Thai Journal of Mathematics
Volume 22 Number 1 (2024)
Pages 179–196
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Annual Meeting in Mathematics 2023

A Fibonacci Galerkin Method for Solving Certain Types of Boundary Value Problems

Sanoe Koonprasert[1,2], **Sekson Sirisubtawee**[1,2,*], **Pruchayaporn Srisutat**[1], **Pattarawadee Petkongkaew**[1] **and Elvin J. Moore**[1,2]

[1] *Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*
e-mail : sanoe.k@sci.kmutnb.ac.th (S. Koonprasert); sekson.s@sci.kmutnb.ac.th (S. Sirisubtawee); ppruchayaa@gmail.com (P. Srisutat); rose.petkongkaew2000@gmail.com (P. Petkongkaew); elvin.j@sci.kmutnb.ac.th (E. Moore)

[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*

Abstract In the Fibonacci sequence, the first two numbers are 0 and 1, and the next numbers are equal to the sum of the previous two numbers. In this paper, the concept of the Fibonacci sequence is extended to polynomial functions which can be effectively applied in various function approximations. Based on these Fibonacci-based polynomials and the Galerkin method, we develop a Fibonacci Galerkin method (FGM) to solve some types of boundary value problems (BVPs) such as a linear singular two-point BVP and a nonlinear multi-point BVP. The FGM process constructs a residual function for a BVP by utilizing an approximate solution formed by the method and then evaluating the integral of the product between residual functions and weight functions over a domain. Equating the value of the integral close to zero, one obtains an analytical solution of the BVP. As examples, we apply the method to certain types of BVPs including a linear singular two-point BVP, a nonlinear multi-point BVP and a regular two-point BVP whose exact solutions are given. Their semi-analytical solutions are obtained. By comparing the solutions of the proposed boundary value problems obtained by this technique with their exact solutions, we believe that the technique is highly accurate and effective.

MSC: 65L60; 65N30
Keywords: Fibonacci polynomial; Galerkin method; boundary value problems

Submission date: 02.06.2023 / Acceptance date: 31.08.2023

1. Introduction

Nonlinear ordinary differential equations (NODEs) have been used as models for a variety of problems in many fields such as engineering [1], chemistry [2], physics [3], and biology [4, 5]. In most cases, it is very difficult to find exact solutions for these nonlinear

*Corresponding author.

problems using analytical methods [6]. Therefore, numerical methods have been used to find approximate solutions for these problems including initial value problems (IVPs) and boundary value problems (BVPs). There have been many different methods developed for solving differential equations, for example, finite element methods [7], finite difference methods [8], shooting method [9], Raleigh method [10] and spectral methods [11, 12].

Spectral methods [11, 12] are a type of numerical method for solving differential equations that use a combination of interpolation and approximation techniques to represent a solution as a sum of simple functions. These simple functions are often orthogonal functions, such as Fourier series [13], Chebyshev polynomials [14] and Legendre polynomials [15]. Spectral methods are particularly effective for problems for which the solutions are smooth as the methods can then converge exponentially fast to exact solutions and they require relatively few function evaluations. Under these conditions, the spectral methods become efficient, reliable, and effective for solving problems that need high-precision solutions or that have complex geometries such as fluid flows with irregular boundaries.

Another advantage of spectral methods is their versatility because they can be utilized to solve a wide range of differential equations including linear and nonlinear problems, initial value problems, and boundary value problems. They can also be adapted to handle partial differential equations. In addition, spectral methods are known to be highly parallelizable and therefore suitable for implementation on modern computing architectures. Generally, spectral methods are powerful and flexible tools for solving differential equations and their combination of accuracy and efficiency makes them a popular choice in many computational fields.

A basic idea of spectral methods is to assume an approximate solution written in terms of a finite sum of a product of simple functions such as polynomials or elementary functions with unknown coefficients. Here, we assume that an analytic function $f(x)$ is a solution of a given problem and that its approximation $\tilde{f}(x)$ can be expressed as

$$\tilde{f}(x) = \sum_{n=0}^{N} a_n \psi_n(x), \tag{1.1}$$

where the set of $\psi_n(x)$ contains special trial polynomial (or function) bases and a_n are unknown constant coefficients. In general, the main idea of these methods is to convert the given ODEs to a system of algebraic equations of the unknown coefficients. Then, depending upon the linearity or nonlinearity of the system of algebraic equations, it is usually easy to solve the system using analytical or numerical methods to obtain values for the unknown coefficients and therefore obtain the solution of the original problem.

Smooth or orthogonal polynomials are a commonly used type of basis functions in spectral methods. These polynomials are defined on a given interval. For example, Chebyshev polynomials are an important kind of polynomial that is commonly used in spectral methods. They are usually defined on the interval $[-1, 1]$. Shifted Chebyshev polynomials are a variation of Chebyshev polynomials that are defined on a different interval such as $[0, 1]$. Legendre polynomials are also another kind of polynomial type that is commonly used in spectral methods. They are also defined on the interval $[-1, 1]$. Shifted Legendre polynomials are a variation of Legendre polynomials that are constructed on a different interval such as $[0, 1]$.

Many researchers have used a variety of types of polynomials in spectral methods in order to solve ODEs, IVPs and BVPs. In this paper, we are interested in exploring the use of Fibonacci polynomials as a basis for the Galerkin method [16] for solving boundary value problems. The Galerkin method is a well-known numerical technique that has been used for many years for solving these types of problems. Since the method uses a suitable set of basis functions for approximating solutions to the problem, the accuracy of the method depends on the choice of the basis functions. The Galerkin method has many interesting properties that are useful and appropriate for the basis functions used in deriving the coefficients of the Galerkin approximation. Consequently, seeking new basis functions such as Fibonacci polynomials could lead to more accurate and efficient methods of solution.

The Fibonacci polynomials [17] satisfy the following recurrence relation

$$F_{n+2}(x) - xF_{n+1}(x) - F_n(x) = 0, \quad -\infty < x < \infty, \tag{1.2}$$

with initial conditions

$$F_0(x) = 1,\ F_1(x) = x. \tag{1.3}$$

The Fibonacci polynomials are of interest because of their mathematical applications such as the golden ratio, Fibonacci numbers and quasi-crystals. These applications can provide insights into the behavior of the solutions to boundary value problems and inspire new research directions. Therefore, the use of Fibonacci polynomials as a basis function for the Galerkin method is a promising approach to solving boundary value problems.

In this paper, we aim to develop a new type of Galerkin method for solving some boundary value problems based on the Fibonacci polynomials. As far as the authors are aware, the Fibonacci polynomials have never been used as a set of basis functions for the Galerkin method. Even though we will only test the FGM on BVPs in this paper, we believe that the method could be successfully used for solving related problems. The remaining parts of this paper are organized as follows. In Section 2, we define Fibonacci polynomials and discuss some of their important properties. In Section 3, we propose the Fibonacci Galerkin method (FGM) and we discuss its applications for solving BVPs in Section 4. Finally, in Section 5, we discuss some conclusions for the paper.

2. Definition and Properties of the Fibonacci Polynomials

In this section, we provide a definition of the k-Fibonacci sequence and Fibonacci polynomials [16, 17]. The characteristics of these polynomials are also given.

Definition 2.1. For any positive real number k, the k-Fibonacci sequence is recursively defined as

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1, \tag{2.1}$$

with the initial conditions

$$F_{k,0} = 1 \text{ and } F_{k,1} = k.$$

If the number k and the sequence $F_{k,n}$ in Definition 2.1 are replaced by the variable x and $F_n(x)$, respectively, then we can define the Fibonacci polynomials as follows.

Definition 2.2. Let n be any nonnegative integer. The Fibonacci polynomials are defined by the following recursive formula

$$F_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ x, & \text{if } n = 1, \\ xF_{n-1}(x) + F_{n-2}(x), & \text{if } n > 1. \end{cases} \tag{2.2}$$

From the recursion formula, the Fibonacci polynomials can be written as

$$\begin{aligned} F_0(x) &= 1, \\ F_1(x) &= x, \\ F_2(x) &= x^2 + 1, \\ F_3(x) &= x^3 + 2x, \\ &\vdots \\ F_n(x) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}, \quad n \geq 0, \end{aligned}$$

where the notation $\lfloor z \rfloor$ represents the largest integer less than or equal to z.

It is worth noticing that the Fibonacci polynomials satisfy the following property.

Proposition 2.1. *If $F_n(x)$, $n = 0, 1, ...$ are the Fibonacci polynomials, then we obtain*

$$\int_0^1 F_n(x)F_m(x)dx = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-i}{i} \binom{m-j}{j} \frac{1}{n+m-2i-2j+1}. \tag{2.3}$$

The next proposition indicates the relationship between the Fibonacci polynomials and its derivatives.

Proposition 2.2. *The following equality*

$$F_n(x) = \frac{1}{n+1} \left[F'_{n+1}(x) + F'_{n-1}(x) \right] \tag{2.4}$$

holds for $n \in \mathbb{N}$. Then, integrating this equation, we obtain the following integral equation

$$\int_0^x F_n(s)ds = \frac{1}{n+1} \left[F_{n+1}(x) + F_{n-1}(x) - F_{n+1}(0) - F_{n-1}(0) \right]. \tag{2.5}$$

If n is odd, then we have $F_{n+1}(0) = F_{n-1}(0) = 1$, and, if n is even, we have $F_{n+1}(0) = F_{n-1}(0) = 0$.

Next, we can approximately expand a function $f \in L^2[0, 1)$ using the Fibonacci polynomials.

Function approximation

Suppose that a continuous function $f(x)$ can be written in terms of the Fibonacci polynomials $F_n(x)$ as [18]

$$f(x) = \sum_{n=0}^{\infty} a_n F_n(x), \tag{2.6}$$

where a_n is the Fibonacci coefficient of the series for $f(x)$. Then, the function $f(x)$ in (2.6) can be approximated using the truncated expansion of N Fibonacci polynomials as

$$f(x) \approx \sum_{n=0}^{N} a_n F_n(x). \tag{2.7}$$

3. Fibonacci Galerkin Method for Solving BVP

In this section, the main steps of the Fibonacci-Galerkin method (FGM) are proposed as follows. Consider the boundary value problem consisting of a general linear second-order differential equation

$$p(x)u''(x) + q(x)u'(x) + r(x)u(x) = F(x), \tag{3.1}$$

and the boundary conditions

$$u(a) = u_a, \quad u(b) = u_b, \tag{3.2}$$

where $p(x), q(x), r(x)$ and $F(x)$ are continuous functions. Suppose that a solution $u(x)$ of the BVP (3.1)-(3.2) can be approximated by $u_{\text{FGM}}(x)$, which is obtained by using the FGM. The approximate solution $u_{\text{FGM}}(x)$ is written as

$$u_{\text{FGM}}(x) = \sum_{i=0}^{N} c_i \phi_i(x), \tag{3.3}$$

where c_i, $i = 0, 1, ..., N$ are the constant coefficients for some positive integer N. The values of the coefficients can be determined later. The function $\phi_i(x)$, $i = 0, 1, ..., N$ are the Fibonacci basis functions, which are recursively obtained by

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ &\vdots \\ \phi_{i+1}(x) &= x\phi_i(x) + \phi_{i-1}(x), \quad i = 1, 2, 3, 4, ... \end{aligned} \tag{3.4}$$

Define the residual function of Eq. (3.1) as

$$\begin{aligned} R(x) &= p(x)u''_{\text{FGM}}(x) + q(x)u'_{\text{FGM}}(x) + r(x)u_{\text{FGM}}(x) - F(x), \\ &= \sum_{i=0}^{N} c_i \left(p(x)\phi''_i(x) + q(x)\phi'_i(x) + r(x)\phi_i(x)\right) - F(x). \end{aligned} \tag{3.5}$$

It is required that the function $u_{\text{FGM}}(x)$ in (3.3) is the approximate solution of the above BVP if the following integral holds

$$\int_a^b W_j(x)R(x)dx = 0, \tag{3.6}$$

where $W_j(x)$ is a weight function. In particular, we select the weight function $W_j(x)$ to be the Fibonacci basis function $\phi_j(x)$, i.e.,

$$W_j(x) = \phi_j(x), \quad j = 0, 1, 2, 3, ..., N-2, \tag{3.7}$$

where $\phi_j(x)$ is defined in (3.4). From (3.5) and (3.6), for $j = 0, 1, 2, 3, ..., N-2$, we have

$$\int_a^b \phi_j(x)\left[\sum_{i=0}^{N} c_i\left(p(x)\phi_i''(x) + q(x)\phi_i'(x) + r(x)\phi_i(x)\right)\right]dx = \int_a^b \phi_j(x)F(x)dx,$$

$$\sum_{i=0}^{N} c_i\left[\int_a^b \left(p(x)\phi_j(x)\phi_i''(x) + q(x)\phi_j(x)\phi_i'(x) + r(x)\phi_j(x)\phi_i(x)\right)dx\right] = \int_a^b \phi_j(x)F(x)dx. \tag{3.8}$$

Using the boundary conditions (3.2) and equation (3.8) for $j = 0, 1, 2, 3, ..., N-2$, we obtain a system of linear equations in variables c_i, $i = 0, 1, ..., N$ as follows:

$$\begin{aligned}
x = a &: \sum_{i=0}^{N} c_i\phi_i(a) = u_a,\\
x = b &: \sum_{i=0}^{N} c_i\phi_i(b) = u_b,\\
\phi_0(x) &: \sum_{i=0}^{N} c_i\left[\int_a^b p(x)\phi_0(x)\phi_i''(x)dx + \int_a^b q(x)\phi_0(x)\phi_i'(x)dx + \int_a^b r(x)\phi_0(x)\phi_i(x)dx\right]\\
&= \int_a^b \phi_0(x)F(x)dx,\\
\phi_1(x) &: \sum_{i=0}^{N} c_i\left[\int_a^b p(x)\phi_1(x)\phi_i''(x)dx + \int_a^b q(x)\phi_1(x)\phi_i'(x)dx + \int_a^b r(x)\phi_1(x)\phi_i(x)dx\right]\\
&= \int_a^b \phi_1(x)F(x)dx,\\
\phi_2(x) &: \sum_{i=0}^{N} c_i\left[\int_a^b p(x)\phi_2(x)\phi_i''(x)dx + \int_a^b q(x)\phi_2(x)\phi_i'(x)dx + \int_a^b r(x)\phi_2(x)\phi_i(x)dx\right]\\
&= \int_a^b \phi_2(x)F(x)dx,\\
\phi_3(x) &: \sum_{i=0}^{N} c_i\left[\int_a^b p(x)\phi_3(x)\phi_i''(x)dx + \int_a^b q(x)\phi_3(x)\phi_i'(x)dx + \int_a^b r(x)\phi_3(x)\phi_i(x)dx\right]\\
&= \int_a^b \phi_3(x)F(x)dx,\\
&\vdots\\
\phi_{N-2}(x) &: \sum_{i=0}^{N} c_i\left[\int_a^b p(x)\phi_{N-2}(x)\phi_i''(x)dx + \int_a^b q(x)\phi_{N-2}(x)\phi_i'(x)dx + \int_a^b r(x)\phi_{N-2}(x)\phi_i(x)dx\right]\\
&= \int_a^b \phi_{N-2}(x)F(x)dx.
\end{aligned} \tag{3.9}$$

After solving the above system for c_i, $i = 0, 1, ..., N$, then the approximate solution of the BVP (3.1)-(3.2) obtained via the FGM can be expressed in (3.3) with the known coefficients.

In the following section, we will show the performance of the proposed method to numerically solve certain BVPs. All of the computational steps have been implemented using MATLAB® R2020b.

4. FGM Solutions for Some Boundary Value Problems

In this section, we illustrate applications of our proposed method to some interesting BVPs including a singular two-point boundary value problem, a nonlinear multi-point boundary value problem and a nonsingular two-point boundary value problem. By constructing the solutions to the problem via the FGM, we can clarify the accuracy, efficiency, and reliability of the obtained solutions compared with the given exact solutions.

Example 4.1. Use of the Fibonacci Galerkin method to find the solution of the following singular two-point boundary value problem:

$$xu''(x) + 2u'(x) + x = 0, \quad 0 < x < 1, \tag{4.1}$$

with the boundary conditions

$$u'(0) = 0 \quad \text{and} \quad u(1) = \frac{1}{6}. \tag{4.2}$$

Solution: It is easy to check that the exact solution of the above problem is $u(x) = \frac{1}{3} - \frac{x^2}{6}$. By the FGM, we assume that the solution to the above problem can be approximated as

$$u(x) \approx u_{\text{FGM}}(x) := \sum_{i=0}^{3} c_i \phi_i(x), \tag{4.3}$$

where c_i is the constant coefficient, which will be calculated at the next step, and $\phi_i(x)$ is the Fibonacci basis function expressed for $i = 0, 1, 2, 3$ as

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ \phi_2(x) &= x^2 + 1, \\ \phi_3(x) &= x^3 + 2x. \end{aligned} \tag{4.4}$$

Now, the solution form and its derivatives are

$$\begin{aligned} u_{\text{FGM}}(x) &= c_0 + c_1 x + c_2(x^2 + 1) + c_3(x^3 + 2x), \\ u'_{\text{FGM}}(x) &= c_1 + 2c_2 x + c_3(3x^2 + 2), \\ u''_{\text{FGM}}(x) &= 2c_2 + 6c_3 x. \end{aligned} \tag{4.5}$$

Using the first two equations of (4.5) and the conditions in (4.2), we have

$$\begin{aligned} c_1 + 2c_3 &= 0, \\ c_0 + c_1 + 2c_2 + 3c_3 &= \frac{1}{6}. \end{aligned} \tag{4.6}$$

Defining the residual function of Eq.(4.1) and substituting the terms in (4.5) into the resulting equation, we have

$$\begin{aligned} R(x) &= xu''_{\text{FGM}}(x) + 2u'_{\text{FGM}}(x) + x, \\ &= x(2c_2 + 6c_3 x) + 2(c_1 + 2c_2 x + c_3(3x^2 + 2)) + x, \\ &= 12c_3 x^2 + (6c_2 + 1)x + 2(c_1 + 2c_3). \end{aligned} \tag{4.7}$$

Using the residual condition (3.6) over the interval $[0, 1]$, we have

$$\int_0^1 W_j(x) \left(12c_3 x^2 + (6c_2 + 1)x + 2(c_1 + 2c_3)\right) dx = 0. \tag{4.8}$$

Choosing the weight function $W_j(x)$ in (4.8) to be $\phi_j(x)$ in (4.4) for $j = 0, 1$, we obtain two more linear equations in c_0, c_1, c_2 and c_3 as:

$$\begin{aligned} \int_0^1 1 \cdot \left(12c_3x^2 + (6c_2+1)x + 2(c_1+2c_3)\right) dx &= 2c_1 + 3c_2 + 8c_3 + \frac{1}{2} = 0, \\ \int_0^1 x \cdot \left(12c_3x^2 + (6c_2+1)x + 2(c_1+2c_3)\right) dx &= c_1 + 2c_2 + 5c_3 + \frac{1}{3} = 0, \end{aligned} \tag{4.9}$$

respectively. From (4.6) and (4.9), we obtain the system of four linear equations in c_i, $i = 0, 1, 2, 3$ as follows

$$\begin{aligned} c_1 + 2c_3 &= 0, \\ c_0 + c_1 + 2c_2 + 3c_3 &= \frac{1}{6}, \\ 2c_1 + 3c_2 + 8c_3 &= -\frac{1}{2}, \\ c_1 + 2c_2 + 5c_3 &= -\frac{1}{3}. \end{aligned} \tag{4.10}$$

Solving system (4.10), we get the coefficients $c_0 = \frac{1}{2}$, $c_1 = 0$, $c_2 = -\frac{1}{6}$, $c_3 = 0$ and hence the solution obtained via the FGM is

$$\begin{aligned} u_{\text{FGM}}(x) &= c_0 + c_1x + c_2(x^2+1) + c_3(x^3+2x), \\ &= \frac{1}{3} - \frac{x^2}{6}. \end{aligned} \tag{4.11}$$

It can be noticed that the solution $u_{\text{FGM}}(x)$ in (4.11) is the exact solution of the problem. The exact solution and the FGM solution are shown in Figure 1. □

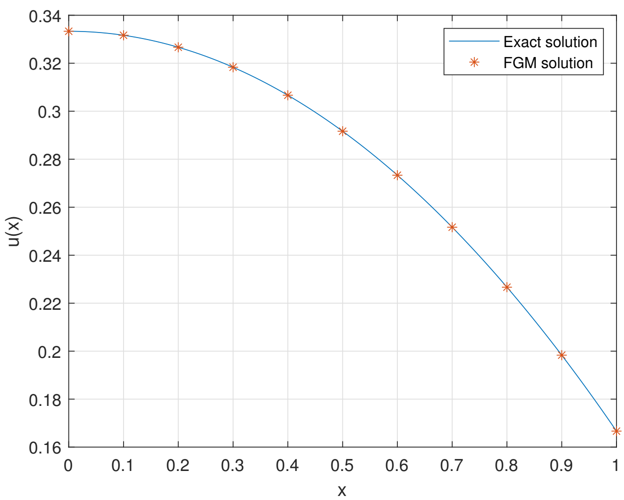

FIGURE 1. Graphical solutions of the BVP (4.1) and (4.2). The solid line represents the exact solution and $*$ represents the FGM solution.

Example 4.2. Use of the Fibonacci Galerkin method to find the solution of the following nonlinear multi-point boundary value problem:

$$(u''(x))^2 - 3u'(x) = 0, \quad -1 < x < 2, \tag{4.12}$$

with the boundary conditions

$$u(-1) = -\frac{1}{4}, \quad u(1) = \frac{7}{4} \quad \text{and} \quad u(2) = 2. \tag{4.13}$$

Solution: It is not difficult to verify that $u(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{2}$ is the exact solution of the above BVP. As observed in the exact solution that the highest degree of the solution is 3, so we at least choose $N = 3$ for the approximate solution of the Fibonacci-Galerkin method. Therefore, the solution of the problem, which is solved by the FGM, can be approximated by

$$u(x) \approx u_{\mathrm{FGM}}(x) := \sum_{i=0}^{3} c_i \phi_i(x), \tag{4.14}$$

where c_i is the constant coefficient, which will be calculated at the next step, and $\phi_i(x)$ is the Fibonacci basis function for $i = 0, 1, 2, 3$ as shown in (4.4). Then, the solution form and its derivatives are

$$\begin{aligned} u_{\mathrm{FGM}}(x) &= c_0 + c_1 x + c_2(x^2 + 1) + c_3(x^3 + 2x), \\ u'_{\mathrm{FGM}}(x) &= c_1 + 2c_2 x + c_3(3x^2 + 2), \\ u''_{\mathrm{FGM}}(x) &= 2c_2 + 6c_3 x. \end{aligned} \tag{4.15}$$

Using the first equation of (4.15) and the conditions in (4.13), we have

$$\begin{aligned} c_0 - c_1 + 2c_2 - 3c_3 &= -\frac{1}{4}, \\ c_0 + c_1 + 2c_2 + 3c_3 &= \frac{7}{4}, \\ c_0 + 2c_1 + 5c_2 + 12c_3 &= 2. \end{aligned} \tag{4.16}$$

Defining the residual function of Eq.(4.12) and substituting the terms in (4.15) into the resulting equation, we have

$$\begin{aligned} R(x) &= (u''_{\mathrm{FGM}}(x))^2 - 3u'_{\mathrm{FGM}}(x), \\ &= (2c_2 + 6c_3 x)^2 - 3(c_1 + 2c_2 x + c_3(3x^2 + 2)), \\ &= (36c_3^2 - 9c_3)x^2 + (24c_2c_3 - 6c_2)x + (4c_2^2 - 3c_1 - 6c_3). \end{aligned} \tag{4.17}$$

Using the residual condition (3.6) over the interval $[-1, 2]$, we have

$$\int_{-1}^{2} W_j(x) \left((36c_3^2 - 9c_3)x^2 + (24c_2c_3 - 6c_2)x + (4c_2^2 - 3c_1 - 6c_3)\right) dx = 0. \tag{4.18}$$

Choosing the weight function $W_j(x)$ in (4.18) to be $\phi_j(x)$ in (4.4) for $j = 0$, we obtain one more linear equation in c_0, c_1, c_2 and c_3 as:

$$\begin{aligned} &\int_{-1}^{2} 1 \cdot \left((36c_3^2 - 9c_3)x^2 + (24c_2c_3 - 6c_2)x + (4c_2^2 - 3c_1 - 6c_3)\right) dx \\ &\qquad = 4c_2^2 + 12c_2c_3 + 36c_3^2 - 3c_1 - 3c_2 - 15c_3 = 0. \end{aligned} \tag{4.19}$$

From (4.16) and (4.19), we obtain the system of nonlinear equations in the variables c_i, $i = 0, 1, 2, 3$ as follows

$$\begin{aligned} c_0 - c_1 + 2c_2 - 3c_3 &= -\frac{1}{4}, \\ c_0 + c_1 + 2c_2 + 3c_3 &= \frac{7}{4}, \\ c_0 + 2c_1 + 5c_2 + 12c_3 &= 2, \\ 4c_2^2 + 12c_2c_3 + 36c_3^2 - 3c_1 - 3c_2 - 15c_3 &= 0. \end{aligned} \tag{4.20}$$

Applying the Newton-Raphson scheme to numerically solve nonlinear system (4.20), we obtain the coefficients $c_0 = \frac{9}{4}$, $c_1 = \frac{1}{4}$, $c_2 = -\frac{3}{4}$, $c_3 = \frac{1}{4}$ and therefore the solution of the problem obtained via the FGM is

$$\begin{aligned} u_{\text{FGM}}(x) &= c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x), \\ &= \frac{9}{4} + \frac{1}{4}x - \frac{3}{4}(x^2+1) + \frac{1}{4}(x^3+2x) \\ &= \frac{x^3}{4} - \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{2}. \end{aligned} \tag{4.21}$$

It can be observed that the obtained solution $u_{\text{FGM}}(x)$ in (4.21) is the exact solution of the problem. The solution graphs of the problem including the exact solution and the FGM solution are shown in Figure 2. $\square$

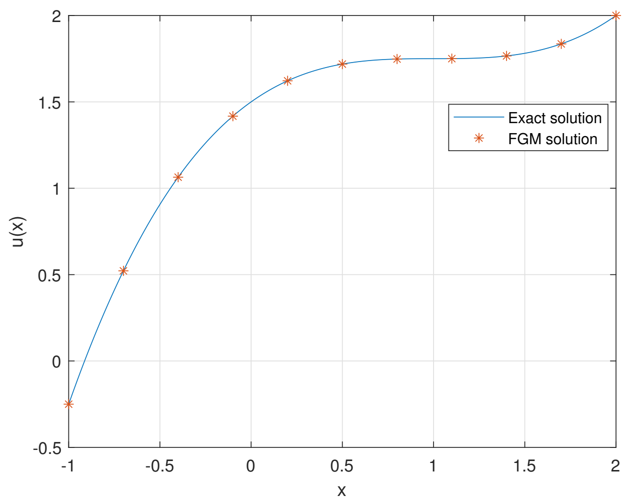

FIGURE 2. Graphical solutions of the BVP (4.12) and (4.13). The sold line represents the exact solution and $*$ represents the FGM solution.

In addition, if we increase the number of solution components, i.e., the value of N, in the solution form in Example 4.2 to be $N = 4, 5, 6, 7$, then we find that the amount of work required in the process for finding the solutions is much more than for $N = 3$. This is because the number of residual conditions is greater and each condition is more complicated. The other conditions for the boundary conditions are also more complex since the approximate solution form has more terms. The last difficult work is to solve the system of nonlinear algebraic equations obtained from all conditions expressed in terms of the coefficients of the solution. However, by solving such a nonlinear system using appropriate software packages, we can eventually get a common set of solution coefficients from which the solution of the BVP is obtained. Therefore, using $N = 3$ in the approximate solution form is sufficient to get the solution of Example 4.2, which is the same as the exact solution.

Next, we want to investigate the effect of increasing the number of terms in an assumed solution for the FGM. In particular, we study the effect of increasing the integer number N in (3.3) on the accuracy of the approximate solution in the following example.

Example 4.3. Use of the Fibonacci Galerkin method (FGM) to construct the analytical solution of the nonsingular two-point boundary value problem:

$$u''(x) = -\frac{2}{x}u'(x) + \frac{2}{x^2}u(x) + \frac{\sin(\ln x)}{x^2}, \quad 1 < x < 2, \tag{4.22}$$

with the boundary conditions

$$u(1) = 1 \quad \text{and} \quad u(2) = 2. \tag{4.23}$$

It can be found in [19] that the exact solution of the above BVP is

$$u(x) = k_1 x + \frac{k_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x), \tag{4.24}$$

where

$$\begin{aligned} k_2 &= \frac{1}{70}\left(8 - 12\sin(\ln 2) - 4\cos(\ln 2)\right) \approx -0.03921, \\ k_1 &= \frac{11}{10} - k_2 \approx 1.13921. \end{aligned}$$

Solution: By the FGM, the approximate analytical solution to the above BVP can be written as

$$u_{\text{FGM}}(x) := \sum_{i=0}^{N} c_i \phi_i(x), \tag{4.25}$$

where c_i is the constant coefficient which will be determined at the next step, and $\phi_i(x)$ is the Fibonacci basis function expressed for $i = 0, 1, 2, ..., N$ as

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ \phi_2(x) &= x^2 + 1, \\ \phi_3(x) &= x^3 + 2x, \\ &\vdots \\ \phi_N(x) &= \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} x^{N-2i}, \quad N \geq 0. \end{aligned} \tag{4.26}$$

Now, the solution form and its derivatives are

$$
\begin{aligned}
u_{\mathrm{FGM}}(x) &= c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x) + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} x^{N-2i}, \\
u'_{\mathrm{FGM}}(x) &= c_1 + 2c_2 x + c_3(3x^2+2) + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} (N-2i) \binom{N-i}{i} x^{N-2i-1}, \\
u''_{\mathrm{FGM}}(x) &= 2c_2 + 6c_3 x + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} (N-2i)(N-2i-1) \binom{N-i}{i} x^{N-2(i+1)}.
\end{aligned} \tag{4.27}
$$

Using the first equation of (4.27) and the conditions in (4.23), we have the following two equations

$$
\begin{aligned}
c_0 + c_1 + 2c_2 + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} &= 1, \\
c_0 + 2c_1 + 5c_2 + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} 2^{N-2i} &= 2.
\end{aligned} \tag{4.28}
$$

Defining the residual function of Eq.(4.22) and substituting the terms in (4.27) into the resulting equation, we have

$$
\begin{aligned}
R(x) &= x^2 u''_{\mathrm{FGM}}(x) + 2x u'_{\mathrm{FGM}}(x) - 2u_{\mathrm{FGM}}(x) - \sin(\ln x), \\
&= x^2 \left(2c_2 + 6c_3 x + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} (N-2i)(N-2i-1) \binom{N-i}{i} x^{N-2(i+1)} \right) \\
&\quad + 2x \left(c_1 + 2c_2 x + c_3(3x^2+2) + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} (N-2i) \binom{N-i}{i} x^{N-2i-1} \right) \\
&\quad - 2 \left(c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x) + \ldots + c_N \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} x^{N-2i} \right) \\
&\quad - \sin(\ln x), \\
&= -2c_0 + (4x^2-2)c_2 + 10x^3 c_3 + \ldots + \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1) x^{N-2i} \right. \\
&\quad \left. \cdot \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x).
\end{aligned} \tag{4.29}
$$

Using the residual condition (3.6) over the interval $[1, 2]$, we have

$$
\begin{aligned}
&\int_1^2 W_j(x) \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3 c_3 + \ldots \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1) x^{N-2i} \cdot \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0.
\end{aligned} \tag{4.30}
$$

Selecting the weight function $W_j(x)$ in (4.30) to be $\phi_j(x)$ in (4.26) for $j = 0, 1, ..., N-2$, we obtain

$$
\begin{aligned}
&\int_1^2 1 \cdot \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3c_3 + ... \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1)x^{N-2i} \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0, \\
&\int_1^2 x \cdot \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3c_3 + ... \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1)x^{N-2i} \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0, \\
&\int_1^2 (x^2+1) \cdot \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3c_3 + ... \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1)x^{N-2i} \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0, \\
&\int_1^2 (x^3+2x) \cdot \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3c_3 + ... \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1)x^{N-2i} \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0, \qquad (4.31) \\
&\vdots \\
&\int_1^2 \left(\sum_{i=0}^{\lfloor \frac{N-2}{2} \rfloor} \binom{N-2-i}{i} x^{N-2-2i} \right) \cdot \Bigg[-2c_0 + (4x^2-2)c_2 + 10x^3c_3 + ... \\
&+ \left(\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N-i}{i} (N-2i-1)x^{N-2i} \left((N-2i)x^4 + 2 \right) \right) c_N - \sin(\ln x) \Bigg] dx = 0,
\end{aligned}
$$

respectively.

For $N = 3$: From (4.28) and the first two equations of (4.31), we obtain the system of four linear equations in the variables c_i, $i = 0, 1, 2, 3$ as follows

$$
\begin{aligned}
c_0 + c_1 + 2c_2 + 3c_3 &= 1, \\
c_0 + 2c_1 + 5c_2 + 12c_3 &= 2, \\
-12c_0 + 44c_2 + 225c_3 &= 3 - 6\cos(\ln 2) + 6\sin(\ln 2), \\
-15c_0 + 60c_2 + 310c_3 &= 5 - 8\cos^2\left(\ln\sqrt{2}\right) + 16\sin\left(\ln\sqrt{2}\right)\cos\left(\ln\sqrt{2}\right).
\end{aligned} \qquad (4.32)
$$

Solving system (4.32) using the MATLAB® program, we get the set of coefficients as follows

$$\{c_0 = 0.04798669084,\ c_1 = 0.6984798944,\ c_2 = 0.1530267129,\ c_3 = -0.01750667035\}.$$

Consequently, the solution of the BVP (4.22)-(4.23) obtained by the FGM is

$$\begin{aligned} u_{\mathrm{FGM}}(x) &= c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x), \\ &= 0.2010134037 + 0.6634665537x + 0.1530267129x^2 - 0.01750667035x^3. \end{aligned} \tag{4.33}$$

<u>For $N = 4$</u> : From (4.28) and the first three equations of (4.31), we obtain the system of five linear equations in the variables c_i, $i = 0, 1, 2, 3, 4$ as follows

$$\begin{aligned} c_0 + c_1 + 2c_2 + 3c_3 + 5c_4 &= 1, \\ c_0 + 2c_1 + 5c_2 + 12c_3 + 29c_4 &= 2, \\ -60c_0 + 220c_2 + 1125c_3 + 4128c_4 &= 15 - 30\cos(\ln 2) + 30\sin(\ln 2), \\ -15c_0 + 60c_2 + 310c_3 + 1155c_4 &= 5 - 8\cos^2\left(\ln\sqrt{2}\right) + 16\sin\left(\ln\sqrt{2}\right)\cos\left(\ln\sqrt{2}\right), \\ -1400c_0 + 5768c_2 + 29925c_3 + 112120c_4 &= 294 - 210\cos(\ln 2) + 210\sin(\ln 2) \\ &\quad - 336\cos^2\left(\ln\sqrt{2}\right) + 1008\sin\left(\ln\sqrt{2}\right)\cos\left(\ln\sqrt{2}\right). \end{aligned} \tag{4.34}$$

Solving system (4.34) using the MATLAB® program, we obtain the set of the coefficients as follows

$$\begin{aligned} \{c_0 = 0.08185298649,\ c_1 = 0.7016060141, c_2 = 0.1028249099, \\ c_3 = 0.01154785298, c_4 = -0.004750475856\}. \end{aligned}$$

Consequently, the solution of the BVP (4.22)-(4.23) obtained by the FGM is

$$\begin{aligned} u_{\mathrm{FGM}}(x) &= c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x) + c_4(x^4+3x^2+1), \\ &= 0.1799274205 + 0.7247017201x + 0.08857348233x^2 \\ &\quad + 0.01154785298x^3 - 0.004750475856x^4. \end{aligned} \tag{4.35}$$

<u>For $N = 5$</u> : From (4.28) and the first four equations of (4.31), we obtain the system of six linear equations in the variables c_i, $i = 0, 1, 2, 3, 4, 5$ as follows

$$\begin{aligned} c_0 + c_1 + 2c_2 + 3c_3 + 5c_4 + 8c_5 &= 1, \\ c_0 + 2c_1 + 5c_2 + 12c_3 + 29c_4 + 70c_5 &= 2, \\ -60c_0 + 220c_2 + 1125c_3 + 4128c_4 + 13320c_5 &= 11.09167125, \\ -15c_0 + 60c_2 + 310c_3 + 1155c_4 + 3780c_5 &= 3.034734605, \\ -1400c_0 + 5768c_2 + 29925c_3 + 112120c_4 + 368865c_5 &= 291.4460466, \\ -13.5c_0 + 58.5c_2 + 305.4285713c_3 + 1154.25c_4 + 3827.492062c_5 &= 2.954229089. \end{aligned} \tag{4.36}$$

Solving system (4.36) using the MATLAB® program, we obtain the set of the coefficients as follows

$$\begin{aligned} \{c_0 = 0.2126400029,\ c_1 = 0.5039892453, c_2 = -0.01444169476,\ c_3 = 0.2286816532, \\ c_4 = -0.09342913915,\ c_5 = 0.01166935967\}. \end{aligned}$$

Consequently, the solution of the BVP (4.22)-(4.23) obtained using the FGM is

$$\begin{aligned} u_{\mathrm{FGM}}(x) &= c_0 + c_1 x + c_2(x^2+1) + c_3(x^3+2x) + c_4(x^4+3x^2+1) + c_5(x^5+4x^3+3x), \\ &= 0.1047691690 + 0.9963606307x - 0.2947291122x^2 + 0.2753590919x^3 \\ &\quad - 0.09342913915x^4 + 0.01166935967x^5. \end{aligned} \tag{4.37}$$

A comparison of the numerical values of the exact solution (4.24) and the FGM solutions as expressed in (4.33) for $N = 3$, (4.35) for $N = 4$ and (4.37) for $N = 5$ at some specific points are shown in Table 1. As shown in the table, the value of the FGM solutions for $N = 3, 4, 5$ are in good agreement with the value of the exact solution at each indicated point and the values become closer to the exact value when the value of N is higher. This can also be seen from the values of the absolute errors between the exact solution and the FGM solutions for $N = 3, 4, 5$ as shown in Table 2. At the same point indicated in Table 2, all of the absolute errors for $N = 5$ are the lowest values when compared with the values for $N = 3$ and $N = 4$. Moreover, the solution graphs of the problem including the exact solution and the FGM solutions for $N = 3, 4, 5$ are plotted in Figure 3.

Since the exact solution u_{exact} of Example 4.3 is known, then the computational order of convergence (COC) of the sequence $\{u^N_{\mathrm{FGM}}\}_{N\geq 0}$, which is defined by [20]

$$\bar{\rho}_N = \frac{\ln |e_{N+1}/e_N|}{\ln |e_N/e_{N-1}|}, \tag{4.38}$$

where $e_N = u^N_{\mathrm{FGM}} - u_{\mathrm{exact}}$, can be calculated for some positive integer N. The term u^N_{FGM} denotes an approximate solution obtained using the FGM with the number of solution components N as shown in (3.3). For example, the value $\bar{\rho}_7$ of u_{FGM} in Example 4.3 will be studied at $x = 1.3$ and $x = 1.6$. First, the values of u^6_{FGM}, u^7_{FGM}, u^8_{FGM} evaluated at those two points must be calculated using the same procedure as mentioned above, but a harder endeavor is required. Secondly, the errors e_8, e_7, e_6 at the specified points can be obtained using the obtained approximate solutions. Finally, the values of the COC for $N = 7$, i.e., $\bar{\rho}_7$, of the sequence of the FGM solutions evaluated at $x = 1.3$ and $x = 1.6$ by using (4.38) are shown in Table 3. As seen in Table 3, the values of $\bar{\rho}_7$ of the FGM solutions for $x = 1.3$ and $x = 1.6$ are approximately 6.871 and 2.855, respectively. □

x	u_{exact}	u_{FGM} $(N=3)$	u_{FGM} $(N=4)$	u_{FGM} $(N=5)$
1.1	1.0926293	1.0926876	1.0926882	1.0926506
1.2	1.1870848	1.1872802	1.1872194	1.1871150
1.3	1.2833824	1.2836729	1.2835316	1.2833943
1.4	1.3814460	1.3817606	1.3815517	1.3814336
1.5	1.4811594	1.4814383	1.4811951	1.4811357
1.6	1.5823925	1.5826010	1.5823656	1.5823754
1.7	1.6850140	1.6851435	1.6849559	1.6850130
1.8	1.7888985	1.7889608	1.7888471	1.7889090
1.9	1.8939295	1.8939480	1.8939090	1.8939373

TABLE 1. Comparison between the exact solution and the FGM solutions for $N = 3, 4, 5$ at some specific interior points in $[1, 2]$.

x	$\lvert u_{\text{exact}} - u_{\text{FGM}} \rvert$ $(N=3)$	$\lvert u_{\text{exact}} - u_{\text{FGM}} \rvert$ $(N=4)$	$\lvert u_{\text{exact}} - u_{\text{FGM}} \rvert$ $(N=5)$
1.1	5.83×10^{-5}	5.89×10^{-5}	2.13×10^{-5}
1.2	1.95×10^{-4}	1.35×10^{-4}	3.01×10^{-5}
1.3	2.91×10^{-4}	1.49×10^{-4}	1.19×10^{-5}
1.4	3.15×10^{-4}	1.06×10^{-4}	1.24×10^{-5}
1.5	2.79×10^{-4}	3.56×10^{-5}	2.37×10^{-5}
1.6	2.08×10^{-4}	2.69×10^{-5}	1.71×10^{-5}
1.7	1.30×10^{-4}	5.81×10^{-5}	9.29×10^{-7}
1.8	6.23×10^{-5}	5.15×10^{-5}	1.04×10^{-5}
1.9	1.85×10^{-5}	2.05×10^{-5}	7.79×10^{-6}

TABLE 2. Absolute errors between the exact solution and the FGM solutions for $N = 3$, 4, 5 at some specific interior points in $[1, 2]$.

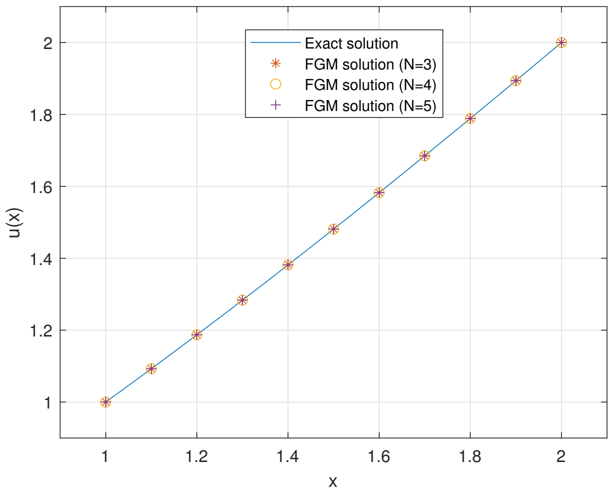

FIGURE 3. Graphical solutions of the BVP (4.22)-(4.23). The solid line represents the exact solution and the symbols $*$, o, $+$ denote the FGM solutions for $N = 3$, 4, 5, respectively.

x	$\lvert e_8/e_7 \rvert$	$\lvert e_7/e_6 \rvert$	$\bar{\rho}_7$
1.3	0.7696060	0.9626058	6.8713629
1.6	0.6304141	0.8507705	2.8548363

TABLE 3. Computational order of convergence $\bar{\rho}_7$ of the sequence of the FGM solutions in Example 4.3 when evaluated at $x = 1.3$ and $x = 1.6$ via (4.38).

5. Conclusions

In this paper, we proposed a Fibonacci Galerkin method (FGM) for constructing approximate analytical solutions of certain types of boundary value problems such as the linear singular two-point BVP shown in Example 4.1, the nonlinear multi-point BVP shown in Example 4.2 and the regular two-point BVP shown in Example 4.3. The exact solutions of the proposed problems were provided for comparison. The method is based on using the Galerkin method with Fibonacci basis functions as weight functions. The accuracy and reliability of the method can be guaranteed using the idea that the values of weighted residual integrals on the problem's domain are close to or equal to zero. As a result, the accuracy of the approximate analytical solutions of the proposed BVPs can be improved by choosing the number of the solution components N as studied in Example 4.3. The technique does not require a translation of the original domain of the problem and can avoid some critical issues due to singular points of the problem. Some illustrative problems, for example, a linear singular two-point BVP, a nonlinear multi-point BVP and a regular two-point BVP with a transcendental-function solution were given and it was shown that the method performed well for such problems. For some examples shown, the solutions obtained by the FGM were the same as the exact solutions. We believe that the FGM can be an efficient method for other kinds of BVPs.

Acknowledgements

The authors are grateful to anonymous referees for their valuable comments and several constructive suggestions, which have significantly improved this manuscript.

References

[1] A.F. Fareed, M.S. Semary, H.N. Hassan, Two semi-analytical approaches to approximate the solution of stochastic ordinary differential equations with two enormous engineering applications, Alexandria Engineering Journal 61 (12) (2022) 11935–11945.

[2] G. Scholz, F. Scholz, First-order differential equations in chemistry, ChemTexts 1 (1) (2015) 1–12.

[3] I. Alhama, F. Alhama, A.S. Meca, The network method for a fast and reliable solution of ordinary differential equations: Applications to non-linear oscillators, Computers and Electrical Engineering 38 (6) (2012) 1524–1533.

[4] A. Mahata, B. Roy, S.P. Mondal, S. Alam, Application of ordinary differential equation in glucose-insulin regulatory system modeling in fuzzy environment, Ecological Genetics and Genomics 3–5 (2017) 60–66.

[5] J.J. Nieto, Solution of a fractional logistic ordinary differential equation, Applied Mathematics Letters 123 (2022) 1–5.

[6] M.S.E. Omaba, New analytical method of solution to a nonlinear singular fractional Lane-Emden type equation, AIMS Mathematics 7 (10) (2022) 19539–19552.

[7] T. Saeed, I. Abbas, M. Marin, A GL model on thermo-elastic interaction in a poroelastic material using finite element method, Symmetry 12 (3) (2020) 488.

[8] A.M. Vargas, Finite difference method for solving fractional differential equations at irregular meshes, Mathematics and Computers in Simulation 193 (2022) 204–216.

[9] T. Salahuddin, Carreau fluid model towards a stretching cylinder: Using Keller box and shooting method, Ain Shams Engineering Journal 11 (2) (2020) 495–500.

[10] T.L. Schmitz, A. Honeycutt, Analytical solutions for fixed-free beam dynamics in thin rib machining, Journal of Manufacturing Processes 30 (2017) 41–50.

[11] D. Baleanu, B. Shiri, H.M. Srivastava, M.A.l. Qurashi, A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel, Advances in Difference Equations 2018 (353) (2018) 1–23.

[12] P. Shanthraj, P. Eisenlohr, M. Diehl, F. Roters, Numerically robust spectral methods for crystal plasticity simulations of heterogeneous materials, International Journal of Plasticity 66 (2015) 31–45.

[13] C. Lanczos, J. Boyd, Discourse on Fourier Series, Society for Industrial and Applied Mathematics, 2016.

[14] M. Abdelhakem, A. Ahmed, M. El-kady, Spectral monic Chebyshev approximation for higher order differential equations, Mathematical Sciences Letters 8 (2) (2019) 11–17.

[15] H. Sweis, N. Shawagfeh, O.A. Arqub, Fractional crossover delay differential equations of Mittag-Leffler kernel: Existence, uniqueness, and numerical solutions using the Galerkin algorithm based on shifted Legendre polynomials, Results in Physics 41 (2022) 1–13.

[16] W.M. Abd-Elhameed, Y.H. Youssri, A novel operational matrix of Caputo fractional derivatives of Fibonacci polynomials: Spectral solutions of fractional differential equations, Entropy 18 (345) (2016) 1–15.

[17] S. Sabermahani, O. Ordokhani, S.A. Yousefi, Fibonacci wavelets and their applications for solving two classes of time-varying delay problems, Optimal Control Applications and Methods 41 (2) (2019) 1–22.

[18] A.B. Koç, M. Çakmak, A. Kurnaz, K. Uslu, A new Fibonacci type collocation procedure for boundary value problems, Advances in Difference Equations 2013 (262) (2013) 1–11.

[19] R. Burden, D. Faires, Numerical Analysis (7th edition) Thomson Learning, 2001.

[20] M. Grau-Sánchez, M. Noguera, J.M. Gutiérrez, On some computational orders of convergence, Applied Mathematics Letters 23 (4) (2010) 472–478.