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On Common Fixed Points and Best Approximation on Nonconvex Sets

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Abstract : By generalizing a theorem of G.Meinardus [Arch. Rational Mech. Anal. 14(1963), 301-303], B.Brosowski [Mathematica (Cluj) 11(1969), 195-200] proved a result on invariant approximation using fixed point theory. Subsquently, many generalizations of Brosowski's result have appeared. We also obtain several Brosowski-Meinardus type theorems for nonexpansive mappings defined on a class of nonconvex sets containing the class of starshaped sets thereby extending and generalizing various known results on the subject.

Keywords : Affine map, nonexpansive map, best approximation, convex metric space, convex set, starshaped set, contractive jointly continuous family.
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1 Introduction and Preliminaries

W.J.Dotson Jr. [4] proved some results concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. For proving these results, which extended his previous work [3] on starshaped sets, he introduced the following class of nonconvex sets:

Suppose S is a subset of a Banach space E, and let $\mathfrak{F} = \{f_\alpha\}_{\alpha \in S}$ be a family of functions from $[0,1]$ into S, having the property that for each $\alpha \in S$ we have $f_\alpha(1) = \alpha$. Such a family $\mathfrak{F}$ is said to be **contractive** provided there exists a function $\varphi : (0,1) \to (0,1)$ such that for all α and β in S and for all t in $(0,1)$ we have

$$\|f_\alpha(t) - f_\beta(t)\| \leq \varphi(t)\|\alpha - \beta\|.$$

Such a family $\mathfrak{F}$ is said to be **jointly continuous** provided that if $t \to t_\circ$ in $[0,1]$ and $\alpha \to \alpha_\circ$ in S then $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in S.

This notion can easily be extended to metric spaces. Also it is easy to observe that if S is a starshaped (with z as star center) subset of a normed space E and $f_z(t) = tx + (1-t)z$, $x \in S$, $t \in [0,1]$, then $\mathfrak{F} = \{f_x : x \in S\}$ is a contractive jointly continuous family with $\varphi(t) = t$. Thus the class of subsets of E with the property of contractive and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

Ever since Dotson's work [4], efforts have been made by many researchers (see e.g. [9], [10]) to extend results proved on convex sets and starshaped sets to the above class of nonconvex sets. The present paper is also a step in the same direction in which we prove some results on common fixed points and best approximation. To start with, we recall some definitions and known facts to be used in the sequel.

Let G be a self-mapping on a metric space (X,d). Then a self-mapping T on X is said to be (s.t.b.) **G-nonexpansive** if

$$d(Tx,Ty) \leq d(Gx,Gy) \qquad \text{for all } x,y \in X \tag{1.1}$$

If strict inequality holds in (1.1) for distinct points, then T is s.t.b. **G-contractive**.

Clearly, G-nonexpansive and G-contractive maps are continuous, whenever G is continuous.

For a metric space (X,d), a continuous mapping $W : X \times X \times [0,1] \to X$ is s.t.b. a **convex structure** on X if for all $x,y \in X$ and $\lambda \in [0,1]$, we have

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $u \in X$. The metric space (X,d) together with a convex structure is called a **convex metric space** [12].

A subset K of a convex metric space (X,d) is s.t.b. a **convex set** [12] if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$. The set K is said to be **p-starshaped** [5] if there exists a $p \in K$ such that $W(x,p,\lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X,d) is said to satisfy **Property (I)** [5] if for all $x,y,q \in X$ and $\lambda \in [0,1]$,

$$d(W(x,q,\lambda), W(y,q,\lambda)) \leq \lambda d(x,y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [5], [12]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset K of a metric space (X,d) and $x \in X$, an element $y \in K$ is s.t.b. a **best approximant** to x or a **best K-approximant** to x if $d(x,y) = d(x,K) \equiv \inf\{d(x,y) : y \in K\}$. The set of all such $y \in K$ is denoted by $P_K(x)$.

For a convex subset K of a convex metric space (X, d), a mapping $g : K \to X$ is s.t.b. **affine** for all $x, y \in K$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0, 1]$.

Throughout, we shall write $F(S)$ for set of fixed points of a mapping S and $F(T, S)$ for set of fixed points of both T and S.

2 Main Results

Al-Thagafi [1] obtained the following common fixed point theorem for commuting mappings:

Theorem A. Let D be a closed subset of a metric space (X, d) and let I and T be self maps of D with $T(D) \subset I(D)$. If $\overline{T(D)}$ is complete, I is continuous, I and T are commuting and T is I-contraction, then I and T have a unique common fixed point.

Using this fixed point theorem of Al-Thagafi, we have the following theorem on common fixed points of commuting mappings for contractive jointly continuous family:

Theorem 1. Let D be a closed subset of a metric space (X,d), I and T be self maps of D with $T(D) \subseteq I(D)$. Suppose D has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in D\}$ such that $I(f_x(\alpha)) = f_{I(x)}(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. If $\overline{T(D)}$ is compact, I is continuous, I and T are commuting and T is I-nonexpansive then I and T have a common fixed point.

Proof. For each positive integer n, let $k_n = \frac{n}{n+1}$ and define $T_n : D \to D$ as $T_n(x) = f_{Tx}(k_n)$, $x \in D$. Consider

$$\begin{aligned} T_n(I(x)) = f_{T(I(x))}(k_n) &= f_{I(T(x))}(k_n) \text{ as } I \text{ and } T \text{ commute} \\ &= I f_{Tx}(k_n) \\ &= IT_n(x), \ x \in D \end{aligned}$$

i.e. for each n, T_n commutes with I and $T_n(D) \subseteq I(D)$.

Since $\mathfrak{F}$ is contractive and T is I-nonexpansive, we have

$$d(T_n x, T_n y) = d(f_{Tx}(k_n), f_{Ty}(k_n)) \leq \varphi(k_n) d(Tx, Ty) \leq \varphi(k_n) d(Ix, Iy)$$

for every $x, y \in D$. Therefore each T_n is I-contraction. Since $\overline{T(D)}$ is compact, T is I-nonexpansive and I is continuous so $\overline{T_n(D)}$ is also compact. Hence by Theorem A there exists $x_n \in D$ such that $x_n \in F(T_n, I)$ for each n. Since $< T(x_n) >$ is a sequence in the compact set $\overline{T(D)}$, there exists a subsequence $< T(x_{n_j}) >$ with $T(x_{n_j}) \to x_\circ \in \overline{T(D)}$. Since $x_{n_j} = T_{n_j}(x_{n_j}) = f_{Tx_{n_j}}(k_{n_j}) \to f_{x_\circ}(1) = x_\circ$, the continuity of I and T imply $x_\circ \in F(T, I)$.

Corollary 1. Let D be a compact subset of a metric space (X, d), I and T be self maps of D with $T(D) \subseteq I(D)$. Suppose D has a contractive jointly continuous family $\mathfrak{F}$ such that $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. If I and T are commuting, I is continuous and T is I-nonexpansive then I and T have a common fixed point.

Remarks. If D is a p-starshaped subset of a convex metric space (X, d) with Property (I), define the family $\mathfrak{F}$ as $f_x(\alpha) = W(x, p, \alpha)$. Then

$$\begin{aligned} d(f_x(\alpha), f_y(\alpha)) &= d(W(x,p,\alpha), W(y,p,\alpha)) \\ &\leq \alpha\, d(x,y), \end{aligned}$$

so taking $\varphi(\alpha) = \alpha$, $0 < \alpha < 1$, the family is a contractive jointly continuous family. Moreover, if D is a p-starshaped, I is affine and $p \in F(I)$ then $I(f_x(\alpha)) = f_{Ix}(\alpha)$ is satisfied. Consequently, we have:

Corollary 2. Let D be a closed p-starshaped subset of a convex metric space (X, d) satisfying Property (I) and $\overline{T(D)}$ is compact, Suppose S and T are self maps on D with $T(D) \subseteq S(D)$, $p \in F(S)$, S is continuous, S and T are commuting and T is S-nonexpansive then S and T have a common fixed point.

Since a normed linear space is a convex metric space with Property (I), we have:

Corollary 3 (Theorem 2.2 [1]). Let D be a closed subset of a normed linear space X, S and T self maps of D with $T(D) \subseteq S(D)$, $p \in F(S)$. If D is p-starshaped, $\overline{T(D)}$ is compact, S is continuous and linear, S and T are commuting with T is S-nonexpansive, then S and T have a common fixed point.

Remark. Corollary 3 holds even when S is only an affine map.

Taking S as the Identity map, we have:

Corollary 4 (Theorem 3 [2]). Let (X, d) be a convex metric space satisfying property (I) and D a closed and p-starshaped subset of X. If T is a nonexpansive self mapping on D and $\overline{T(D)}$ is compact then T has a fixed point.

Corollary 5 (Theorem 4 [6]). Let D be a closed p-starshaped subset of a normed linear space X and T is nonexpansive self map of D. If $\overline{T(D)}$ is compact, then T has a fixed point.

Remarks. Dotson [3] proved Corollary 5 in Banach spaces, and Guay et al. [5] in convex metric spaces for compact sets D.

Corollary 6 (Theorem 1 [4]). Suppose S is a compact subset of a Banach space E, and suppose there exists a contractive jointly continuous family $\mathfrak{F}$ of functions associated with S. Then any nonexpansive self mapping T of S has a fixed point.

Next we state the following common fixed point theorem for two maps in metric spaces. This theorem for p-normed spaces was proved by Khan et al. [9] and it is easy to see that as in the proof of Theorem 1, the proof given in [9] can easily be extended to metric spaces.

Theorem 2. Let I and T be self maps on a metric space (X, d), $u \in F(T) \cap F(I)$ and S is T-invariant subset of X. Suppose I and T are commuting on $D = P_S(u)$, I is continuous on D, T is I-nonexpansive on $D \cup \{u\}$ and $I(D) = D$. Suppose that D has a contractive jointly continuous family $\mathfrak{F}$ such that $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. Then I and T have a common fixed point on D.

Corollary 1 (Theorem 3 [11]). Let $T, I : X \to X$ be operators, C be a subset of X such that $T : \partial C \to C$, and $x_\circ \in F(T) \cap F(I)$. Further, T and I satisfy $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in D$ and let I be linear, continuous on D, and

$ITx = TIx$ for all x in D. If D is nonempty, compact and starshaped with respect to a point $q \in F(I)$ and if $I(D) = D$, then $D \cap F(T) \cap F(I) \neq \phi$.

Corollary 2 (Theorem 6 [2]). Let (X, d) be a convex metric space satisfying property (I). Let $T, S : X \to X$ be operators, C a subset of X such that $T : \partial C \to C$, and $x_\circ \in F(T) \cap F(S)$. Further, T and S satisfy $d(Tx, Ty) \leq d(Sx, Sy)$ for all x, y in $P_C(x_\circ) \cup \{x_\circ\}$ and let S be continuous and affine on $P_C(x_\circ)$, and $STx = TSx$ for all x in $P_C(x_\circ)$. If $P_C(x_\circ)$ is nonempty, compact and q-starshaped with respect to a point $q \in F(S)$ and if $S(P_C(x_\circ)) = P_C(x_\circ)$, then $P_C(x_\circ) \cap F(T) \cap F(S) \neq \phi$.

Jungck [7] obtained the following common fixed point theorem for G-contractive mappings.

Theorem B. Let F and G be commuting mappings of a compact metric space (X, d) into itself such that $F(X) \subset G(X)$, and G is continuous. If F is G-contractive map on the metric space X, then there is a unique common fixed point of F and G.

Using this fixed point theorem of Jungck, we have the following theorem on common fixed points of commuting mappings in metric spaces:

Theorem 3. Let (X, d) be a metric space, F and $G : X \to X$ be commuting mappings such that F is G-nonexpansive where G satisfies $G^2 = G$. Let C be a subset of X and x a point of X such that both are invariant under F and G. Let $D = P_C(x)$ be the set of best approximant of x in C. If G is continuous on D, $F(D) \subseteq G(D)$ and D is nonempty compact and has jointly continuous contractive family $\mathfrak{F}$ such that $G(f_y(\alpha)) = f_{Gy}(\alpha)$, for all $y \in D$ and $\alpha \in [0, 1]$, then D contains a point invariant under both F and G.

Proof. We first observe that both F and G are self maps on D. Let $y \in D$. Consider

$$d(x, GFy) = d(x, FGy) = d(Fx, FGy) \leq d(x, G^2 y) = d(x, Gy) = d(x, C). \quad (2.1)$$

Also,

$$d(x, GGy) = d(x, Gy) = d(x, C). \quad (2.2)$$

From relations (2.1) and (2.2) we have $F(y) \in D$ and $G(y) \in D$. Thus F and G are self mappings on D.

Define $F_n : D \to D$ as $F_n(x) = f_{F(x)}(t_n)$, $x \in D$ and $< t_n >$ is a sequence of real numbers in $(0, 1)$ such that $t_n \to 1$.

Since F and G commute on D, it follows from the property of the family $\mathfrak{F}$ that $F_n(G(x)) = f_{G(F(x))}(t_n) = G f_{F(x)}(t_n) = G(F_n(x))$ for each $x \in D$. Thus for each n, F_n commutes with G and $F_n(D) \subseteq G(D)$. Since $\mathfrak{F}$ is contractive and F is G-nonexpansive, we get

$$d(F_n y, F_n z) = d(f_{F(y)}(t_n), f_{F(z)}(t_n)) \leq \varphi(t_n) d(F(y), F(z)) \leq \varphi(t_n) d(G(y), G(z))$$

for every $y, z \in D$ and so F_n is G-nonexpansive for each n.

It follows from Theorem B that there is a unique common fixed point, say $x_n \in D$, of F_n and G for each n i.e. $Gx_n = x_n = F_n x_n$ for each n. Since D

is compact, $< x_n >$ has a subsequence $< x_{n_i} > \to x_\circ \in D$ and hence $F(x_{n_i}) \to F(x_\circ)$. The joint continuity of $\mathfrak{F}$ gives

$$x_{n_i} = F_{n_i} x_{n_i} = f_{F x_{n_i}}(t_{n_i}) \to f_{F x_\circ}(1) = F(x_\circ)$$

and so $x_\circ = F(x_\circ)$. Also continuity of G gives $G(x_\circ) = G(\lim x_{n_i}) = \lim G(x_{n_i}) = \lim x_{n_i} = x_\circ$. Thus $x_\circ \in D$ is invariant under both F and G.

Corollary 1 (Theorem 3.1 [10]). Let F and G be commuting operators on a normed linear space X such that F is G-nonexpansive, where G is linear, continuous and satisfies $G^2 = G$. Let C be a subset of X, x a point of X such that both of them are invariant under both F and G. Let $D = \{y \in C : Gy$ is a best C-approximant to $x\}$. If $F(D) \subset G(D)$, and also D is nonempty, compact and G-starshaped with respect to G, then D contains a point invariant under both F and G.

Assuming $G = I$ (the identity mapping), we obtain the following:

Corollary 2 (Theorem 3.4 [10]). Let F be a nonexpansive operator on a normed linear space X. Let C be an F-invariant subset of X and x an F-invariant point. If the set of best C-approximants to x is nonempty, compact and for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions, then it contains an F-invariant point.

We shall be using the following result of Jungck [8] to prove our next theorem:

Theorem C. Let f be a continuous self map of a compact metric space (X, d). If $f(x) \neq f(y)$ implies $d(fx, fy) < d(gx, hy)$ for some $g, h \in C_f$, where C_f is the set of all maps $g : X \to X$ such that $gf = fg$, then there is a unique point $a \in X$ such that $a = f(a)$. Infact $a = h(a)$ for all $h \in C_f$.

Using Theorem C, we have:

Theorem 4. Let (X, d) be a metric space, I and $T : C \to C$ be commuting maps where C is a compact subset of X and has jointly continuous contractive family $\mathfrak{F}$ satisfying $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. If for each $x, y \in C$, there exists $I = I(x, y)$, $J = J(x, y) \in C_T$ such that

$$d(Tx, Ty) \le d(Ix, Jy) \tag{2.3}$$

then there exists $a \in C$ such that $a = Ta$ and $a = Ia$ for all continuous $I \in C_T$.

Proof. Let $< t_n >$ be a sequence of real numbers in $(0, 1)$ such that $t_n \to 1$. Let $T_n : C \to C$ be defined as $T_n(x) = f_{Tx}(t_n)$. Since I and T commute, it follows from the property of the family $\mathfrak{F}$ that

$$\begin{aligned} T_n(I(x)) = f_{T(I(x))}(t_n) &= f_{I(T(x))}(t_n) \text{ as } I \text{ and } T \text{ commute} \\ &= I f_{Tx}(t_n) \\ &= IT_n(x) \end{aligned}$$

for each $x \in C$. Thus for each n, $T_n I = IT_n$.

Now fix n. By hypothesis, for each $x, y \in C$ there exists $I, J \in C_T$ such that

$d(Tx, Ty) \leq d(Ix, Jy)$. So,

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{Tx}(t_n), f_{Ty}(t_n)) \\ &\leq \varphi(t_n) d(Tx, Ty) \\ &\leq \varphi(t_n) d(Ix, Jy) \end{aligned}$$

Therefore, for all $x, y \in C$, $T_n(x) \neq T_n(y)$ implies $d(T_n x, T_n y) \leq d(Ix, Jy)$ for some $I, J \in C_T$. Since T_n is continuous, by Theorem C there is unique $x_n \in C$ such that for all $I \in C_T$, $x_n = T_n x_n = I x_n$. Since C is compact, $< x_n >$ has a subsequence $< x_{n_i} > \to a \in C$. Now

$$a = \lim x_{n_i} = \lim T_{n_i} x_{n_i} = \lim f_{T x_{n_i}}(t_{n_i}) \to f_{Ta}(1) = Ta$$

Also, $a = \lim x_{n_i} = \lim I x_{n_i} = I \lim x_{n_i} = Ia$ for all continuous $I \in C_T$. Thus $a = Ta$ and $a = Ia$ for all continuous $I \in C_T$.

Corollary. Let C be a compact subset of a metric space (X, d), $T, I : C \to C$ be continuous commuting maps, C has jointly continuous contractive family $\mathfrak{F}$ satisfying $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. If for $x, y \in C$, there exists $n = n(x, y)$, $m = m(x, y)$ in $\mathcal{N} \cup \{0\}$ such that

$$d(Tx, Ty) \leq d(I^m x, I^n y) \qquad (2.4)$$

then there exists $a \in C$ such that $a = Ta$ and $a = Ia$.

Proof. For each n,

$$\begin{aligned} T_n(I^n x) = f_{T(I^n x)}(t_n) &= f_{TI(I^{n-1}x)}(t_n) \\ &= f_{IT(I^{n-1}x)}(t_n) = I f_{T(I^{n-1}x)}(t_n) = ... = I^n(T_n x) \end{aligned}$$

i.e. $T_n I^n = I^n T_n$ for each n and $I^n : C \to C$. Therefore (2.4) implies that members of $\mathfrak{F}$ satisfy (2.3) and so the result follows from Theorem 4.

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