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More on Continuity in Rough Set Theory

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Abstract It is generally known that in rough set theory, we can define the topology imposed by the lower approximation operator in the approximation space (U, R). As a result, any approximation can be viewed as a topological space. In this article, we present the sufficient condition for a map between two approximation spaces to be a continuous map. Some elementary properties of the maps are stated.

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1. Introduction

Rough Set Theory is a mathematical framework that deals with the uncertainty and vagueness present in real-world data. It was introduced by Zdzislaw Pawlak [1] in the early 1980s and has since been applied to various fields, including computer science, data mining, artificial intelligence, and decision-making.

The core idea of rough set theory is to represent knowledge in terms of sets of objects and their properties. These sets are classified into different equivalence classes based on their similarity or dissimilarity. The theory provides a way to approximate the boundary between different classes of objects, which is often referred to as the "rough set boundary".

One application of rough set theory is in the analysis of continuous maps. A continuous map is a function that maps one set of points to another, where small changes in the input result in small changes in the output. Rough set theory provides a way to analyze the behavior of continuous maps in terms of their approximate boundaries, which can be useful in understanding the stability and predictability of such maps.

Pawlak [2] introduced the concept of rough relations and rough functions between two specified approximation spaces in 1994. But his work was so difficult to grasp. Later in 2015, Qui [3] introduced a new concept on the map between two approximation spaces, namely rough mappings, which is a generalized version of Pawlak's. In this paper, we

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introduce a new concept of a map between two approximation spaces, which we believe is simpler than the two earlier works.

Kondo [4] showed in 2006 that the reflexive relation R on the set U induced a topology given by the lower approximation. Then it is not surprising to consider the continuous map between approximation spaces. In this article, we prove that a map between two approximation spaces has a certain sufficient condition to become a continuous map.

Definition 1.1. Let U be a finite set and R be an equivalence relation on U. A pair (U, R) is called the *approximation space.*

Each $x \in U$, the equivalence class containing x, denoted by $[x]_R$ is called a *granule.* We denote the set of all granules by U/R.

Definition 1.2. Let (U, R) be an approximation space. Let X be a subset of U. We define the *lower* and *upper approximation* of X as follows.

$$R_*(X) = \{x \in U \mid [x]_R \subseteq X\}.$$
$$R^*(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

We say that X is *exact* or *crisp* if $R_*(X) = R^*(X)$ and
we say that X is *rough* if $R_*(X) \neq R^*(X)$.
The boundary of X is defined to be $BN_R(X) = R^*(X) - R_*(X)$.

Here are some basic properties of approximations.

Theorem 1.3. [1] *Let (U, R) be an approximation space. Let $X, Y \subseteq U$. Then*

(1) $R_*(X) \subseteq X \subseteq R^*(X)$.
(2) $R_*(\emptyset) = R^*(\emptyset)$ *and* $R_*(U) = R^*(U)$.
(3) $R^*(X \cup Y) = R^*(X) \cup R^*(Y)$.
(4) $R_*(X \cap Y) = R_*(X) \cap R_*(Y)$.
(5) $R^*(X \cap Y) \subseteq R^*(X) \cap R^*(Y)$.
(6) $R_*(X \cup Y) \supseteq R_*(X) \cup R_*(Y)$.
(7) If $X \subseteq Y$, *then* $R_*(X) \subseteq R_*(Y)$ *and* $R^*(X) \subseteq R^*(Y)$.
(8) $R_*(U - X) = U - R^*(X)$.
(9) $R^*(U - X) = U - R_*(X)$.
(10) $R_*R_*(X) = R^*R_*(X) = R_*(X)$.
(11) $R^*R^*(X) = R_*R^*(X) = R^*(X)$.

The following theorem is a handy tool for determining if the given sets are exact or rough.

Lemma 1.4. *Let (U, R) be an approximation space.*
For $X \subseteq U$, $X = R_*(X)$ if and only if $X = R^*(X)$.

Proof. Let $X \subseteq U$. Assume that $X = R_*(X)$.
By Theorem 1.3(10), we obtain that $R^*(X) = R^*(R_*(X)) = R_*(X) = X$.
Conversely, assume that $R^*(X) = X$.
By Theorem 1.3(11), we obtain that $R_*(X) = R_*(R^*(X)) = R^*(X) = X$. ∎

Example 1.5. Application to data analysis : An information system is a pair (U, A) where U is a non-empty finite set of objects called the *universe*, and A is non-empty finite set of *attributes* such that $a : U \to V_a$ for every $a \in A$. The set V_a is called the values set of a.

Any subset B of A determines an equivalence relation $I(B)$ on U, which will be called an *indiscernibility* relation, and is defined as follows: for $x, y \in U$

$$xI(B)y \text{ if and only if } a(x) = a(y) \text{ for every } a \in B.$$

Typically, a table can be used to represent an information system.
The Table 1 above displays the student grades in three subjects.

TABLE 1. Student Grade

Student	**Mathematics** (MA)	**Science** (SC)	**Computer** (CP)
s1	A	A	B
s2	A	A	A
s3	A	A	B
s4	B	B	A
s5	B	B	A
s6	A	C	B

In this case, we have that $U = \{s1, s2, s3, s4, s5, s6\}$, $A = \{MA, SC, CP\}$, $V_{MA} = \{A, B\}, V_{SC} = \{A, B, C\}$ and $V_{CP} = \{A, B\}$.
For $B = \{MA, SC\} \subset A$, we obtain that

$$U/I(B) = \{\{s1, s2, s3\}, \{s4, s5\}, \{s6\}\} \text{ and } U/I(A) = \{\{s1, s3\}, \{s2\}, \{s4, s5\}, \{s6\}\}.$$

For $X = \{s1, s2, s3, s4\}$ and $Y = \{s1, s2, s3\}$, we can easily see that $I(A)_*(X) = \{s1, s2, s3\}$ but $I(A)^*(X) = \{s1, s2, s3, s4, s5\}$ and $I(A)_*(Y) = \{s1, s2, s3\} = I(A)^*(Y)$. Hence, X is rough but Y is exact.

2. Main Results

2.1. Approximation Spaces to Topological Spaces

Definition 2.1. Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map. We say that f is an *order-preserving map* or *granule-preserving map* if for all $x, y \in U$,

$$xRy \implies f(x)Sf(y).$$

Definition 2.2. Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$. A map f is called a *rough isomorphism* if
(i) f is granule-preserving , bijective and
(ii) f^{-1} is granule-preserving.

Remark 2.3. From Example 1.5, if we let $B \subset A$, then the identity map $i : (U, I(A)) \to (U, I(B))$ is granule-preserving and bijective but the inverse is not granule-preserving. Thus in the definition 2.2, the condition (i) does not imply the condition (ii).

Definition 2.4. Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map. We say that
f is an *exact map* if for any exact set $X \subseteq U$, $f(X)$ is exact in V and
f is a *rough map* if for any rough set $X \subseteq U$, $f(X)$ is rough in V.

Lemma 2.5. *Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map. If f is a granule-preserving map, then $f([x]_R) \subseteq [f(x)]_S$, $\forall x \in U$.*

Proof. Let f be a granulepreserving map.
Assume that $y \in f([x]_R)$. That is $y = f(z)$ for some $z \in [x]_R$.
Then xRz. Since f is a granulepreserving map, $f(x)Sf(z)$.
Therefore, $y = f(z) \in [f(x)]_R$. Hence, $f([x]_R) \subseteq [f(x)]_S$. ∎

As stated in the introduction, Kondo [4] showed that for the approximation space (U, R), we can define a topology induced by R, namely τ_R given by

$$\tau_R = \{X \subseteq U \mid R_*(X) = X\}.$$

We will now discuss the continuity of a map between two approximation spaces, but first let us recall the definition of a continuous map, which is as follows.

Definition 2.6. Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map.

We say that f is *continuous* if for every open set B in V, i.e. $S_*(B) = B$, $f^{-1}(B)$ is open in U, i.e. $R_*(f^{-1}(B)) = f^{-1}(B)$.

Definition 2.7. Let (U, τ_R) and (V, τ_S) be two topological space spaces and $f : U \to V$ be a map.

We say that f is a *homeomorphism* if
(i) f is continuous, bijective and
(ii) f^{-1} is continuous.

Theorem 2.8. *Let (U, R) and (V, S) be two approximation space and $f : U \to V$ be a map. Then f is a granule-preserving map if and only if f is a continuous map.*

Proof. Assume that f is a granule-preserving map.
Let B be an open set in V. Then $S_*(B) = B$.
We want to show that $f^{-1}(B)$ is open in U
which is enough to show that $f^{-1}(B) \subseteq R_*(f^{-1}(B))$.
Assume that $x \in f^{-1}(B)$. So $f(x) \in B = S_*(B)$.
Therefore, $[f(x)]_S \subseteq B$.
By Lemma 2.5, $f([x]_R) \subseteq [f(x)]_S \subseteq B$, so $[x]_R \subseteq f^{-1}(B)$.
Hence, $x \in R_*(f^{-1}(B))$. Therefore, $f^{-1}(B) \subseteq R_*(f^{-1}(B))$.
Conversely, assume that f is continuous and let $x, y \in U$.
Assume that $f(x)$ is not S-related to $f(y)$.
Then $[f(x)]_S \cap [f(y)]_S = \emptyset$.
Since $S_*([f(x)]_S) = [f(x)]_S$ and $S_*([f(y)]_S) = [f(y)]_S$.
Then $[f(x)]_S$ and $[f(y)]_S$ are open. By the continuity of f,
we obtain that $x \in \{x\} \subseteq f^{-1}(f(\{x\})) \subseteq f^{-1}([f(x)]_S) = R_*(f^{-1}([f(x)]_S))$.
Therefore, $[x]_R \subseteq f^{-1}([f(x)]_S)$.
Hence, $f([x]_R) \subseteq f(f^{-1}([f(x)]_S)) \subseteq [f(x)]_S$.
Similarly, $f([y]_R) \subseteq [f(y)]_S$.
Thus, $f([x]_R) \cap f([y]_R) \subseteq [f(x)]_S \cap [f(y)]_S = \emptyset$. Then $f([x]_R) \cap f([y]_R) = \emptyset$.
If xRy, then $[x]_R = [y]_R$.
Therefore, $f(x) \in f([x]_R) \subseteq f([x]_R) \cap f([y]_R) = \emptyset$, which is a contradiction.
Thus x is not R-related to y, so f is a granule-preserving map. ∎

Corollary 2.9. *Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$. Then f is a rough isomorphism if and only if f is a homeomorphism.*

Proof. Clearly from Theorem 2.8. ∎

Theorem 2.10. *Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map. If f is a surjective rough map, then f is continuous.*

Proof. Let B be an open subset in V. Then $S_*(B) = B$.
We want to show that $f^{-1}(B)$ is open in U which is enough to show that $f^{-1}(B) \subseteq R_*(f^{-1}(B))$.
Let $x \in f^{-1}(B)$. Then $f(x) \in B = S_*(B)$, so $[f(x)]_S \subseteq B$.
Let $y \in [x]_R$. Then xRy.
If $y \notin f^{-1}(B)$, then $f^{-1}(B)$ is rough, so $f(f^{-1}(B))$ is rough by the assumption.
Since f is surjective, $f(f^{-1}(B)) = B$. Thus, B is rough.
Therefore, there exists $z \in V$, $z \in S^*(B)$ but $z \notin S_*(B) = B$.
Then $[z]_S \cap B \neq \emptyset$. Thus, there exists $w \in [z]_S$ and $w \in B$.
By Theorem 1.3(10), we obtain that $S^*(S_*(B)) = B$.
Thus, $z \in S^*(\{w\}) \subseteq S^*(B) = S^*(S_*(B)) = B$.
This contradicts to the fact that $z \notin B$.
Hence, $y \in f^{-1}(B)$. Therefore, $[x]_R \subseteq f^{-1}(B)$. Therefore, $x \in R_*(f^{-1}(B))$.
Thus, $f^{-1}(B) \subseteq R_*(f^{-1}(B))$ as required. ∎

Theorem 2.11. *Let (U, R) and (V, S) be two approximation spaces and $f : U \to V$ be a map. If f is an injective continuous map, then f is a rough map.*

Proof. Let A be a rough subset of U. Suppose that $f(A)$ is exact, so by Theorem 1.3(1), $S_*(f(A)) = f(A)$ or $f(A)$ is open in V.
Since f is continuous, $f^{-1}(f(A))$ is open in U. By the injectivity of f, we have that $f^{-1}(f(A)) = A$.
Hence, A is open in U or A is exact, which is a contradiction
Therefore, $f(A)$ is rough. This implies that f is a rough map. ∎

The surjective property of a map cannot be dropped from Theorem 2.10 as we can see in the following example.

Example 2.12. Let $U = \{x, y, z\}$ and $V = \{1, 2, 3, 4\}$.
Let R and S be the equivalence relations such that $U/R = \{\{x, y\}, \{z\}\}$ and $V/S = \{\{1, 4\}, \{2, 3\}\}$.
Notice that $\{x\}, \{y\}, \{x, z\}$ and $\{y, z\}$ are rough subsets of U.
Define the function $f : U \to V$ by $f(x) = 1$, $f(y) = 2$ and $f(z) = 2$.
We have that $f(\{x\}) = \{1\}, f(\{y\}) = \{2\}, f(\{x, z\}) = \{2\}, f(\{y, z\}) = \{1, 2\}$.
Hence, f is rough. Clearly, the map f is not a granule-preserving map. As a result, according to the Theorem 2.8, it is not continuous.

Moreover, you can show that the injective property is necessary for Theorem 2.11, as shown in the following.

Example 2.13. Let $U = \{1, 2, 3, 4\}$ and $V = \{x, y\}$.
Let R and S be the equivalence relations such that
$U/R = \{\{1, 2\}, \{3, 4\}\}$ and $V/S = \{\{x\}, \{y\}\}$.
Define the function $f : U \to V$ by $f(x) = 1$, $f(y) = 2$ and $f(z) = 2$.
We have that $f(\{1\}) = \{x\}, f(\{2\}) = \{x\}, f(\{3\}) = \{y\}$ and $f(\{4\}) = \{y\}$.
Hence, f is not injective but continuous. Notice that $A = \{2, 3\}$ is rough but $f(A) = \{x, y\}$ is exact. Therefore, f is not a rough map.

Lastly, we cannot replace "rough map" with "exact map" in Theorem 2.10 and Theorem 2.11 by illustrating a bijective discontinuous exact map as follows.

Example 2.14. Let $U = V = \{1, 2\}$.
Let R and S be the equivalence relations such that
$U/R = \{\{1, 2\}\}$ and $V/S = \{\{1\}, \{2\}\}$.
Define the function $f : U \to V$ by $f(1) = 1$ and $f(2) = 2$.
Hence, f is bijective discontinuous exact map.

3. Conclusion

We showed that the granule-preserving map is continuous as well as vice versa. Moreover, we also showed that the surjective rough map is a continuous map. But on the other hand, the injective continuous map is a rough map.

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References

[1] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences 11 (1982) 341–356.

[2] Z. Pawlak, Rough functions, Bull. Polish Acad. Sci. Tech. 35 (1987) 249–251.

[3] D. Qiu, L. Shu, Rough mappings, IEEE International Conference on Information Reuse and Integration, Las Vegas, NV, USA (2007) 496–501.

[4] M. Kondo, On the structure of generalized rough sets, Information Sciences 176 (5) (2006) 589–600.