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Some Properties of a Trinomial Random Walk Conditioned on End Points

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Abstract Given a sequence of trinomial random variables $\{X_i\}_{i=1}^{\infty}$ and define $S_n = \sum_{i=1}^{n} X_i$ and $S_0 = 0$, we study some properties of X_i conditioned on $S_n = 0$. The mathematical expressions of expectation, variance and covariance were investigated. We found that the a finite sequence $(X_1, X_2, \ldots, X_n)$ conditioned on $S_n = 0$ is exchangeable. Moreover, the expectation of X_i is zero and the covariance of X_i and X_j where $i \neq j$ is nonpositive. Furthermore, we extend the previous setting to a rescaled trinomial random walk. Some properties on the extension were derived.

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1. Introduction

One of the most basic stochastic processes is called a simple random walk which has many applications to areas of physics, economics, and finance. For example, the application of random walks to finance and economics was summarized in [9] where the author used a continuous time random walk to model the ruin theory of insurance companies and also to understand the dynamics of prices in financial markets. The random walk models give us a first approximation to the theory of Brownian motion. Let $\{X_n \,, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables such that

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = p, \quad 0 < p < 1,$$

for all i. The sequence $\{S_n \,, n \geq 0\}$ defined by $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ is called the simple random walk. Some properties of X_i conditioned on $S_{2n} = 0$ have been studied extensively. Moreover, an invariance principle for the random walk conditioned on $S_{2n} = 0$ has been studied by many authors in the literature (see [4, 7] for more details).

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Let restrict our consideration to a trinomial random variable. For $0 < p < \frac{1}{2}$, the trinomial random walk $\{S_n\,,\, n \geq 0\}$ can be written as $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$, where X_i, $i = 1, 2, \ldots$ are i.i.d. random variables such that

$$X_i = \begin{cases} -1, & \text{with probability } p \\ 0, & \text{with probability } 1 - 2p \\ 1, & \text{with probability } p. \end{cases}$$

The main purpose of this paper is to explore properties of a finite sequence $(X_1, \ldots, X_n)$ conditioned on $S_n = 0$. We then investigate properties of a rescaled trinomial random conditioned on $S_n = 0$.

The rest of the paper is organized as follows. Section 2 covers all necessary background for this field where the main results are discussed in Section 3.

2. Preliminaries

Let us first state the definition of negative quadrant dependency which was introduced by Lehmann [6] in 1966.

Definition 2.1. A pair of random variables, X and Y, is said to be *negative quadrant dependent (NQD)* if

$$\mathbb{P}\left(X \leq x, Y \leq y\right) \leq \mathbb{P}\left(X \leq x\right) P\left(Y \leq y\right), \ \forall x, y \in \mathbb{R}.$$

The next definition is a concept of dependency which is stronger than NQD. This definition was introduced by Newman [8].

Definition 2.2. A sequence of random variables X_n is *linearly negative quadrant dependent (LNQD)* if for any disjoint $A, B \subseteq \mathbb{N}$, and a positive sequence $\{\lambda_i\}$, the random variables $\Sigma_{i \in A} \lambda_i X_i$ and $\Sigma_{j \in B} \lambda_j X_j$ are negative quadrant dependent.

The strongest dependency, which was introduced by Joag-Dev and Proschan (1983) in [[5], Definition 2.1], is defined as follows.

Definition 2.3. A finite set of random variables $\{X_1, X_2, \ldots, X_n\}$ is said to be *negatively associated* if for any pair A and B of disjoint subsets of $\{1, 2, \ldots, n\}$ the following holds

$$\mathrm{Cov}\big(f(X_i,\ i \in A), g(X_j,\ j \in B)\big) \leq 0$$

for all coordinate-wise increasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$, whenever the respective covariances exist. An infinite sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be *negatively associated* if every finite subfamily is negatively associated.

Next, let us recall the statistical concept of finite exchangeability. The exchangeability was introduced by de Finetti and he proved his famous representation theorem called de Finettis Theorem for an infinite sequence of binary random variables (see [2] for more details).

Definition 2.4. A finite sequence $(X_1, X_2, \ldots, X_n)$ of random variables is *exchangeable* if

$$(X_1, \ldots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)})$$

for all $\pi \in S(n)$ where $S(n)$ is the group of permutations of $\{1, 2, \ldots, n\}$ and $\stackrel{d}{=}$ denotes equality in distribution. An infinite sequence of random variables is exchangeable if any finite sub-sequence is exchangeable.

We next recall the definition of Pólya frequency function of order 2 (see [3] and [10] for more details).

Definition 2.5. A probability density function on the real line $f(x)$ is said to be a *Pólya frequency function of order* 2 *(PF_2)* if $x_2 \geq x_1, z_2 \geq z_1$ implies

$$\det \begin{pmatrix} f(x_1 - z_1) & f(x_1 - z_2) \\ f(x_2 - z_1) & f(x_2 - z_2) \end{pmatrix} \geq 0.$$

Similarly, the discrete analog for an integer-valued random variable is defined as follows.

Definition 2.6. An integer-valued random variable X is said to be PF_2 if for all pair of integers $m_2 \geq m_1, n_2 \geq n_1$ implies

$$\det \begin{pmatrix} \mathbb{P}(X = m_1 - n_1) & \mathbb{P}(X = m_1 - n_2) \\ \mathbb{P}(X = m_2 - n_1) & \mathbb{P}(X = m_2 - n_2) \end{pmatrix} \geq 0.$$

The next result shows a connection between PF_2 densities and negative association.

Theorem 2.7. [10] *Let $X_1, X_2, \ldots, X_n$ be n independent random variables with PF_2 densities. Define $S_n = \sum\limits_{i=1}^{n} X_i$. Then $X_1, X_2, \ldots, X_n$ conditioned on $S_n = s$ are negatively associated for almost all s.*

Let $\{X_n\,,\, n \geq 1\}$ be a sequence of i.i.d. random variables such that for each $i \in \mathbb{N}$,

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = p, \quad 0 < p < 1.$$

The following corollary states that $\mathrm{Cov}(X_i, X_j \mid S_{2n} = 0) < 0$ for all $i \neq j$. In fact, the finite sequence $(X_1, X_2, \ldots, X_{2n})$ is negatively associated under the conditional probability $\mathbb{P}(\,\cdot \mid S_{2n} = 0)$.

Corollary 2.8. [4] *Let $\{X_n\,,\, n \geq 1\}$ be a sequence of i.i.d. random variables such that*

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = p, \quad 0 < p < 1,$$

for all i. Then

$$\mathrm{Cov}(X_i, X_j \mid S_{2n} = 0) = -\frac{1}{2n-1}$$

for all $i \neq j$.

3. Main Results

Throughout this section, we let X_i be i.i.d. random variables such that for each $i \in \mathbb{N}$,

$$X_i = \begin{cases} -1, & \text{with probability } p \\ 0, & \text{with probability } 1 - 2p \\ 1, & \text{with probability } p \end{cases}$$

where $0 < p < \frac{1}{2}$. The trinomial random walk started at 0 is the sequence $\{S_n\}_{n \geq 0}$ where $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$ for $n \geq 1$. Let $\mathbb{P}_{0,n}$ denote the conditional probability $\mathbb{P}(\,\cdot \mid S_n = 0)$ and $\mathbb{E}_{0,n}$ denote the conditional expectation $\mathbb{E}[\,\cdot \mid S_n = 0]$.

3.1. Expectation, Covariance and Variance

In this subsection, we will find the mean, the covariance and the variance of a finite sequence $(X_1, X_2, \dots, X_n)$ under $\mathbb{P}_{0,n}$.

It is easy to see that the exchangeability under a probability $\mathbb{P}_{0,n}$ comes from the assumption that $\{X_i\}_{1\le i\le n}$ are i.i.d. random variables.

Remark 3.1. A finite sequence $(X_1, X_2, \dots, X_n)$ under $\mathbb{P}_{0,n}$ is exchangeable.

The next remark follows immediately from exchangeability. Its proof follows the same reason as Corollary 3.1 of [4].

Remark 3.2. For $i = 1, 2, \dots, n$, $\mathbb{E}_{0,n}[X_i] = 0$.

As a consequence of Remark 3.1 and Remark 3.2, the variance of X_i under $P_{0,n}$, where $i = 1, 2, \dots, n$, is obtained as follows.

Lemma 3.3. *For $i \in \{1, 2, \dots, n\}$,*

$$\mathrm{Var}_{0,n}(X_i) = 1 - (1-2p) \cdot \frac{\mathbb{P}(S_{n-1} = 0)}{\mathbb{P}(S_n = 0)}.$$

Proof. By Remark 3.1 and Remark 3.2, it suffices to show that

$$\mathrm{Var}_{0,n}(X_1) = \mathbb{E}_{0,n}\left[X_1^2\right] = 1 - (1-2p) \cdot \frac{\mathbb{P}(S_{n-1} = 0)}{\mathbb{P}(S_n = 0)}.$$

We first note that

$$\begin{aligned}
\mathbb{P}(S_n = 0) &= \mathbb{P}(S_{n-1} = 0, X_n = 0) + \mathbb{P}(S_{n-1} = 1, X_n = -1) + \mathbb{P}(S_{n-1} = -1, X_n = 1) \\
&= \mathbb{P}(S_{n-1} = 0, X_n = 0) + 2\mathbb{P}(S_{n-1} = -1, X_n = 1) \\
&= (1-2p)\mathbb{P}(S_{n-1} = 0) + 2p\mathbb{P}(S_{n-1} = -1). \qquad (3.1)
\end{aligned}$$

Then, by (3.1),

$$\begin{aligned}
\mathbb{E}_{0,n}\left[X_1^2\right] &= \mathbb{P}_{0,n}(X_1 = -1) + \mathbb{P}_{0,n}(X_1 = 1) \\
&= 2\mathbb{P}_{0,n}(X_1 = 1) \\
&= \frac{2\mathbb{P}(X_1 = 1, S_n = 0)}{\mathbb{P}(S_n = 0)} \\
&= \frac{2\mathbb{P}(X_1 = 1)\mathbb{P}(S_{n-1} = -1)}{\mathbb{P}(S_n = 0)} \\
&= 2p \cdot \frac{\mathbb{P}(S_{n-1} = -1)}{\mathbb{P}(S_n = 0)} \\
&= 1 - (1-2p) \cdot \frac{\mathbb{P}(S_{n-1} = 0)}{\mathbb{P}(S_n = 0)}.
\end{aligned}$$

■

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 3.4. *Assume that $0 < p \le \frac{1}{3}$. Then the finite sequence $(X_1, X_2, \dots, X_n)$ is negatively associated under $\mathbb{P}_{0,n}$, whence NQD under $\mathbb{P}_{0,n}$.*

Proof. It suffices to show that X_1 is PF_2. We proceed with proofs by cases.

Case 1: $m_2 = 1, m_1 = -1, n_2 = 1, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -2)\\ \mathbb{P}(X_1 = 2) & \mathbb{P}(X_1 = 0)\end{pmatrix} = \det\begin{pmatrix}1-2p & 0\\ 0 & 1-2p\end{pmatrix} \ge 0.$$

Case 2: $m_2 = 1, m_1 = -1, n_2 = 1, n_1 = 0$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = -1) & \mathbb{P}(X_1 = -2)\\ \mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = 0)\end{pmatrix} = \det\begin{pmatrix}p & 0\\ p & 1-2p\end{pmatrix} \ge 0.$$

Case 3: $m_2 = 1, m_1 = -1, n_2 = 0, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -1)\\ \mathbb{P}(X_1 = 2) & \mathbb{P}(X_1 = 1)\end{pmatrix} = \det\begin{pmatrix}1-2p & p\\ 0 & p\end{pmatrix} \ge 0.$$

Case 4: $m_2 = 1, m_1 = 0, n_2 = 1, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = -1)\\ \mathbb{P}(X_1 = 2) & \mathbb{P}(X_1 = 0)\end{pmatrix} = \det\begin{pmatrix}p & p\\ 0 & 1-2p\end{pmatrix} \ge 0.$$

Case 5: $m_2 = 1, m_1 = 0, n_2 = 1, n_1 = 0$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -1)\\ \mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = 0)\end{pmatrix} = \det\begin{pmatrix}1-2p & p\\ p & 1-2p\end{pmatrix} = (1-3p)(1-p),$$

which is not negative when $0 < p \le \frac{1}{3}$.

Case 6: $m_2 = 1, m_1 = 0, n_2 = 0, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = 0)\\ \mathbb{P}(X_1 = 2) & \mathbb{P}(X_1 = 1)\end{pmatrix} = \det\begin{pmatrix}p & 1-2p\\ 0 & p\end{pmatrix} \ge 0.$$

Case 7: $m_2 = 0, m_1 = -1, n_2 = 1, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -2)\\ \mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = -1)\end{pmatrix} = \det\begin{pmatrix}1-2p & 0\\ p & p\end{pmatrix} \ge 0.$$

Case 8: $m_2 = 0, m_1 = -1, n_2 = 1, n_1 = 0$.We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = -1) & \mathbb{P}(X_1 = -2)\\ \mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -1)\end{pmatrix} = \det\begin{pmatrix}p & 0\\ 1-2p & p\end{pmatrix} \ge 0.$$

Case 9: $m_2 = 0, m_1 = -1, n_2 = 0, n_1 = -1$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = -1)\\ \mathbb{P}(X_1 = 1) & \mathbb{P}(X_1 = 0)\end{pmatrix} = \det\begin{pmatrix}1-2p & p\\ p & 1-2p\end{pmatrix} = (1-3p)(1-p),$$

which is not negative when $0 < p \le \frac{1}{3}$.

Case 10: $m_1 = m_2$ or $n_1 = n_2$. We have

$$\det\begin{pmatrix}\mathbb{P}(X_1 = m_1 - n_1) & \mathbb{P}(X_1 = m_1 - n_2)\\ \mathbb{P}(X_1 = m_2 - n_1) & \mathbb{P}(X_1 = m_2 - n_2)\end{pmatrix} = 0.$$

Then X_1 is PF_2. Therefore, the claim follows by applying Theorem 2.7. ■

Note that Corollary 3.4 shows that $\mathrm{Cov}_{0,n}(X_i, X_j) \le 0$ for $i \ne j$ when $0 < p \le \frac{1}{3}$. In fact, for $0 < p < \frac{1}{2}$, two random variables X_i and X_j have a nonpositive covariance under $\mathbb{P}_{0,n}$ for $i \ne j$. We have the following result.

Corollary 3.5. *Let $i, j \in \{1, 2, \ldots, n\}$. Then*

$$\mathrm{Cov}_{0,n}(X_i, X_j) = -\frac{1}{n-1}\mathbb{E}_{0,n}[X_1^2]$$

for all $i \neq j$.

Proof. We follow the same procedure as Corollary 2.8. Note that $0 \leq \mathbb{E}_{0,n}[X_1^2] \leq 1$. Applying Remark 3.1 and Remark 3.2, we have

$$\begin{aligned}
\mathbb{E}_{0,n}[X_1^2] &= \frac{1}{n}\mathbb{E}_{0,n}\left[\sum_{k=1}^{n} X_k^2\right] \\
&= \frac{1}{n}\mathbb{E}_{0,n}[S_n^2] - \frac{2}{n}\sum_{\substack{i<j\\ i,j=1}}^{n} \mathbb{E}_{0,n}[X_iX_j] \\
&= -\frac{2}{n}\binom{n}{2}\mathbb{E}_{0,n}[X_1X_2] \\
&= -(n-1)\mathbb{E}_{0,n}[X_1X_2] \\
&= -(n-1)\mathrm{Cov}_{0,n}(X_1, X_2)
\end{aligned}$$

It implies that, for all $i \neq j$,

$$\mathrm{Cov}_{0,n}(X_i, X_j) = \frac{-1}{n-1}\mathbb{E}_{0,n}[X_1^2].$$

■

Letting $t \in (0, 1)$, the following result shows that the variance of $S_{\lfloor nt \rfloor}$ can be computed as follows.

Corollary 3.6. *For $t \in (0, 1)$,*

$$\mathrm{Var}_{0,n}\left(S_{\lfloor nt \rfloor}\right) = \frac{\lfloor nt \rfloor (n - \lfloor nt \rfloor)}{n-1}\mathbb{E}_{0,n}[X_1^2].$$

Proof. Applying Remark 3.1, Remark 3.2 and Corollary 3.5, we obtain

$$\begin{aligned}
\mathrm{Var}_{0,n}\left(S_{\lfloor nt \rfloor}\right) &= \mathrm{Var}_{0,n}\left(\sum_{i=1}^{\lfloor nt \rfloor} X_i\right) \\
&= \lfloor nt \rfloor \mathrm{Var}_{0,n}(X_1) + 2\sum_{i<j}\mathrm{Cov}_{0,n}(X_i, X_j) \\
&= \lfloor nt \rfloor \mathrm{Var}_{0,n}(X_1) + 2\binom{\lfloor nt \rfloor}{2}\mathrm{Cov}_{0,n}(X_i, X_j) \\
&= \lfloor nt \rfloor \mathrm{Var}_{0,n}(X_1) - 2\binom{\lfloor nt \rfloor}{2}\left(\frac{1}{n-1}\right)\mathrm{Var}_{0,n}(X_1) \\
&= \frac{\lfloor nt \rfloor \left(n - \lfloor nt \rfloor\right)}{n-1}\mathrm{Var}_{0,n}(X_1) \\
&= \frac{\lfloor nt \rfloor \left(n - \lfloor nt \rfloor\right)}{n-1}\mathbb{E}_{0,n}[X_1^2].
\end{aligned}$$

■

Under a suitable scaling, the rescaled trinomial random walk has mean zero under $\mathbb{P}_{0,n}$. Fix $0 \le t \le 1$. As $n \to \infty$, the variance of the scaled trinomial random walk at time t converges to $t(1-t)$ under $\mathbb{P}_{0,n}$. Thus, we state the following remark.

Remark 3.7. Assume $0 < \mathbb{E}_{0,n}[X_1^2] = \sigma < \infty$.

(1) Using Remark 3.2, the mean of $\frac{1}{\sqrt{\sigma n}} S_{\lfloor nt \rfloor}$ is zero under $\mathbb{P}_{0,n}$.

(2) Applying Corollary 3.6, $\mathrm{Var}_{0,n}\left(\frac{1}{\sqrt{\sigma n}} S_{\lfloor nt \rfloor}\right) \longrightarrow t(1-t)$ as $n \to \infty$.

3.2. Some Properties on Exchangeable Pairs

For each $n \in \mathbb{N}$, we now define

$$W_n(t) = \frac{1}{\sigma_n} S_{\lfloor nt \rfloor},\ 0 \le t \le 1, \tag{3.2}$$

where $\sigma_n^2 = \dfrac{\lfloor nt \rfloor (n - \lfloor nt \rfloor)}{n-1}$. Let $\tau = (i\ j)$ be a permutation that i and j are each chosen independently and uniformly from $\{1, 2, \ldots, \lfloor nt \rfloor\}$ and $\{\lfloor nt \rfloor + 1, \ldots, n\}$, respectively. Define

$$W_n^\tau = W_n + \frac{X_j - X_i}{\sigma_n}. \tag{3.3}$$

By abuse of notation, we will denote W_n^τ by W_{ij}^n.

We devote this subsection to explore some properties of W_n if we perturb W_n by a small amount to get another random variable W_{ij}^n without changing the distribution under $\mathbb{P}_{0,n}$.

Recall that a pair of random variables (W, W') is called *exchangeable* if (W, W') and (W', W) are equal in distribution. The exchangeable pair approach of Stein can prove a central limit theorem for W (see [1] for more details).

From our construction, $W_n \stackrel{d}{=} W_{ij}^n$ under $\mathbb{P}_{0,n}$. Let $0 < t < 1$. Let C and M be random variables that count the number of zeros on $[0, \lfloor nt \rfloor]$, and $[0, n]$, respectively, for the trinomial random walk conditioned on $S_n = 0$. We assume further that the number of zeros on $[0, n]$ is m where $0 \le m \le n$ and the number of zeros on $[0, \lfloor nt \rfloor]$ is c where $0 \le c \le \lfloor nt \rfloor$.

Imagine a particle performing the trinomial random walk on the integer points of the real line, where it in each step moves to the right, moves to the left, or stays the same. The particle will reach the point $S_{\lfloor nt \rfloor}$ in $\lfloor nt \rfloor$ steps. Observe that each path containing c steps to stay the same in $\lfloor nt \rfloor$ steps contains $\frac{S_{\lfloor nt \rfloor} + \lfloor nt \rfloor - c}{2}$ steps to the right and $\frac{\lfloor nt \rfloor - S_{\lfloor nt \rfloor} - c}{2}$ steps to the left. Since the particle must reach 0 in n steps, the particle will reach the point $-S_{\lfloor nt \rfloor}$ in $n - \lfloor nt \rfloor$ steps. Since each path contains $m - c$ steps to stay the same in $n - \lfloor nt \rfloor$ steps, it contains $\frac{n - \lfloor nt \rfloor - S_{\lfloor nt \rfloor} - m + c}{2}$ steps to the right and $\frac{n - \lfloor nt \rfloor + S_{\lfloor nt \rfloor} - m + c}{2}$ steps to the left.

The following table summarizes the observations of counting the number of $+1, -1$ and 0 when $C = c$ and $M = m$.

Interval	$\#(+1)$	$\#(-1)$	$\#0$
On $[0, \lfloor nt \rfloor]$	$\dfrac{S_{\lfloor nt \rfloor} + \lfloor nt \rfloor - c}{2}$	$\dfrac{\lfloor nt \rfloor - S_{\lfloor nt \rfloor} - c}{2}$	c
On $[\lfloor nt \rfloor + 1, n]$	$\dfrac{n - \lfloor nt \rfloor - S_{\lfloor nt \rfloor} - m + c}{2}$	$\dfrac{n - \lfloor nt \rfloor + S_{\lfloor nt \rfloor} - m + c}{2}$	$m - c$

Recall that i and j are each chosen independently and uniformly from $\{1, 2, \ldots, \lfloor nt \rfloor\}$ and $\{\lfloor nt \rfloor + 1, \ldots, n\}$, respectively. The following remark shows how to calculate the probability of swapping of two numbers and will be used in the proof of our main results.

Remark 3.8. Based on the information presented in the above table, the following statements hold for every sufficiently large n.

$$\begin{aligned}
(1)\ &\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{1}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \mathbb{P}_{0,n}\left(X_i = -1, X_j = 0 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&\quad + \mathbb{P}_{0,n}\left(X_i = 0, X_j = 1 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \left(\frac{\lfloor nt \rfloor - S_{\lfloor nt \rfloor} - c}{2\lfloor nt \rfloor}\right)\left(\frac{m - c}{n - \lfloor nt \rfloor}\right) + \left(\frac{c}{\lfloor nt \rfloor}\right)\left(\frac{n - \lfloor nt \rfloor - S_{\lfloor nt \rfloor} - m + c}{2(n - \lfloor nt \rfloor)}\right)
\end{aligned}$$

$$\begin{aligned}
(2)\ &\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{2}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \mathbb{P}_{0,n}\left(X_i = -1, X_j = 1 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \left(\frac{\lfloor nt \rfloor - S_{\lfloor nt \rfloor} - c}{2\lfloor nt \rfloor}\right)\left(\frac{n - \lfloor nt \rfloor - S_{\lfloor nt \rfloor} - m + c}{2(n - \lfloor nt \rfloor)}\right)
\end{aligned}$$

$$\begin{aligned}
(3)\ &\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{-1}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \mathbb{P}_{0,n}\left(X_i = 1, X_j = 0 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&\quad + \mathbb{P}_{0,n}\left(X_i = 0, X_j = -1 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right) \\
&= \left(\frac{\lfloor nt \rfloor + S_{\lfloor nt \rfloor} - c}{2\lfloor nt \rfloor}\right)\left(\frac{m - c}{n - \lfloor nt \rfloor}\right) + \left(\frac{c}{\lfloor nt \rfloor}\right)\left(\frac{n - \lfloor nt \rfloor + S_{\lfloor nt \rfloor} - m + c}{2(n - \lfloor nt \rfloor)}\right)
\end{aligned}$$

$$\begin{aligned}
(4)\ \mathbb{P}_{0,n} & \left(W_{ij}^n - W_n = \frac{-2}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right) \\
&= \mathbb{P}_{0,n} \left(X_i = 1, X_j = -1 \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right) \\
&= \left(\frac{\lfloor nt \rfloor + S_{\lfloor nt \rfloor} - c}{2\lfloor nt \rfloor} \right) \left(\frac{n - \lfloor nt \rfloor + S_{\lfloor nt \rfloor} - m + c}{2(n - \lfloor nt \rfloor)} \right).
\end{aligned}$$

The following remark gives us conditions that c and m must satisfy in order for the conditional event to occur.

Remark 3.9. In order to ensure that $\mathbb{P}_{0,n} \left(S_{\lfloor nt \rfloor}, C = c, M = m \right) > 0$, we have that c and m are integers such that $0 \le c \le \lfloor nt \rfloor$, $c \le m \le c + n - \lfloor nt \rfloor$ and $m \equiv n \mod 2$.

To be more easily understandable, we provide an example below.

Example 3.10. Assume $n = 10$ and $\lfloor nt \rfloor = 5$. If $S_5 = 0$, then the number of 0 on $[0, 5]$ can be $1, 3$ or 5 and the number of 0 on $[6, 10]$ can be $1, 3$ or 5. Then the possible ordered pairs (c, m) are $(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (3, 8), (5, 6), (5, 8), (5, 10)$. If $S_5 = 1$, then the number of 0 on $[0, 5]$ can be $0, 2$ or 4 and the number of 0 on $[6, 10]$ can be $0, 2$ or 4. Then the possible ordered pairs (c, m) are $(0, 0), (0, 2), (0, 4), (2, 2), (2, 4), (2, 6), (4, 4), (4, 6), (4, 8)$. In the remaining cases, we follow the same arguments. Therefore, the set of all possible ordered pairs (c, m) is

$$\Big\{ (0,0), (0,2), (0,4), (1,2), (1,4), (1,6), (2,2), (2,4), (2,6),$$
$$(3,4), (3,6), (3,8), (4,4), (4,6), (4,8), (5,6), (5,8), (5,10) \Big\}.$$

The following theorems are our main results of this subsection. Theorem 3.11 gives us how to find the mean value of $W_{ij}^n - W_n$ given the event $\{S_{\lfloor nt \rfloor}, C = c, M = m\}$ under $\mathbb{P}_{0,n}$.

Theorem 3.11. *Let W_n and W_{ij}^n be random elements constructed as in (3.2) and (3.3), repectively. Then, for each $t \in (0, 1)$,*

$$\mathbb{E}_{0,n} \left[W_{ij}^n - W_n \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right] = -\frac{n}{\lfloor nt \rfloor (n - \lfloor nt \rfloor)} W_n.$$

Proof. Using the definition of the conditional expectation, we have

$$\begin{aligned}
&\mathbb{E}_{0,n} \left[W_{ij}^n - W_n \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right] \\
&= \frac{2}{\sigma_n} \mathbb{P}_{0,n} \left(W_{ij}^n - W_n = \frac{2}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right) \\
&\quad + \frac{1}{\sigma_n} \mathbb{P}_{0,n} \left(W_{ij}^n - W_n = \frac{1}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right) \\
&\quad - \frac{1}{\sigma_n} \mathbb{P}_{0,n} \left(W_{ij}^n - W_n = \frac{-1}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right) \\
&\quad - \frac{2}{\sigma_n} \mathbb{P}_{0,n} \left(W_{ij}^n - W_n = \frac{-2}{\sigma_n} \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\sigma_n}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&\quad + \frac{1}{\sigma_n}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&\quad - \frac{1}{\sigma_n}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&\quad - \frac{2}{\sigma_n}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right).
\end{aligned}
\tag{3.4}
$$

Calculating the sum of the first and fourth terms in (3.4), we have

$$
\begin{aligned}
&\frac{2}{\sigma_n}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right.\\
&\qquad \left. - \frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&= \frac{1}{2\sigma_n\lfloor nt\rfloor(n-\lfloor nt\rfloor)}\Big(n\lfloor nt\rfloor - \lfloor nt\rfloor^2 - \lfloor nt\rfloor S_{\lfloor nt\rfloor} - m\lfloor nt\rfloor + c\lfloor nt\rfloor - nS_{\lfloor nt\rfloor} + \lfloor nt\rfloor S_{\lfloor nt\rfloor}\\
&\qquad + S_{\lfloor nt\rfloor}^2 + mS_{\lfloor nt\rfloor} - cS_{\lfloor nt\rfloor} - cn + c\lfloor nt\rfloor + cS_{\lfloor nt\rfloor} + cm - c^2 - \big(n\lfloor nt\rfloor\\
&\qquad - \lfloor nt\rfloor^2 + \lfloor nt\rfloor S_{\lfloor nt\rfloor} - m\lfloor nt\rfloor + c\lfloor nt\rfloor + nS_{\lfloor nt\rfloor} - \lfloor nt\rfloor S_{\lfloor nt\rfloor}\\
&\qquad + S_{\lfloor nt\rfloor}^2 - mS_{\lfloor nt\rfloor} + cS_{\lfloor nt\rfloor} - cn + c\lfloor nt\rfloor - cS_{\lfloor nt\rfloor} + cm - c^2\big)\Big)\\
&= \frac{1}{2\sigma_n\lfloor nt\rfloor(n-\lfloor nt\rfloor)}\left(-2nS_{\lfloor nt\rfloor} + 2mS_{\lfloor nt\rfloor}\right)\\
&= \frac{S_{\lfloor nt\rfloor}}{\sigma_n\lfloor nt\rfloor(n-\lfloor nt\rfloor)}(m-n) = \frac{m-n}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}W_n.
\end{aligned}
\tag{3.5}
$$

Calculating the sum of the second and third terms in (3.4), we obtain

$$
\begin{aligned}
&\frac{1}{\sigma_n}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&- \frac{1}{\sigma_n}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right)\\
&= \frac{1}{2\sigma_n\lfloor nt\rfloor(n-\lfloor nt\rfloor)}\Big(m\lfloor nt\rfloor - c\lfloor nt\rfloor - mS_{\lfloor nt\rfloor} + cS_{\lfloor nt\rfloor} - cm - c^2 + cn - c\lfloor nt\rfloor - cS_{\lfloor nt\rfloor}\\
&\qquad - cm + c^2 - \big(m\lfloor nt\rfloor - c\lfloor nt\rfloor + mS_{\lfloor nt\rfloor} - cS_{\lfloor nt\rfloor} - cm + c^2 + cm\\
&\qquad + c^2 + cn - cm + c^2 + cn - c\lfloor nt\rfloor + cS_{\lfloor nt\rfloor} - cm + c^2\big)\Big)\\
&= \frac{-2mS_{\lfloor nt\rfloor}}{2\sigma_n\lfloor nt\rfloor(n-\lfloor nt\rfloor)} = -\frac{m}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}W_n.
\end{aligned}
\tag{3.6}
$$

Adding (3.5) and (3.6) gives

$$
\mathbb{E}_{0,n}\left[W_{ij}^n - W_n \,\middle|\, S_{\lfloor nt\rfloor}, C = c, M = m\right] = -\frac{n}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}W_n.
$$

■

The following theorem refers to a statistical measurement of the distance of W_{ij}^n and W_n given the event $\{S_{\lfloor nt\rfloor}, C = c, M = m\}$ under $\mathbb{P}_{0,n}$.

Theorem 3.12. *Let W_n and W_{ij}^n be random elements constructed as in (3.2) and (3.3), repectively. Then, for each $t \in (0,1)$,*

$$\mathbb{E}_{0,n}[(W_{ij}^n - W_n)^2|S_{\lfloor nt\rfloor}, C=c, M=m] = \frac{2c\lfloor nt\rfloor - cn - 2\lfloor nt\rfloor^2 - m\lfloor nt\rfloor + 2n\lfloor nt\rfloor}{\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)} + \frac{2W_n^2}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}.$$

Consequently,

$$\mathbb{E}_{0,n}[(W_{ij}^n - W_n)^2|S_{\lfloor nt\rfloor}, C, M] = \frac{2C\lfloor nt\rfloor - Cn - 2\lfloor nt\rfloor^2 - M\lfloor nt\rfloor + 2n\lfloor nt\rfloor}{\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)} + \frac{2W_n^2}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}.$$

Proof. Using the definition of the conditional expectation, we have

$$\begin{aligned}
&\mathbb{E}_{0,n}[(W_{ij}^n - W_n)^2|S_{\lfloor nt\rfloor}, C=c, M=m] \\
&= \frac{4}{\sigma_n^2}\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{2}{\sigma_n} \;\middle|\; S_{\lfloor nt\rfloor}, C=c, M=m\right) \\
&\quad + \frac{1}{\sigma_n^2}\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{1}{\sigma_n} \;\middle|\; S_{\lfloor nt\rfloor}, C=c, M=m\right) \\
&\quad + \frac{4}{\sigma_n^2}\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{-2}{\sigma_n} \;\middle|\; S_{\lfloor nt\rfloor}, C=c, M=m\right) \\
&\quad + \frac{1}{\sigma_n^2}\mathbb{P}_{0,n}\left(W_{ij}^n - W_n = \frac{-1}{\sigma_n} \;\middle|\; S_{\lfloor nt\rfloor}, C=c, M=m\right) \\
&= \frac{4}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right) \\
&\quad + \frac{1}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right) \\
&\quad + \frac{4}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right) \\
&\quad + \frac{1}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right).
\end{aligned} \tag{3.7}$$

Calculating the sum of the first and third terms in (3.7), we have

$$\begin{aligned}
&\frac{4}{\sigma_n^2}\Bigg(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)} \\
&\quad + \frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{4\sigma_n^2(\lfloor nt\rfloor)(n-\lfloor nt\rfloor)}\Big(n\lfloor nt\rfloor - \lfloor nt\rfloor^2 - \lfloor nt\rfloor S_{\lfloor nt\rfloor} - m\lfloor nt\rfloor + c\lfloor nt\rfloor - nS_{\lfloor nt\rfloor} + \lfloor nt\rfloor S_{\lfloor nt\rfloor} \\
&\qquad + S_{\lfloor nt\rfloor}^2 + mS_{\lfloor nt\rfloor} - cS_{\lfloor nt\rfloor} - cn + c\lfloor nt\rfloor + cS_{\lfloor nt\rfloor} + cm - c^2 \\
&\qquad + n\lfloor nt\rfloor - \lfloor nt\rfloor^2 + \lfloor nt\rfloor S_{\lfloor nt\rfloor} - m\lfloor nt\rfloor + c\lfloor nt\rfloor + nS_{\lfloor nt\rfloor} - \lfloor nt\rfloor S_{\lfloor nt\rfloor} \\
&\qquad + S_{\lfloor nt\rfloor}^2 - mS_{\lfloor nt\rfloor} + cS_{\lfloor nt\rfloor} - cn + c\lfloor nt\rfloor - cS_{\lfloor nt\rfloor} + cm - c^2\Big) \\
&= \frac{1}{\sigma_n^2(\lfloor nt\rfloor)((n-\lfloor nt\rfloor))}\Big(2n\lfloor nt\rfloor - 2\lfloor nt\rfloor^2 - 2m\lfloor nt\rfloor + 4c\lfloor nt\rfloor + 2S_{\lfloor nt\rfloor}^2 - 2cn + 2cm - 2c^2\Big).
\end{aligned}
\tag{3.8}
$$

Calculating the sum of the second and fourth terms in (3.7), we obtain

$$
\begin{aligned}
&\frac{1}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor - S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right) \\
&+ \frac{1}{\sigma_n^2}\left(\frac{\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - c}{2\lfloor nt\rfloor}\cdot\frac{m-c}{n-\lfloor nt\rfloor} + \frac{c}{\lfloor nt\rfloor}\cdot\frac{n-\lfloor nt\rfloor + S_{\lfloor nt\rfloor} - m + c}{2(n-\lfloor nt\rfloor)}\right) \\
&= \frac{1}{2\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)}\Big(m\lfloor nt\rfloor - c\lfloor nt\rfloor - mS_{\lfloor nt\rfloor} + cS_{\lfloor nt\rfloor} - cm + c^2 + cn - c\lfloor nt\rfloor - cS_{\lfloor nt\rfloor} \\
&- cm + c^2 + m\lfloor nt\rfloor - c\lfloor nt\rfloor + mS_{\lfloor nt\rfloor} - cS_{\lfloor nt\rfloor} - cm + c^2 + cn - c\lfloor nt\rfloor + cS_{\lfloor nt\rfloor} - cm + c^2\Big) \\
&= \frac{1}{2\sigma_n^2\lfloor nt\rfloor((n-\lfloor nt\rfloor))}\Big(2m\lfloor nt\rfloor - 4c\lfloor nt\rfloor - 4cm + 4c^2 + 2cn\Big) \\
&= \frac{1}{\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)}(m\lfloor nt\rfloor - 2c\lfloor nt\rfloor - 2cm + 2c^2 + cn).
\end{aligned}
\tag{3.9}
$$

By (3.7), (3.8) and (3.9), we have

$$
\begin{aligned}
&\mathbb{E}_{0,n}[(W_{ij}^n - W_n)^2 | S_{\lfloor nt\rfloor}, C = c, M = m] \\
&= \frac{1}{\sigma_n^2(\lfloor nt\rfloor)((n-\lfloor nt\rfloor))}(2n\lfloor nt\rfloor - 2\lfloor nt\rfloor^2 - 2m\lfloor nt\rfloor + 4c\lfloor nt\rfloor + 2S_{\lfloor nt\rfloor}^2 - 2cn + 2cm - 2c^2) \\
&+ \frac{1}{\sigma_n^2(\lfloor nt\rfloor)(n-\lfloor nt\rfloor)}(m\lfloor nt\rfloor - 2c\lfloor nt\rfloor - 2cm + 2c^2 + cn) \\
&= \frac{2n\lfloor nt\rfloor - 2\lfloor nt\rfloor^2 - m\lfloor nt\rfloor + 2c\lfloor nt\rfloor + 2S_{\lfloor nt\rfloor}^2 - cn}{\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)} \\
&= \frac{2n\lfloor nt\rfloor - 2\lfloor nt\rfloor^2 - m\lfloor nt\rfloor + 2c\lfloor nt\rfloor - cn}{\sigma_n^2\lfloor nt\rfloor(n-\lfloor nt\rfloor)} + \frac{2W_n^2}{\lfloor nt\rfloor(n-\lfloor nt\rfloor)}.
\end{aligned}
$$

■

4. Conclusion

Let $0 < p < \frac{1}{2}$. Suppose $X_1, X_2, \dots$ is a sequence of i.i.d. random variables with

$$
X_i = \begin{cases} -1, & \text{with probability } p \\ 0, & \text{with probability } 1-2p \\ 1, & \text{with probability } p. \end{cases}
$$

The finite sequence $(X_1, X_2, \dots, X_n)$ under $\mathbb{P}_{0,n}$ satisfies many interesting properties and can be summarized as follows.

(1) The finite sequence $(X_1, X_2, \dots, X_n)$ under $\mathbb{P}_{0,n}$ is exchangeable.
(2) $\mathbb{E}_{0,n}[X_i] = 0$ for $i = 1, 2, \dots, n$.
(3) For $0 < p \le \frac{1}{3}$, the finite sequence $(X_1, X_2, \dots, X_n)$ is negatively associated under $\mathbb{P}_{0,n}$, hence $\mathrm{Cov}_{0,n}(X_i, X_j) \le 0$ for all $i \ne j$. In fact,

$$\mathrm{Cov}_{0,n}(X_i, X_j) = -\frac{1}{n-1}\mathbb{E}_{0,n}[X_1^2]$$

for all $i \ne j$.
(4) $\mathrm{Var}_{0,n}(X_i) = 1 - (1-2p) \cdot \dfrac{\mathbb{P}(S_{n-1} = 0)}{\mathbb{P}(S_n = 0)}$ for all $i = 1, 2, \dots, n$.

Let W_n and W_{ij}^n be random elements constructed as in (3.2) and (3.3), repectively. Applying all the above properties to the rescaled trinomial random walk, we can show that

$$\frac{1}{\sqrt{n\mathrm{Var}_{0,n}(X_1)}}\mathbb{E}_{0,n}\left[W_n(t)\right] = 0$$

and

$$\frac{1}{\sqrt{n\mathrm{Var}_{0,n}(X_1)}}\mathrm{Var}_{0,n}\left(W_n(t)\right) \longrightarrow t(1-t)$$

as $n \to \infty$. Moreover, the mean value of $W_{ij}^n - W_n$ and the distance between W_{ij}^n and W_n given the event $\{S_{\lfloor nt \rfloor}, C = c, M = m\}$ under $\mathbb{P}_{0,n}$ were derived, which are very helpful in studying the convergence of the rescaled trinomial random walk returning to the origin. We have the following results.

(1)

$$\mathbb{E}_{0,n}\left[W_{ij}^n - W_n \,\middle|\, S_{\lfloor nt \rfloor}, C = c, M = m\right] = -\frac{n}{\lfloor nt \rfloor(n - \lfloor nt \rfloor)} W_n.$$

(2)

$$\begin{aligned}\mathbb{E}_{0,n}\left[(W_{ij}^n - W_n)^2 | S_{\lfloor nt \rfloor}, C = c, M = m\right] &= \frac{2c\lfloor nt \rfloor - cn - 2\lfloor nt \rfloor^2 - m\lfloor nt \rfloor + 2n\lfloor nt \rfloor}{\sigma_n^2 \lfloor nt \rfloor(n - \lfloor nt \rfloor)} \\ &\quad + \frac{2W_n^2}{\lfloor nt \rfloor(n - \lfloor nt \rfloor)}.\end{aligned}$$

The study of an invariance principle for the rescaled trinomial random walk returning to the origin is left as a future work.

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