Thai Journal of Mathematics
Volume 22 Number 1 (2024)
Pages 217–225

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Annual Meeting in Mathematics 2023

Arithmetic Functions Associated with Exponentially Odd and Exponentially Even Integers

Vichian Laohakosol[1] **and Pussadee Yangklan**[2,*]

[1] *Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand*
e-mail : fscivil@ku.ac.th

[2] *King Mongkut's University of Technology Thonburi, Ratchaburi Learning Park, Ratchaburi 70150, Thailand*
e-mail : pussadee.yang@kmutt.ac.th

Abstract An exponentially even integer is a positive integer whose prime factorization contains only even prime powers, while an exponentially odd integer is a positive integer whose prime factorization contains only odd prime powers. We investigate here the problem of counting the number of positive integers that are semi-prime to an exponentially even integer, and to an exponentially odd integer. Basic properties of the functions involved are established.

MSC: 11A25
Keywords: exponentially odd integer; exponentially even integer; unitary convolution

Submission date: 02.06.2023 / Acceptance date: 31.08.2023

1. Introduction

An arithmetic function, [1], [11], is a complex-valued function defined on the set of positive integers, $\mathbb{N}$. Over the set of arithmetic functions $\mathcal{A}$, the operations of addition $+$, and unitary convolution $\sqcup$, of two elements $f, g \in \mathcal{A}$ are defined respectively, by

$$(f+g)(n) = f(n) + g(n), \qquad (f \sqcup g)(n) = \sum_{d \| n} f(d)g(n/d),$$

where $d \| n$ signifies the *unitary divisor* d of n, i.e., $d|n$ and $\gcd(d, n/d) = 1$. The identity element with respect to the unitary convolution is the function

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

A function $f \in \mathcal{A} \setminus \{0\}$ is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. It is well-known that the structure $(\mathcal{A}, +, \sqcup)$ is a commutative ring with zero divisor and the unitary convolution of two multiplicative functions is a multiplicative

*Corresponding author.

function. Recall that the *Euler totient* function $\varphi(n)$ counts the number of positive integers $\leq n$ which are relatively prime to n, and has a representation [1, Theorem 2.3]

$$\varphi(n) = \sum_{d|n} \mu(d)n/d, \tag{1.1}$$

where $\mu \in \mathcal{A}$ is the well-known *Möbius* function. For $x \in \mathbb{R}, x \geq 1$ and $n \in \mathbb{N}$, the Legendre totient $\varphi(x,n)$, is defined to be the number of positive integers $\leq x$ which are relatively prime to n, and has a representation, [2, equation (3.8)],

$$\varphi(x,n) = \sum_{d|n} \mu(d) \lfloor x/d \rfloor ; \tag{1.2}$$

clearly, $\varphi(n,n) = \varphi(n)$. Extending the preceding discussion, the function $\varphi_k(n)$ $(k \in \mathbb{N})$ ([7], [8], [10, chapter 2, p.278], [11]) is defined to be the number of positive integers $m \leq n$ such that $\gcd(m,n)$ is kth power-free (i.e., not divisible by any kth power of an integer > 1). This function can be expressed as

$$\varphi_k(n) = \sum_{d|n} \mu_k(d)n/d, \tag{1.3}$$

where μ_k is the Klee's Möbius function defined by $\mu_k(n) = \mu(n^{1/k})$ if $n = h^k$ for some $h \in \mathbb{N}$, and 0 otherwise ; clearly, $\mu_1 = \mu,\ \varphi_1(n) = \varphi(n)$. For $a, b, k \in \mathbb{N}$, let $(a,b)_k$ denote the greatest among the common kth power divisors of a and b. If $(a,b)_k = 1$, we say that a is relatively k-prime to b. Suryanarayana [12] defined the function $\varphi_k(x,n)$ $(x \geq 1)$ to be the number of positive integers $\leq x$ which are relatively k-prime to n, and proved that

$$\varphi_k(x,n) = \sum_{d|n} \mu_k(d) \lfloor x/d \rfloor ; \tag{1.4}$$

clearly $\varphi_k(n,n) = \varphi_k(n),\ \varphi_1(x,n) = \varphi(x,n)$. For $a, b \in \mathbb{Z}$ with $b > 0$, denote by $(a,b)^*$ the greatest divisor of a which is a unitary divisor of b ; when $(a,b)^* = 1$, i.e., when the greatest divisor of a which is a unitary divisor of b is 1, the integer a is said to be **semi-prime** to b. In [2], Cohen defined *the unitary Euler totient* $\varphi^*(n)$ to be the number of positive integers $\leq n$ that are semi-prime to n, and proved that ([2, Corollary 2.4.1])

$$\varphi^*(n) = \sum_{d||n} \bar{\mu}(d)n/d = (\bar{\mu} \sqcup \zeta_1)(n) = \prod_{p|n} (p^{\nu_p(n)} - 1) \tag{1.5}$$

where $\bar{\mu}(n) = (-1)^{\omega(n)}$, $\omega(n)$ being the number of distinct prime factors of $n > 1$ and $\omega(1) = 0$, $\zeta_1(n) = n$, and $\nu_p(n)$ is the highest power of p that divides n. In the same paper, Cohen introduced the concept of an **exponentially odd** integer which is a positive integer ≥ 2 whose prime factorization takes the form $p_1^{o_1} \cdots p_s^{o_s}$ with all powers o_i being odd positive integers. Denote by E_o the set of all exponentially odd numbers; since a positive integer belongs to E_o whenever its greatest unitary square divisor is 1, it then makes sense to adopt the convention that $1 \in E_o$. In [2], Cohen proved that

$$(\bar{\mu} \sqcup \chi_o)(n) = K(n) := \begin{cases} \bar{\mu}(\sqrt{n}) = \bar{\mu}(n) = (-1)^{\omega(n)} & \text{if } n \text{ is a square} \\ 0 & \text{otherwise;} \end{cases} \tag{1.6}$$

where χ_o is the characteristic function of E_o; note that being a unitary convolution of two multiplicative functions, the function K is itself multiplicative. The following interesting identities related to the unitary Euler totient and greatest common divisor function are derived in [15]

$$\varphi^*(k) = \sum_{m \mod k} (m,k)^* \cos\left(\frac{2\pi m}{k}\right), \qquad \sum_{m \mod k} (m,k)^* = \sum_{d||k} \varphi^*(d)k/d.$$

In [6], Haukkanen defined the unitary analogue of the Legendre totient function $\varphi^*(x,n)$. For $x \geq 1$, the function $\varphi^*(x,n)$, which counts the number of positive integers $a \leq x$ such that $(a,n)^* = 1$, can be written as

$$\varphi^*(x,n) = \sum_{d||n} \bar{\mu}(d) \left\lfloor \frac{x}{d} \right\rfloor. \tag{1.7}$$

Clearly, $\varphi^*(n,n) = \varphi^*(n)$. Rao in [9] gave the following extension. For $n,m,k \in \mathbb{N}$, let $(m,n^k)_k^*$ denote the largest unitary divisor of n^k that divides m and is a kth power; note that $(m,n)_1^* = (m,n)^*$. Rao's extension of φ^* is the function $\varphi_k^*(n)$ defined to be the number of positive integers $m \leq n^k$ such that $(m,n^k)_k^* = 1$. Its convolution representation is

$$\varphi_k^*(n) = \sum_{d||n} \bar{\mu}(d) \, (n/d)^k; \tag{1.8}$$

clearly, $\varphi_1^*(n) = \varphi^*(n)$. An integer $n \in \mathbb{N}$ is said to be *k-full* ($k \in \mathbb{N}$, $k \geq 2$) if it has no prime factor of multiplicity $< k$, and denote by Q_k the set of k-full integers. In 1963, Cohen [3] introduced the function $\hat{\varphi}_k(n)$ which counts the number of integers $\leq n$ that are relatively prime to the maximal divisor of n contained in Q_k. In 1964, Cohen [4] defined $\bar{\varphi}_k^*(n)$ ($k \in \mathbb{N}$) to be the number of integers $\leq n$ which are semi-prime to the maximal unitary divisor of n contained in Q_k. For $m,n \in \mathbb{N}$, let $(m,n)^{**}$ denote the greatest common unitary divisor of m and n, i.e., $(m,n)^{**} = \max\{d \in \mathbb{N} \,|\, d||m,\ d||n\}$. Note that $(m,n)^{**} \leq (m,n)^* \leq \gcd(m,n)$, and it is known, [13], that $\varphi(n) \leq \varphi^*(n) \leq \varphi^{**}(n)$. In [5], the bi-unitary analogue of Euler's totient function $\varphi^{**}(n)$ is defined to be the number of positive integers $m \leq n$ such that $(m,n)^{**} = 1$. Its convolution representation is

$$\varphi^{**}(n) = \sum_{d||n} \overline{\mu}(d)\varphi(d, n/d). \tag{1.9}$$

Pushing further these earlier works as well as complementing Cohen's concept of an exponentially odd integer, we introduce here the notion of an *exponentially even* integer which is defined to be a positive integer ≥ 2, whose prime factorization contains only even prime powers, i.e., a perfect square ≥ 2. Adding to the set E_o of exponentially odd integers, let E_e be the set of all exponentially even integers. *We adopt the convention that* 1 *belongs to both* E_o *and* E_e. For a positive integer $n > 1$, its unique prime factorization can be written according to odd and even powers of primes (henceforth referred to as its unique **odd-even prime representation**) under the form

$$n = p_1^{o_1} \cdots p_s^{o_s} \cdot q_1^{e_1} \cdots q_r^{e_r},$$

where $p_1, \ldots, p_s, q_1, \ldots, q_r$ are distinct primes, $o_1, \ldots, o_s$ are odd positive integers, and $e_1, \ldots, e_r$ are even positive integers; we adopt the convention that if there is no such odd or even prime powers, i.e., if there is no such s or r, the corresponding part is taken to be 1. Define the *e-odd part* of n by

$$\alpha(n) = \begin{cases} p_1^{o_1} \cdots p_s^{o_s} & \text{if there is such an } s \in \mathbb{N}, \\ 1 & \text{if there is no such } s, \end{cases}$$

and the *e-even part* of n by

$$\beta(n) = \begin{cases} q_1^{e_1} \cdots q_r^{e_r} & \text{if there is such an } r \in \mathbb{N}, \\ 1 & \text{if there is no such } r. \end{cases}$$

Clearly, $\beta(n) = n$ when n is perfect square, while $\alpha(n) = n$ when n is itself exponentially odd, and $\alpha(1) = 1 = \beta(1)$, $\gcd(\alpha(n), \beta(n)) = 1$, $n = \alpha(n)\beta(n)$. The concepts of exponentially odd and exponentially even integers are closely connected to that of maximal unitary divisors. Indeed, it is easy to see that for each $n \in \mathbb{N}$, its e-odd part $\alpha(n)$ ($\in E_o$) is the maximal unitary divisor of n that is a perfect square, and that its e-even part $\beta(n)$ ($\in E_e$) is the maximal unitary divisor of n that is exponentially odd. Recently in [14], we defined the *odd-phi* function, $\varphi^*(\alpha; n)$ to be the number of integers $\leq n$ which are semi-prime to $\alpha(n)$, and analogously the *even-phi* function, $\varphi^*(\beta; n)$ to be the number of integers $\leq n$ which are semi-prime to $\beta(n)$. It is shown in [14] that both of these functions are multiplicative and have convolution representations of the form

$$\varphi^*(\alpha; n) = \sum_{d||n} K(d)\, n/d, \qquad \varphi^*(\beta; n) = \sum_{d||n} T(d)\, n/d, \tag{1.10}$$

where K is as in (1.6) and the multiplicative function T is defined at prime powers by

$$T(p^a) = \begin{cases} -1 & \text{if } a \text{ is odd}, \\ 0 & \text{if } a \text{ is even} \geq 2. \end{cases}$$

For $n \in \mathbb{N}$, we clearly see that

$$T(n) = \begin{cases} \bar{\mu}(n) = (-1)^{\omega(n)} & \text{if } n \in E_o, \\ 0 & \text{if } n \notin E_o. \end{cases} \tag{1.11}$$

The objective of the present work is to establish basic algebraic properties of the following two counting functions related and/or complementing those of (1.1), (1.2), (1.3), (1.4), (1.5), (1.7), (1.8), (1.9), (1.10), namely,

- $\varphi^*(\alpha; x, n)$ which counts the number of positive integers $\leq x$ that are semi-prime to $\alpha(n)$, the e-odd part of n and $\varphi^*(\alpha; n, n) =: \varphi^*(\alpha; n)$;
- $\varphi^*(\beta; x, n)$ which counts the number of positive integers $\leq x$ that are semi-prime to $\beta(n)$, the e-even part of n and $\varphi^*(\beta; n, n) =: \varphi^*(\beta; n)$.

2. Basic Properties

Recall that for $a \in \mathbb{Z}, b \in \mathbb{N}$, the symbol $(a, b)^*$ refers to the greatest divisor of a which is a unitary divisor of b. Clearly, if $\gcd(a, b) = 1$, then $(a, b)^* = 1$, but the converse is not necessarily true. For example, $(2, 12)^* = 1$, but $\gcd(2, 12) = 2$. The next three lemmas collect some basic properties of this function.

Lemma 2.1. *Let $a, b \in \mathbb{N}$ whose unique prime representations are so arranged as*

$$a = k_a p_1^{B_1} \cdots p_u^{B_u} q_1^{y_1} \cdots q_v^{y_v}, \quad b = k_b p_1^{b_1} \cdots p_u^{b_u} q_1^{Y_1} \cdots q_v^{Y_v},$$

where p_i, q_j are distinct primes, and $k_a, k_b \in \mathbb{N}$ are such that

$$1 = \gcd(k_a, k_b) = \gcd(k_a, p_i) = \gcd(k_a, q_j) = \gcd(k_b, p_i) = \gcd(k_b, q_j),$$
$$b_i \leq B_i, \; y_j < Y_j \qquad (i = 1, \ldots, u; \; j = 1, \ldots, v).$$

These two unique prime factorization of a and b are written with the same set of primes that actually appear in any of a or b. This allows zero exponent for some prime that *appears in one but not both factorization and also allow the integers a and b to be 1. Then*

$$\text{A)} \quad (a,b)^* = \begin{cases} p_1^{b_1}\cdots p_u^{b_u} & \text{if there is such an integer } u,\\ 1 & \text{if there is no such an integer } u.\end{cases}$$

$$\text{B)} \quad (a,b)^* = \ell \Longleftrightarrow (a/\ell, b/\ell)^* = 1$$

Proof. A) If $(1,m)^* = (1,n)^*$, it is obviously that $(1,mn)^* = 1$ for all $m,n \in \mathbb{N}$. If either a or b is equal to 1, then the result is trivial. Assume now that both a and b are > 1. If there is no such u, then $a = k_a q_1^{y_1}\cdots q_v^{y_v}$, $b = k_b q_1^{Y_1}\cdots q_v^{Y_v}$. Since each $Y_j > y_j$, it follows that $(a,b)^* = 1$. If there is such a u, then $a = k_a p_1^{B_1}\cdots p_u^{B_u} q_1^{y_1}\cdots q_v^{y_v}$, $b = k_b p_1^{b_1}\cdots p_u^{b_u} q_1^{Y_1}\cdots q_v^{Y_v}$. Since each $b_i \le B_i$ and each $y_j < Y_j$, we easily see that $(a,b)^* = p_1^{b_1}\cdots p_u^{b_u}$.

B) follows immediately from part A). ■

Lemma 2.2. *Let $a,m,n \in \mathbb{N}$.*

A) If $(a,m)^* = 1 = (a,n)^*$, then $(a,mn)^* = 1$.

B) If $\gcd(m,n) = 1$ and $(a,mn)^* = 1$, then $(a,m)^* = 1 = (a,n)^*$.

C) We have

$$\gcd(a,m) = (a,m)^* \Longleftrightarrow \text{either } i)\ a \text{ or } m \text{ equals } 1$$
$$\text{or } ii)\ \operatorname{ord}_p(a) \ge \operatorname{ord}_p(m) \text{ for all primes } p \mid \gcd(a,m).$$

Proof. For $a = 1$, the three result are initial. Henceforth, assume $a > 1$ let $a = q_1^{\alpha_1}\cdots q_v^{\alpha_v} > 1$ be its unique prime representation where q_j's are distinct primes and each $\alpha_j \in \mathbb{N}$.

A) For the case $(1,m)^* = 1 = (1,n)^*$, the proof is trivial. If $(a,m)^* = 1 = (a,n)^*$, then Lemma 2.1 indicates that "$m = k_m\, q_{m_1}^{A_1}\cdots q_{m_w}^{A_w}$ or $m = k_m$" and "$n = k_n\, q_{n_1}^{B_1}\cdots q_{n_u}^{B_u}$ or $n = k_n$" for distinct $m_i, n_j \in \{1,\ldots,v\}$ and $A_i > \alpha_i$, $B_j > \alpha_j$, $\gcd(k_m,q_i) = \gcd(k_n,q_j) = 1$. The product mn either contains a prime factor q_j of power $> \alpha_j$, or does not contain such a prime factor. In either case, Lemma 2.1 implies that $(a,mn)^* = 1$.

B) If $\gcd(m,n) = 1$ and $(1,mn)^* = 1$ then $(1,m)^* = 1 = (1,n)^*$. If $\gcd(m,n) = 1$ and $(a,mn)^* = 1$, then by Lemma 2.1 "$m = k_m\, q_1^{A_1}\cdots q_y^{A_y}$ or $m = k_m$" and "$n = k_n\, q_{y+1}^{A_{y+1}}\cdots q_z^{A_z}$ or $n = k_n$" for some $y,z \in \{1,\ldots,v\}$ with $y + z \le v$ and $A_i > \alpha_i$, $\gcd(k_m,q_i) = \gcd(k_n,q_j) = \gcd(k_m,k_n) = 1$. The product mn either contains a prime factor q_j of power $> \alpha_j$, or does not contain such a prime factor. In either case, Lemma 2.1 implies that $(a,m)^* = 1 = (a,n)^*$.

C) i) If $m = 1$, the result is trivial.
ii) Assume that $\gcd(a,m) = (a,m)^* = p_1^{b_1}\cdots p_v^{b_v}$ for dictinct primes p_i and each $b_i \in \mathbb{N}$. By Lemma 2.1, we have $a = k_a p_1^{B_1}\cdots p_v^{B_v}$ and $m = k_m p_1^{b_1}\cdots p_v^{b_v}$, where $b_i \le B_i$, $\gcd(k_a,k_m) = \gcd(p_i,k_a) = \gcd(p_i,k_m) = 1$ for $i \in \{1,\ldots,v\}$ yielding $\operatorname{ord}_p(a) \ge \operatorname{ord}_p(m)$. Conversely, assume that $\operatorname{ord}_p(a) \ge \operatorname{ord}_p(m)$ for all primes $p \mid \gcd(a,m)$. Then we can write $a = \ell_a p_1^{B_1}\cdots p_v^{B_v}$ and $m = \ell_m p_1^{b_1}\cdots p_v^{b_v}$, where $B_j \ge b_j$, $\gcd(p_j,\ell_a) = \gcd(p_j,\ell_m) = 1$ yielding $\gcd(a,m) = p_1^{b_1}\cdots p_v^{b_v} = (a,m)^*$. ■

Lemma 2.3. *For a fixed $a \in \mathbb{N}$, the function $(a,n)^*$ is multiplicative in $n \in \mathbb{N}$.*

Proof. Let the unique prime representations (with corresponding parameters as given in Lemma 2.1) of m, n and a be

$$m = k_m p_1^{A_1} \cdots p_u^{A_u} q_1^{b_1} \cdots q_v^{b_v}, \quad n = k_n r_1^{C_1} \cdots r_y^{C_y} s_1^{d_1} \cdots s_z^{d_z}$$
$$a = k_a p_1^{a_1} \cdots p_u^{a_u} q_1^{B_1} \cdots q_v^{B_v} r_1^{c_1} \cdots r_y^{c_y} s_1^{D_1} \cdots s_z^{D_z},$$

If $\gcd(m,n) = 1$, then by Lemma 2.1 A)

$$(a,m)^* = q_1^{b_1} \cdots q_v^{b_v}, \quad (a,n)^* = s_1^{d_1} \cdots s_z^{d_z} \quad \text{and} \quad (a,mn)^* = q_1^{b_1} \cdots q_v^{b_v} s_1^{d_1} \cdots s_z^{d_z}$$

showing at once that $(a,mn)^* = (a,m)^*(a,n)^*$. ∎

Some interesting unitary connections between the set of exponentially even and exponentially odd integers are now established. To do so, define the constant 1-function $U \in \mathcal{A}$ by $U(n) = 1$ $(n \in \mathbb{N})$; clearly U is a multiplicative function.

Lemma 2.4. *Let E_e, E_o be the respective sets of exponentially even and exponentially odd integers, whose characteristic functions are, respectively, χ_e, χ_o. Then*

$$\chi_e \sqcup \chi_o = U, \quad T \sqcup U = \chi_e.$$

Proof. To verify the first relation, we start with $(\chi_e \sqcup \chi_o)(1) = \chi_e(1)\chi_o(1) = 1 = U(1)$. For $n > 1$, since χ_e and χ_o are multiplicative, we need to show that $(\chi_e \sqcup \chi_o)(p^a) = U(p^a)$. This follows from

$$(\chi_e \sqcup \chi_o)(p^a) = \chi_e(1)\chi_o(p^a) + \chi_e(p^a)\chi_o(1) = \begin{cases} 1 + 0 = 1 & \text{if } a \text{ is odd,} \\ 0 + 1 = 1 & \text{if } a \text{ is even} \end{cases} = U(p^a).$$

To prove the second relation, we first check $(T \sqcup U)(1) = T(1)U(1) = 1 = \chi_e(1)$. For $n > 1$, since T and U are multiplicative, it suffices to verify the relation at prime powers p^a, i.e., $(T \sqcup U)(p^a) = \chi_e(p^a)$. This follows from

$$(T \sqcup U)(p^a) = T(1) + T(p^a) = \begin{cases} 1 - 1 = 0 & \text{if } a \text{ is odd} \\ 1 + 0 = 1 & \text{if } a \text{ is even} \end{cases} = \chi_e(p^a).$$

∎

The functions T and K, as defined in (1.11) and (1.6), respectively, satisfy the following inversion formulae.

Theorem 2.5. *Let $f, g \in \mathcal{A}$. Then*

$$f = g \sqcup T \iff g = f \sqcup \chi_o, \quad f = g \sqcup K \iff g = f \sqcup \chi_e.$$

Proof. By Lemma 2.4, we have

$$f = g \sqcup T \iff f \sqcup U = g \sqcup \chi_e \iff f \sqcup \chi_o \sqcup U = g \sqcup U \iff f \sqcup \chi_o = g,$$

which is the first relation. The proof of the second relation is similar, i.e., from Lemma 2.4 and (1.6), we have

$$f = g \sqcup K \iff f \sqcup U = g \sqcup \chi_o \iff f \sqcup \chi_e \sqcup U = g \sqcup U \iff f \sqcup \chi_e = g.$$

∎

3. Counting Formulae

Unitary convolution formulae in the last theorem enable us to relate the functions T and K to the two counting functions $\varphi^*(\alpha; x, n)$ and $\varphi^*(\beta; x, n)$ mentioned in the introduction.We first note, [14], that $\varphi^*(\alpha; x, n)$ and $\varphi^*(\beta; x, n)$ are not multiplicative in n, but the functions $\varphi^*(\alpha; n)$ and $\varphi^*(\beta; n)$ are multiplicative.

Theorem 3.1. *For $n \in \mathbb{N}$ and real number $x \geq 2$, we have*

$$\varphi^*(\alpha; x, n) = \sum_{d||n} T(d) \lfloor x/d \rfloor, \quad \varphi^*(\beta; x, n) = \sum_{d||n} K(d) \lfloor x/d \rfloor.$$

Proof. Since the e-odd part $\alpha(1) = 1$, from its definition we get

$$\varphi^*(\alpha; x, 1) := \sum_{\substack{m \leq x \\ (m, \alpha(1))^* = 1}} 1 = \sum_{\substack{m \leq x \\ (m, 1)^* = 1}} 1 = \lfloor x \rfloor = \sum_{d||1} T(d) \lfloor x/d \rfloor.$$

Let n be a positive integer > 1 with odd-even prime representation $n = p_1^{o_1} \cdots p_s^{o_s} q_1^{e_1} \cdots q_r^{e_r}$. Then $\alpha(n) = p_1^{o_1} \cdots p_s^{o_s}$. By the inclusion-exclusion principle, we have

$$\varphi^*(\alpha; x, n) := \sum_{\substack{m \leq x \\ (m, \alpha(n))^* = 1}} 1 = \sum_{\substack{m \leq x \\ (m, p_1^{o_1} \cdots p_s^{o_s})^* = 1}} 1 = \lfloor x \rfloor - \sum_{\substack{m \leq x \\ (m, p_1^{o_1} \cdots p_s^{o_s})^* \neq 1}} 1 \tag{3.1}$$

$$= \lfloor x \rfloor - \sum_{i=1}^{s} \left\lfloor \frac{x}{p_i^{o_i}} \right\rfloor + \sum_{\substack{i,j=1 \\ i<j}}^{s} \left\lfloor \frac{x}{p_i^{o_i} p_j^{o_j}} \right\rfloor + \cdots + (-1)^s \left\lfloor \frac{x}{p_1^{o_1} \cdots p_s^{o_s}} \right\rfloor. \tag{3.2}$$

Since $T(p_1^{o_1} \cdots p_s^{o_s} q_1^{e_1} \cdots q_r^{e_r}) = 0$ if there is such an integer $r \in \mathbb{N}$, we get

$$\sum_{d||n} T(d) \lfloor x/d \rfloor = \sum_{d||p_1^{o_1} \cdots p_s^{o_s}} T(d) \lfloor x/d \rfloor \tag{3.3}$$

$$= T(1)\lfloor x \rfloor + \sum_{i=1}^{s} T(p_i^{o_i}) \left\lfloor \frac{x}{p_i^{o_i}} \right\rfloor + \sum_{\substack{i,j=1 \\ i<j}}^{s} T(p_i^{o_i} p_j^{o_j}) \left\lfloor \frac{x}{p_i^{o_i} p_j^{o_j}} \right\rfloor + \cdots \tag{3.4}$$

$$+ T(p_1^{o_1} \cdots p_s^{o_s}) \left\lfloor \frac{x}{p_1^{o_1} \cdots p_s^{o_s}} \right\rfloor \tag{3.5}$$

$$= \lfloor x \rfloor - \sum_{i=1}^{s} \left\lfloor \frac{x}{p_i^{o_i}} \right\rfloor + \sum_{\substack{i,j=1 \\ i<j}}^{s} \left\lfloor \frac{x}{p_i^{o_i} p_j^{o_j}} \right\rfloor + \cdots + (-1)^s \left\lfloor \frac{x}{p_1^{o_1} \cdots p_s^{o_s}} \right\rfloor = \varphi^*(\alpha; x, n). \tag{3.6}$$

To verify the second relation, we first look at the case $n = 1$. Since $\beta(1) = 1$, we have

$$\varphi^*(\beta; x, 1) = \sum_{\substack{m \leq x \\ (m, \beta(1))^* = 1}} 1 = \sum_{\substack{m \leq x \\ (m, 1)^* = 1}} 1 = \lfloor x \rfloor = \sum_{d||1} K(d) \lfloor x/d \rfloor.$$

Let n be a positive integer > 1 with odd-even prime representation $n = p_1^{o_1} \cdots p_s^{o_s} q_1^{e_1} \cdots q_r^{e_r}$. Then $\beta(n) = q_1^{e_1} \cdots q_r^{e_r}$. Following the same proof as in (3.1) - (3.5), replacing odd prime parts by even prime parts, we obtain

$$\begin{aligned}\varphi^*(\beta;x,n) &= \lfloor x\rfloor - \sum_{i=1}^{r}\left\lfloor \frac{x}{q_i^{e_i}}\right\rfloor + \sum_{\substack{i,j=1\\ i<j}}^{r}\left\lfloor \frac{x}{q_i^{e_i}q_j^{e_j}}\right\rfloor + \cdots + (-1)^r\left\lfloor \frac{x}{q_1^{e_1}\cdots q_r^{e_r}}\right\rfloor \\ &= \sum_{d||n} K(d)\lfloor x/d\rfloor .\end{aligned}$$

■

Remark 3.2. I. Since $\varphi^*(\alpha;n,n) = \varphi^*(\alpha;n)$ and $\varphi^*(\beta;n,n) = \varphi^*(\beta;n)$, Theorem 3.1 gives

$$\varphi^*(\alpha;n) = \sum_{d||n} T(d)n/d, \quad \text{and} \quad \varphi^*(\beta;n) = \sum_{d||n} K(d)n/d,$$

the identities that have already appeared in [14, Theorem 3.4].

II. Using the definitions of the odd-phi function, the even-phi function and the unitary Euler's totient, we see that $\varphi^*(\alpha;n) = \varphi^*(\alpha(n))$ and $\varphi^*(\beta;n) = \varphi^*(\beta(n))$. These relations enable to obtain an alternative proof of Theorem 3.1 as follows:

$$\varphi^*(\alpha;x,n) = \varphi^*(x,\alpha(n)) = \sum_{d||\alpha(n)} (-1)^{\omega(d)}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{\substack{d||\alpha(n)\\ d\in E_o}} (-1)^{\omega(d)}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{d||n} T(d)\left\lfloor \frac{x}{d}\right\rfloor,$$

$$\varphi^*(\beta;x,n) = \varphi^*(x,\beta(n)) = \sum_{d||\beta(n)} (-1)^{\omega(d)}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{\substack{d||\alpha(n)\\ d\in E_e}} (-1)^{\omega(d)}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{d||n} K(d)\left\lfloor \frac{x}{d}\right\rfloor.$$

Next, we prove some results related to (1.5) by specializing the values of n.

Corollary 3.3. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $x \geq 2$.*
i) If $n \in E_o$, then

$$\varphi^*(\alpha;x,n) = \sum_{d||n} \bar{\mu}(d)\lfloor x/d\rfloor, \ \varphi^*(\alpha;n) = \varphi^*(n); \ \varphi^*(\beta;x,n) = \lfloor x\rfloor, \ \varphi^*(\beta;n) = n.$$

ii) If $n \in E_e$, then

$$\varphi^*(\alpha;x,n) = \lfloor x\rfloor, \ \varphi^*(\alpha;n) = n; \quad \varphi^*(\beta;x,n) = \sum_{d||n} \bar{\mu}(d)\lfloor x/d\rfloor, \ \varphi^*(\beta;n) = \varphi^*(n).$$

Proof. The results hold trivially for $n = 1$.

i) For $n > 1$, since $n \in E_o$, we have $\alpha(n) = n, \beta(n) = 1, T(n) = \bar{\mu}(n)$ and $K(n) = 0$, and the assertions are immediate consequences of Theorem 3.1.

The proof for ii) is similar. ■

Corollary 3.4. *We have*

$$\varphi^*(\alpha;n)\varphi^*(\beta;n) = n\varphi^*(n).$$

Proof. Clearly, $\varphi^*(\alpha; 1)\varphi^*(\beta; 1) = 1 \times 1 = 1 \times \varphi^*(1)$. Let $n > 1$ with odd-even prime representation $n = p_1^{o_1} \cdots p_s^{o_s} q_1^{e_1} \cdots q_r^{e_r}$. By [14, Lemma 3.1] and the relation (1.5), we have

$$\varphi^*(\alpha; n) = \prod_{i=1}^{s} (p_i^{o_i} - 1)\, q_1^{e_1} \cdots q_r^{e_r}, \ \varphi^*(\beta; n) = p_1^{o_1} \cdots p_s^{o_s} \prod_{j=1}^{r} \left(q_j^{e_j} - 1\right),$$

and the desired identity is immediate. ∎

Acknowledgements

This work was supported by KMUTT Research Fund.

References

[1] T.M. Apotsol, Introduction to Analytic Number Theory, Springer, New York, 1976.

[2] E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, Math. Zeit. 74 (1960) 66–80.

[3] E. Cohen, Some analogues of certain arithmetical functions, Riv. Mat. Univ. Parma. 4 (1963) 115–125.

[4] E. Cohen, Arithmetical notes. X. A class of totients, Proc. Amer. Math. Soc. 15 (1964) 534–539.

[5] P. Haukkanen, Basic properties of the bi-unitary convolution and the semi-unitary convolution, Indian Journal of Mathematics 40 (3) (1998) 305–315.

[6] P. Haukkanen, On an inequality related to the Legendre totient function, J. Inequalities in Pure and Applied Math. 3 (3) (2002) Article no. 37.

[7] V.L. Klee, A generalization of Euler's $\varphi-$function, Amer. Math. Monthly 55 (1948) 358–359.

[8] K. Nageswara Rao, A note of an extension of Euler's function, Math. Student 29 (1961) 33–35.

[9] K. Nageswara Rao, On the unitary analogue of certain totients, Monatsh. Math. 70 (1966) 149–154.

[10] J. Sandor, B. Crstici, Handbook of Number Theory II, Kluwer, Dordrecht, 2004.

[11] R. Sivaramakrishnan, Classical Theory of Arithmetic Functions, Marcel Dekker, New York-Basel, 1989.

[12] D. Suryanarayana, The number of k-ary divisors of an integer, Monatsh. Math. 72 (1968) 445–450.

[13] L. Toth, On the bi-unitary analogues of Euler's arithmetical function and the gcd-sum function, Journal of Integer Sequences 12 (2009) 1–10.

[14] P. Yangklan, V. Laohakosol, P. Ruengsinsub, S. Mavecha, Unitary analogues of some arithmetic functions, Thai J. Math. (Special Issue: The 14th IMT-GT ICMSA 2018) (2020) 127–134.

[15] P. Yangklan, V. Laohakosol, Generalized unitary convolution, Ramanujan sums and applications, Lecture Notes in Networks and Systems 518 (2022) 69–92.