provide the easiest case $a = 3$ as follows:

Theorem 4.5. *The equation $3^x - 3^y = z^3$ has infinitely many nonnegative integer solutions. More precisely, all nonnegative integer solutions (x, y, z) are in the following forms $(k, k, 0)$ or $(3k + 2, 3k, 2 \cdot 3^k)$ for all nonnegative integers k.*

Proof. It is clear that if $x = y$, then all nonnegative integer solutions in this case are in the form $(x, y, z) = (x, x, 0)$. If $x > y$, then we consider

$$z^3 = 3^x - 3^y = 3^y \left(3^{x-y} - 1\right).$$

Then $3 \mid y$ and $3^{x-y} - 1$ is cubic. By Lemma 2.3, it implies that all nonnegative integer solutions (x, y, z) are $(3k + 2, 3k, 2 \cdot 3^k)$ for all nonnegative integer k. ∎

Finally, we also end this section with the equations

$$3^x - p^y = z^3 \qquad \text{and} \qquad p^x - 3^y = z^3,$$

where p is prime. Note that for prime p, it is easy to see that the number $\sqrt[3]{p-1}$ is an integer if and only if $p = 2$. We immediately obtain the following consequence.

Corollary 4.6. *Let p be prime.*

(1) The equation $p^x - 1 = z^3$ has exactly three nonnegative integer solutions (x, z, p), namely, $(0, 0, p), (1, 1, 2), (2, 2, 3)$.
(2) The equation $3^x - 3^y = z^3$ has infinitely many nonnegative integer solutions (x, y, z) in the forms $(k, k, 0)$ or $(3k + 2, 3k, 2 \cdot 3^k)$ for all nonnegative integer k.
(3) If $p \neq 3$ and $p \not\equiv \pm 1 \mod 9$, then the nonnegative integer solutions (x, y, z, p) of the equation $p^x - 3^y = z^3$ are $(0, 0, 0, p), (1, 0, 1, 2)$ or $y = 1$.
(4) If $p \neq 3$ and $p \not\equiv \pm 1 \mod 9$, then the equation $3^x - p^y = z^3$ has exactly four nonnegative integer solutions (x, y, z, p), namely, $(0, 0, 0, p), (1, 1, 1, 2), (2, 0, 2, p)$, $(2, 3, 1, 2)$.

Proof. The first item follows from Lemma 2.3; the second one follows from Theorem 4.5; the third item follows from Theorem 4.2 and the final item follows from Theorem 4.4. ∎

5. Conclusions

In this section, we shall discuss the main results in two previous sections. In Section 3, we obtain that the following equations have no integer solutions:

(1) $3^x + a^y = z^3$ when a positive integer a (where $\gcd(a,3) = 1$) has one of the followings:
- a is cubic;
- $a \equiv 2,4,5,7 \pmod 9$, $\sqrt[3]{a+1}$ is not an integer and satisfies property P.

(2) $3^x + \left(3^n a^3\right)^y = z^3$ when a positive integer $a = 2^k a_o$ (where $\gcd(a_o,6) = 1$) has one of the followings:
- $a \equiv 1,19 \pmod{24}$;
- $a_o < 4^k$ and $a \equiv 1 \pmod 3$.

(3) $3^x + (3^n a)^y = z^3$ when a positive integer $a = 2^k a_o$ (where $\gcd(a_o,6) = 1$) has one of the followings:
- $a \equiv 25,43,49,67 \pmod{72}$ satisfies property P, or
- $a_o < 4^k$, $a \equiv 1 \pmod 3$ and the integer a satisfies property P.

In the case $a = p$ is prime, we obtain that the equation $3^x + p^y = z^3$ has no integer solution when p satisfies one of the followings:

- $p = 3$, or
- $p \neq 7$ and $p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod 6$ such that $3 \nmid M$.

Now, we consider the prime p such that $3^x + p^y = z^3$ has a nonnegative integer solution. From Corollary 3.8, if $p \neq 3, 7$, $p \not\equiv \pm 1 \pmod 9$ and the equation has a nonnegative integer solution, then $x = 1$. According to the computation, there is no integer $y \geq 2$ and the prime number p such that $3 + p^y = z^3$ when $z \leq 100,000$. In the other word, if $z \leq$ 100,000, the equation $3 + p^y = z^3$ has at most one nonnegative integer solution and that solution is $(y,z,p) = \left(1, \sqrt[3]{p+3}, p\right)$. In particular, there are 4,389 primes such that the equation has a nonnegative integer solution. The following are those first twenty primes:

5	61	509	997	2,741
4,093	17,573	39,301	54,869	63,997
405,221	511,997	1,191,013	1,330,997	1,560,893
1,906,621	2,515,453	3,944,309	5,639,749	6,229,501

Hence, the following conjectures are stated from the observation:

Conjecture 5.1. Let p be prime.

(1) The equation $3 + p^y = z^3$ has at most one nonnegative integer solution for fixed prime p and the nonnegative integer solution is $(y,z,p) = \left(1, \sqrt[3]{p+3}, p\right)$.

(2) There are infinitely many primes p such that $\sqrt[3]{p+3}$ is an integer.

Furthermore, in this section, we also consider the other two equations which are ingredients to prove Theorems 3.6 and 3.7. Those equations are $1 + 3^x a^y = z^3$ (Lemma 3.4) and $1 + 3^x\left(2^k y\right) = z^3$ (Lemma 3.5). If we remove the condition "$a \equiv 1,3 \pmod 8$) in Lemma 3.4, we cannot conclude that the equation $1 + 3^x a^y = z^3$ has no nonnegative integer solutions, for example,

x	y	z	a
2	1	4	7
3	1	10	37
2	1	16	455
2	1	22	1,183
4	1	28	271

Similarly, if we remove the condition "$a < 4^k$" in Lemma 3.5, we cannot conclude that the equation $1 + 3^x\left(2^k a\right)^y = z^3$ has no nonnegative integer solutions, for example,

x	y	z	k	a
2	1	7	1	19
2	1	13	2	61
3	1	19	1	127
2	1	25	3	217
2	1	31	1	1,655

Next, we shall discuss the condition "$p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod 6$" in the property P. This condition is effective in dealing with the case $x = 1$ of the equation $3^x + a^y = z^3$. Hence, we shall note the first fifty primes $p = \frac{1}{4}\left(L^2 + 27M^2\right) \equiv 1 \pmod 6$ and $3 \nmid M$ as follows:

7	13	19	31	37	43	61	67	73	79
97	103	109	127	139	151	157	163	181	193
199	211	223	229	241	271	277	283	307	313
331	337	349	367	373	379	397	409	421	433
439	457	463	487	499	523	541	547	571	577

In Section 4, we get all nonnegative integer solutions of the following equations:

- $a^{3x} - 3^y = z^3$ and $3^x - a^{3y} = z^3$ when $3 \nmid a$;
- $3^x - a^y = z^3$ when $3 \nmid a$ and $a \not\equiv \pm 1 \pmod 9$.

By the way for the equation $a^x - 3^y = z^3$, we cannot conclude the explicit nonnegative integer solutions when $y = 1$.

In the case $a = p$ is prime, we obtain all nonnegative integer solutions of the equation $3^x - p^y = z^3$ when $p \not\equiv \pm 1 \pmod 9$. For the equation $p^x - 3^y = z^3$ we cannot conclude when $y = 1$. By the computation (again!) for $z \leq 100{,}000$, the equation $p^x - 3 = z^3$ has at most one nonnegative integer solution for fixed p and the nonnegative integer solution is $(x, z, p) = \left(1, \sqrt[3]{p-3}, p\right)$. In this situation, there are 4,453 primes such that the equation has a nonnegative integer solution. The following are those first twenty primes:

3	11	67	4,099	10,651
17,579	32,771	125,003	140,611	238,331
262,147	405,227	438,979	636,059	830,587
1,000,003	1,124,867	1,191,019	1,906,627	2,744,003

Similarly, the following conjectures are also stated from the observation:

Conjecture 5.2. Let p be prime.

(1) The equation $p^x - 3 = z^3$ has at most one nonnegative integer solution for fixed prime p and the nonnegative integer solution is $(x, z, p) = \left(1, \sqrt[3]{p-3}, p\right)$.

(2) There are infinitely many primes p such that $\sqrt[3]{p-3}$ is an integer.

However, in Section 4 we have no results for the case $3 \mid a$ except for $a = 3$ because we cannot use the same techniques used in Section 3. Furthermore, in this article, we do not study when $a \equiv \pm 1 \pmod 9$ is not cubic. These are still open problems and we invite readers to study these in the future.

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