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Completeness of Low-Dimensional Leibniz Algebras

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Dedicated to Professor Kailash C. Misra on the occasion of his 70th birthday.

Abstract Leibniz algebras are generalizations of Lie algebras. By using the classification results of low-dimensional non-Lie nilpotent and non-nilpotent solvable Leibniz algebras obtained earlier, we define a basis of the derivation algebra $\mathrm{Der}(\mathbf{A})$ of each Leibniz algebra $\mathbf{A}$ and study their properties. It is known that for a Leibniz algebra $\mathbf{A}$ if the Lie algebra $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$ is complete, then $\mathbf{A}$ is a complete Leibniz algebra. We show that the converse holds when $\mathbf{A}$ is a complete solvable Leibniz algebra with $\dim(\mathbf{A}) \leq 3$. It is also known that for the derivation algebra of a complete Lie algebra is complete. However, our results show that this is not true for Leibniz algebras.

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1. Introduction

A Leibniz algebra, named after Gottfried Wilhelm Leibniz, was first studied by Bloh [1] and later popularized by Loday [2] as a generalization of Lie algebras. Several studies of Leibniz algebras have been associated to various areas such as differential geometry, noncommutative geometry, algebraic K-theory, algebraic topology and quantum physics.

Given any Leibniz algebra $\mathbf{A}$, we denote $\operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{[x,x] \mid x \in \mathbf{A}\}$. A linear map $d : \mathbf{A} \to \mathbf{A}$ is called a derivation of a Leibniz algebra $\mathbf{A}$ if $d[x,y] = [d(x),y]+[x,d(y)]$ for all $x, y \in \mathbf{A}$. The vector space of derivations $\mathrm{Der}(\mathbf{A})$ is a Lie algebra under the commutator bracket. As in case of Lie algebras, derivations play a crucial role in understanding the structure of Leibniz algebras and their representations. The aim of this paper is to discuss properties of derivations and the completeness of low-dimensional non-Lie Leibniz algebras. We use a generalization to Leibniz algebras for the concept of completeness. Precisely, a Leibniz algebra $\mathbf{A}$ is called complete [3] if $Z(\mathbf{A}/\operatorname{Leib}(\mathbf{A})) = \{0\}$ and for each

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$d \in \mathrm{Der}(\mathbf{A})$, there exists $x \in \mathbf{A}$ such that $\mathrm{im}(d - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ where L_x is the left multiplication operator defined by $L_x(y) = [x, y]$ for all $y \in \mathbf{A}$.

This paper is organized as follows. In Section 2, we review important notions and properties of Lie algebras and Leibniz algebras. In Section 3, we revisit the classifications of two and three-dimensional non-Lie Leibniz algebras given in [4]. We define a basis for $\mathrm{Der}(\mathbf{A}), \mathrm{Leib}(\mathbf{A})$ and $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ of each algebra and study their properties. We also determine which of these Leibniz algebras are complete and which are not complete. Boyle, Misra and Stitzinger [3] proved that for a Leibniz algebra $\mathbf{A}$ if the Lie algebra $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is complete, then $\mathbf{A}$ is a complete Leibniz algebra. We show that the converse holds for every complete solvable Leibniz algebra of dimension 3. In [5], Meng proved that for a complete Lie algebra $\mathbf{L}$, $\mathrm{Der}(\mathbf{L})$ is complete. However, we prove that this is not true for Leibniz algebras. In particular, we show that there exists a complete solvable Leibniz algebra $\mathbf{A}$ for which $\mathrm{Der}(\mathbf{A})$ is not complete. We also show that for a complete solvable Leibniz algebra $\mathbf{A}$ with $\dim(\mathbf{A}) \leq 3$, $\mathrm{Der}(\mathbf{A})/I$ is complete where $I = \{d \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$.

Throughout the paper, all algebras are assumed to be finite-dimensional over an algebraically closed field $\mathbb{F}$ with characteristic zero.

2. Preliminaries

In this section, we recall the basic definitions for Lie algebras from [5] and [6]. For Leibniz algebras, we closely follow the notations in [3] and [4].

Definition 2.1. A *Lie algebra* $\mathbf{L}$ over a field $\mathbb{F}$ is a vector space equipped with a bilinear map, called bracket, $[\ ,\ ] : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$ such that $[x, x] = 0$ for all $x \in \mathbf{L}$ and satisfying the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathbf{L}$.

Let I be a subspace of a Lie algebra $\mathbf{L}$. Then I is a *subalgebra* if $[I, I] \subseteq I$, an *ideal* if $[\mathbf{L}, I] \subseteq I$. For a Lie algebra $\mathbf{L}$, $\mathbf{L}$ is called *abelian* if $[x, y] = 0$ for all $x, y \in \mathbf{L}$, and the set $Z(\mathbf{L}) = \{x \in \mathbf{L} \mid [x, y] = 0 \text{ for all } y \in \mathbf{L}\}$ is called the *center* of $\mathbf{L}$. Clearly, $Z(\mathbf{L})$ is an abelian ideal of $\mathbf{L}$.

Definition 2.2. For a given Lie algebra $(\mathbf{L}, [\ ,\ ])$, the series of ideals

$$\mathbf{L}^{(0)} \supseteq \mathbf{L}^{(1)} \supseteq \mathbf{L}^{(2)} \supseteq \dots \text{ where } \mathbf{L}^{(0)} = \mathbf{L} \text{ and } \mathbf{L}^{(i+1)} = [\mathbf{L}^{(i)}, \mathbf{L}^{(i)}],$$

$$\mathbf{L}^{0} \supseteq \mathbf{L}^{1} \supseteq \mathbf{L}^{2} \supseteq \dots \text{ where } \mathbf{L}^{0} = \mathbf{L} \text{ and } \mathbf{L}^{i+1} = [\mathbf{L}, \mathbf{L}^{i}]$$

are called the *derived series* and the *lower central series* of $\mathbf{L}$, respectively. The Lie algebra is said to be *solvable* (resp. *nilpotent*) if $\mathbf{L}^{(m)} = 0$ (resp. $\mathbf{L}^{m} = 0$) for some non-negative integer m.

It is clear that if $\mathbf{L}$ is nilpotent, then $\mathbf{L}$ is solvable. It is not true that if a Lie algebra is solvable, then it is nilpotent.

Definition 2.3. A linear map $d : \mathbf{L} \to \mathbf{L}$ of a Lie algebra $(\mathbf{L}, [\ ,\ ])$ is said to be a *derivation* if

$$d([x, y]) = [d(x), y] + [x, d(y)]$$

for all $x, y \in \mathbf{L}$. The set of all derivations of $\mathbf{L}$ is denoted by $\mathrm{Der}(\mathbf{L})$.

It is known that $\mathrm{Der}(\mathbf{L})$ is a Lie algebra under the commutator bracket $[d_1, d_2] = d_1d_2 - d_2d_1$ for all $d_1, d_2 \in \mathrm{Der}(\mathbf{L})$. For $x \in \mathbf{L}$, define the map $ad_x : \mathbf{L} \to \mathbf{L}$ by $ad_x(y) = [x, y]$ for all $y \in \mathbf{L}$. Then we have that $ad_x \in \mathrm{Der}(\mathbf{L})$ for all $x \in \mathbf{L}$.

Definition 2.4. A derivation $d \in \mathrm{Der}(\mathbf{L})$ is called an *inner derivation* if there exists $x \in \mathbf{L}$ such that $d = ad_x$. An *outer derivation* is any derivation which is not inner.

Definition 2.5. [5] A Lie algebra $\mathbf{L}$ is said to be *complete* if

(1) the center $Z(\mathbf{L}) = \{0\}$, and
(2) every derivation of $\mathbf{L}$ is inner.

It is known that a nonzero nilpotent Lie algebra is never complete since it has a nontrivial center and it also has outer derivations. The following result from [5] will be useful in the next section.

Theorem 2.6. [5] *The non-abelian Lie algebra of dimension 2 is a complete Lie algebra.*

Definition 2.7. A *(left) Leibniz algebra* $\mathbf{A}$ over a field $\mathbb{F}$ is a vector space equipped with a bilinear map, called bracket, $[\ ,\ ] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

for all $x, y, z \in \mathbf{A}$.

For a Leibniz algebra $\mathbf{A}$ and $x \in \mathbf{A}$, we define the *left multiplication operator* $L_x : \mathbf{A} \to \mathbf{A}$ and the *right multiplication operator* $R_x : \mathbf{A} \to \mathbf{A}$ by $L_x(y) = [x, y]$ and $R_x(y) = [y, x]$ respectively for all $y \in \mathbf{A}$. Then the vector space of left multiplication operators $L(\mathbf{A}) = \{L_x | x \in \mathbf{A}\}$ is a Lie algebra under the commutator bracket. A right Leibniz algebra is a vector space equipped with a bilinear map satisfying $[[x, y], z] = [x, [y, z]] + [[x, z], y]$ for all $x, y, z \in \mathbf{A}$. Throughout this work Leibniz algebra always refers to (left) Leibniz algebra. As the following example shows a (left) Leibniz algebra is not necessarily a (right) Leibniz algebra.

Example 2.8. [4] Let $\mathbf{A}$ be a 2-dimensional algebra with the following brackets:

$$[x, x] = 0, [x, y] = 0, [y, x] = x, [y, y] = x.$$

Then $\mathbf{A}$ is a (left) Leibniz algebra, but it is not a (right) Leibniz algebra, since $[[y, y], y] \neq [y, [y, y]] + [[y, y], y]$.

Any Lie algebra is clearly a Leibniz algebra. A Leibniz algebra $\mathbf{A}$ satisfying the condition that $[x, x] = x^2 = 0$ for all $x \in \mathbf{A}$, is a Lie algebra since in this case the Leibniz identity becomes the Jacobi identity. Given any Leibniz algebra $\mathbf{A}$, we denote $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{[x, x] \mid x \in \mathbf{A}\}$. The Leibniz algebra $\mathbf{A}$ is said to be *abelian* if $[\mathbf{A}, \mathbf{A}] = \mathbf{0}$. The *left center* of $\mathbf{A}$ is denoted by $Z^l(\mathbf{A}) = \{x \in \mathbf{A} \mid [x, y] = 0 \text{ for all } y \in \mathbf{A}\}$ and the *right center* of $\mathbf{A}$ is denoted by $Z^r(\mathbf{A}) = \{x \in \mathbf{A} \mid [y, x] = 0 \text{ for all } y \in \mathbf{A}\}$. The *center* of $\mathbf{A}$ is $Z(\mathbf{A}) = Z^l(\mathbf{A}) \cap Z^r(\mathbf{A})$.

Let I be a subspace of a Leibniz algebra $\mathbf{A}$. Then I is a *subalgebra* if $[I, I] \subseteq I$, a *left* (resp. *right*) *ideal* if $[\mathbf{A}, I] \subseteq I$ (resp. $[I, \mathbf{A}] \subseteq I$). I is an *ideal* of $\mathbf{A}$ if it is both a left ideal and a right ideal. In particular, $\mathrm{Leib}(\mathbf{A})$ is an abelian ideal of $\mathbf{A}$. For any ideal I of $\mathbf{A}$ we define the quotient Leibniz algebra in the usual way. In fact, $\mathrm{Leib}(\mathbf{A})$ is the minimal ideal such that $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a Lie algebra with respect to the bracket $[\ ,\ ] : \mathbf{A}/\mathrm{Leib}(\mathbf{A}) \times \mathbf{A}/\mathrm{Leib}(\mathbf{A}) \to \mathbf{A}/\mathrm{Leib}(\mathbf{A})$ defined by $[x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})] = [x, y] + \mathrm{Leib}(\mathbf{A})$ for all $x, y \in \mathbf{A}$.

Definition 2.9. For a given Leibniz algebra $(\mathbf{A}, [\ ,\ ])$, the series of ideals

$$\mathbf{A}^{(0)} \supseteq \mathbf{A}^{(1)} \supseteq \mathbf{A}^{(2)} \supseteq \dots \text{ where } \mathbf{A}^{(0)} = \mathbf{A} \text{ and } \mathbf{A}^{(i+1)} = [\mathbf{A}^{(i)}, \mathbf{A}^{(i)}],$$
$$\mathbf{A}^{0} \supseteq \mathbf{A}^{1} \supseteq \mathbf{A}^{2} \supseteq \dots \text{ where } \mathbf{A}^{0} = \mathbf{A} \text{ and } \mathbf{A}^{i+1} = [\mathbf{A}, \mathbf{A}^{i}]$$

are called the *derived series* and the *lower central series* of $\mathbf{A}$, respectively. The Leibniz algebra is said to be *solvable* (resp. *nilpotent*) if $\mathbf{A}^{(m)} = 0$ (resp. $\mathbf{A}^{m} = 0$) for some non-negative integer m.

Definition 2.10. A linear map $d : \mathbf{A} \to \mathbf{A}$ of a Leibniz algebra $(\mathbf{A}, [\ ,\ ])$ is said to be a *derivation* if

$$d([x, y]) = [d(x), y] + [x, d(y)]$$

for all $x, y \in \mathbf{A}$. The set of all derivations of $\mathbf{A}$ is denoted by $\mathrm{Der}(\mathbf{A})$. An ideal I of $\mathbf{A}$ is said to be a *characteristic ideal* of $\mathbf{A}$ if $d(I) \subseteq I$ for all $d \in \mathrm{Der}(\mathbf{A})$.

As in case of Lie algebras, $\mathrm{Der}(\mathbf{A})$ is a Lie algebra under the commutator bracket. The following result from [3] will also be useful in the next section.

Proposition 2.11. [3] *For a finite dimensional Leibniz algebra $\mathbf{A}$ over field $\mathbb{C}$, $\mathrm{Leib}(\mathbf{A})$ is a characteristic ideal.*

Definition 2.12. A derivation $d \in \mathrm{Der}(\mathbf{A})$ is called an *inner derivation* if there exists $x \in \mathbf{A}$ such that $\mathrm{im}(d - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$. An *outer derivation* is any derivation which is not inner.

Definition 2.13. [3] A Leibniz algebra $\mathbf{A}$ is said to be *complete* if

(1) the center $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$, and
(2) every derivation of $\mathbf{A}$ is inner.

Thus, if a Lie algebra $\mathbf{A}$ is complete as a Leibniz algebra, then it is also complete as a Lie algebra since $\mathrm{Leib}(\mathbf{A}) = \{0\}$ in this case. Note that another notion of complete Leibniz algebras was also defined in [7]. In [3], it is shown that by Definition 2.13, the signature results from Lie theory would carry over to Leibniz algebras. In particular, a semisimple Leibniz algebra over a field of characteristic zero would not be complete in the sense of [7], but will be complete by Definition 2.13. With this in mind, we use the definition of inner derivations and completeness of Leibniz algebras as in [3].

3. Main Results

We first observe that if $\mathbf{A}$ is a non-Lie Leibniz algebra, then $\mathrm{Leib}(\mathbf{A}) \neq \{0\}$ and $\mathrm{Leib}(\mathbf{A}) \neq \mathbf{A}$. Thus, there is no one-dimensional non-Lie Leibniz algebra. It is known that if $\mathbf{A}$ is a non-Lie nilpotent Leibniz algebra and $\dim(\mathbf{A}) \leq 3$, then $\mathbf{A}$ is solvable (see [4]). In this section, we assume $\mathbb{F} = \mathbb{C}$ and revisit the classifications of two and three-dimensional non-Lie Leibniz algebras given in [4]. We define a basis for $\mathrm{Der}(\mathbf{A})$, $\mathrm{Leib}(\mathbf{A})$, and $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ of each algebra and study their properties.

3.1. Two-Dimensional Leibniz Algebras

Let $\mathbf{A}$ be a non-Lie Leibniz algebra and $\dim(\mathbf{A}) \leq 2$. By [4], $\mathbf{A}$ is isomorphic to a cyclic Leibniz algebra generated by x with $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x^2\}$ and either $[x, x^2] = 0$ (hence $\mathbf{A}$ is nilpotent) or $[x, x^2] = x^2$ (hence $\mathbf{A}$ is solvable). Let $\mathcal{B}$ denote the ordered basis for $\mathbf{A}$ given by $\mathcal{B} = \{x, x^2\}$. To find a basis for $\mathrm{Der}(\mathbf{A})$, let $d \in \mathrm{Der}(\mathbf{A})$ and define the action of d on the basis vectors as follows: $d(x) = \alpha_1 x + \alpha_2 x^2$ and $d(x^2) = \beta_1 x + \beta_2 x^2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$.

Case 1: $\mathbf{A} = \operatorname{span}\{x, x^2\}$ with $[x, x^2] = 0$. By the derivation property, we have that

$$d([x,x]) = 2\alpha_1 x^2,\ d([x,x^2]) = \beta_1 x^2,\ d([x^2,x]) = \beta_1 x^2,\ d([x^2,x^2]) = 0.$$

By the linear independence of the basis vectors, we obtain that $\beta_1 = 0$ and $\beta_2 = 2\alpha_1$. This implies that

$$[d]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 2\alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $\operatorname{Der}(\mathbf{A}) = \operatorname{span}\{d_1, d_2\}$, where

$$\begin{aligned} d_1(x) &= x, & d_1(x^2) &= 2x^2 \\ d_2(x) &= x^2, & d_2(x^2) &= 0. \end{aligned}$$

Since $d_2 = L_x$, d_2 is inner. Since $d_1(x) = x$ and $L_w(x) \neq x$ for all $w = \alpha x + \beta x^2 \in \mathbf{A}$, there is no element $w \in \mathbf{A}$ such that $\operatorname{im}(d_1 - L_w) \subseteq \operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{x^2\}$ which implies d_1 is not inner. We observe that $[d_1, d_2] = d_2$. Thus, $\operatorname{Der}(\mathbf{A})^{(1)} = \operatorname{Der}(\mathbf{A})^1 = [\operatorname{Der}(\mathbf{A}), \operatorname{Der}(\mathbf{A})] = \operatorname{span}\{d_2\}$, $\operatorname{Der}(\mathbf{A})^{(2)} = [\operatorname{Der}(\mathbf{A})^{(1)}, \operatorname{Der}(\mathbf{A})^{(1)}] = \{0\}$ and $\operatorname{Der}(\mathbf{A})^2 = [\operatorname{Der}(\mathbf{A}), \operatorname{Der}(\mathbf{A})^1] = \operatorname{span}\{d_2\} = \operatorname{Der}(\mathbf{A})^1$. Therefore, $\operatorname{Der}(\mathbf{A})$ is solvable but not nilpotent.

Case 2: $\mathbf{A} = \operatorname{span}\{x, x^2\}$ with $[x, x^2] = x^2$. We have that

$$\begin{aligned} d([x,x]) &= (2\alpha_1 + \alpha_2)x^2, & d([x,x^2]) &= (\alpha_1 + \beta_1 + \beta_2)x^2, \\ d([x^2,x]) &= \beta_1 x^2, & d([x^2,x^2]) &= \beta_1 x^2. \end{aligned}$$

Hence, $\alpha_1 = 0, \beta_1 = 0$ and $\beta_2 = \alpha_2$. This implies that

$$[d]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Hence, $\operatorname{Der}(\mathbf{A}) = \operatorname{span}\{d\}$, where $d(x) = x^2$ and $d(x^2) = x^2$. Since $d = L_x$, d is inner. Clearly, $\operatorname{Der}(\mathbf{A})$ is abelian in this case.

Therefore, we obtain the following theorem.

Theorem 3.1. *Let* $\mathbf{A}$ *be a non-Lie Leibniz algebra and* $\dim(\mathbf{A}) = 2$. *Then* $\mathbf{A}$ *is nilpotent if and only if* $\operatorname{Der}(\mathbf{A})$ *is not nilpotent.*

Remark 3.2. We observe that by suitable change of basis our derivation algebras coincide with results given in [8, page 2192–2193]. However, our results show that any two-dimensional non-Lie Leibniz algebra is not complete. This result might be a counterexample to Corollary 3.4 in [8].

Proposition 3.3. *Let* $\mathbf{A}$ *be a non-Lie Leibniz algebra and* $\dim(\mathbf{A}) \leq 2$. *Then* $\mathbf{A}$ *is not complete.*

Proof. Since $\operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{x^2\}$, $\mathbf{A}/\operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{x + \operatorname{Leib}(A)\}$ is a one-dimensional Lie algebra. Thus, $Z(\mathbf{A}/\operatorname{Leib}(\mathbf{A})) = \mathbf{A}/\operatorname{Leib}(\mathbf{A})$ which implies $\mathbf{A}$ is not complete. ∎

3.2. Three-Dimensional Nilpotent Leibniz Algebras

Let $\mathbf{A}$ be a non-Lie nilpotent Leibniz algebra and $\dim(\mathbf{A}) = 3$. By [4], $\mathbf{A}$ is isomorphic to one of the following algebras spanned by the ordered basis $\mathcal{B} = \{x, y, z\}$ defined by the given nonzero multiplications.

Case 1: $[x, x] = y,\ [x, y] = z$,
Case 2: $[x, x] = z$,
Case 3: $[x, x] = z, [y, y] = z$,
Case 4: $[x, y] = z, [y, x] = -z, [y, y] = z$,
Case 5: $[x, y] = z, [y, x] = \alpha z, \alpha \in \mathbb{F} \setminus \{1, -1\}$.

To find a basis for $\mathrm{Der}(\mathbf{A})$, let $d \in \mathrm{Der}(\mathbf{A})$ and define the action of d on the basis vectors as follows: $d(x) = \alpha_1 x + \alpha_2 y + \alpha_3 z, d(y) = \beta_1 x + \beta_2 y + \beta_3 z$ and $d(z) = \gamma_1 x + \gamma_2 y + \gamma_3 z$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}, i = 1, 2, 3$.

Case 1: $[x, x] = y,\ [x, y] = z$. Then,

$$\begin{aligned} &d([x,x]) = 2\alpha_1 y + \alpha_2 z, && d([x,y]) = \beta_1 y + (\alpha_1 + \beta_2) z, && d([x,z]) = \gamma_1 y + \gamma_2 z,\\ &d([y,x]) = \beta_1 y, && d([y,y]) = \beta_1 z, && d([y,z]) = 0,\\ &d([z,x]) = \gamma_1 y, && d([z,y]) = \gamma_1 z, && d([z,z]) = 0. \end{aligned}$$

This implies that

$$[d]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 2\alpha_1 & 0 \\ \alpha_3 & \alpha_2 & 3\alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\}$ where

$$\begin{aligned} &d_1(x) = x, && d_1(y) = 2y, && d_1(z) = 3z,\\ &d_2(x) = y, && d_2(y) = z, && d_2(z) = 0,\\ &d_3(x) = z, && d_3(y) = 0, && d_3(z) = 0. \end{aligned}$$

Since $\mathrm{im}(d_2 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_3 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$, d_2 and d_3 are inner. If $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ for some $w \in \mathbf{A}$, then $x \in \mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{y, z\}$ which is a contradiction. Hence, there is no element $w \in \mathbf{A}$ such that $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ which implies d_1 is not inner. We observe that $[d_1, d_2] = d_2, [d_1, d_3] = 2d_3$ and $[d_2, d_3] = 0$. Thus, $\mathrm{Der}(\mathbf{A})^{(1)} = \mathrm{Der}(\mathbf{A})^1 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})] = \mathrm{span}\{d_2, d_3\}$, $\mathrm{Der}(\mathbf{A})^{(2)} = [\mathrm{Der}(\mathbf{A})^{(1)}, \mathrm{Der}(\mathbf{A})^{(1)}] = \{0\}$ and $\mathrm{Der}(\mathbf{A})^2 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})^1] = \mathrm{span}\{d_2, d_3\} = \mathrm{Der}(\mathbf{A})^1$. Therefore, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that in this case, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{y, z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x + \mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$ because $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 2: $[x, x] = z$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4, d_5\}$ where

$$\begin{aligned} &d_1(x) = x, && d_1(y) = 0, && d_1(z) = 2z,\\ &d_2(x) = y, && d_2(y) = 0, && d_2(z) = 0,\\ &d_3(x) = z, && d_3(y) = 0, && d_3(z) = 0,\\ &d_4(x) = 0, && d_4(y) = y, && d_4(z) = 0,\\ &d_5(x) = 0, && d_5(y) = z, && d_5(z) = 0. \end{aligned}$$

Since $\mathrm{im}(d_3 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_5 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$, d_3 and d_5 are inner. If $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ for some $w \in \mathbf{A}$, then $x \in \mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$ which is a contradiction. Hence, there is no element $w \in \mathbf{A}$ such that $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ which implies d_1 is not inner. Similarly, d_2 and d_4 are not inner. We observe that

$$[d_1, d_2] = -d_2, \quad [d_1, d_3] = d_3, \quad [d_1, d_4] = 0, \quad [d_1, d_5] = 2d_5, \quad [d_2, d_3] = 0,$$
$$[d_2, d_4] = -d_2, \quad [d_2, d_5] = -d_3, \quad [d_3, d_4] = 0, \quad [d_3, d_5] = 0, \quad [d_4, d_5] = -d_5.$$

Thus, $\mathrm{Der}(\mathbf{A})^{(1)} = \mathrm{Der}(\mathbf{A})^1 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})] = \mathrm{span}\{d_2, d_3, d_5\}$, $\mathrm{Der}(\mathbf{A})^{(2)} = [\mathrm{Der}(\mathbf{A})^{(1)}, \mathrm{Der}(\mathbf{A})^{(1)}] = \mathrm{span}\{d_3\}$, $\mathrm{Der}(\mathbf{A})^{(3)} = [\mathrm{Der}(\mathbf{A})^{(2)}, \mathrm{Der}(\mathbf{A})^{(2)}] = \{0\}$ and $\mathrm{Der}(\mathbf{A})^2 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})^1] = \mathrm{span}\{d_2, d_3, d_5\} = \mathrm{Der}(\mathbf{A})^1$. Therefore, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that in this case, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$ because $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 3: $[x, x] = z$, $[y, y] = z$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4\}$ where

$$\begin{aligned} &d_1(x) = x, && d_1(y) = y, && d_1(z) = 2z, \\ &d_2(x) = y, && d_2(y) = -x, && d_2(z) = 0, \\ &d_3(x) = z, && d_3(y) = 0, && d_3(z) = 0, \\ &d_4(x) = 0, && d_4(y) = z, && d_4(z) = 0. \end{aligned}$$

Since $\mathrm{im}(d_3 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_4 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$, d_3 and d_4 are inner. There is no element $w \in \mathbf{A}$ such that $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ which implies d_1 is not inner. Likewise, d_2 is not inner. We observe that

$$\begin{aligned} &[d_1, d_2] = 0, && [d_1, d_3] = d_3, && [d_1, d_4] = d_4, \\ &[d_2, d_3] = d_4, && [d_2, d_4] = -d_3, && [d_3, d_4] = 0. \end{aligned}$$

Thus, $\mathrm{Der}(\mathbf{A})^{(1)} = \mathrm{Der}(\mathbf{A})^1 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})] = \mathrm{span}\{d_3, d_4\}$, $\mathrm{Der}(\mathbf{A})^{(2)} = [\mathrm{Der}(\mathbf{A})^{(1)}, \mathrm{Der}(\mathbf{A})^{(1)}] = \{0\}$ and $\mathrm{Der}(\mathbf{A})^2 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})^1] = \mathrm{span}\{d_3, d_4\} = \mathrm{Der}(\mathbf{A})^1$. Therefore, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that in this case, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$ because $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 4: $[x, y] = z$, $[y, x] = -z$, $[y, y] = z$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4\}$ where

$$\begin{aligned} &d_1(x) = x, && d_1(y) = y, && d_1(z) = 2z, \\ &d_2(x) = z, && d_2(y) = 0, && d_2(z) = 0, \\ &d_3(x) = 0, && d_3(y) = x, && d_3(z) = 0, \\ &d_4(x) = 0, && d_4(y) = z, && d_4(z) = 0. \end{aligned}$$

Since $\mathrm{im}(d_2 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_4 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$, d_2 and d_4 are inner. There is no element $w \in \mathbf{A}$ such that $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ which implies d_1 is not inner. Similarly, d_3 is not inner. We observe that

$$\begin{aligned} &[d_1, d_2] = d_2, && [d_1, d_3] = 0, && [d_1, d_4] = d_4, \\ &[d_2, d_3] = d_4, && [d_2, d_4] = 0, && [d_3, d_4] = 0. \end{aligned}$$

Thus, $\mathrm{Der}(\mathbf{A})^{(1)} = \mathrm{Der}(\mathbf{A})^1 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})] = \mathrm{span}\{d_2, d_4\}$, $\mathrm{Der}(\mathbf{A})^{(2)} = [\mathrm{Der}(\mathbf{A})^{(1)}, \mathrm{Der}(\mathbf{A})^{(1)}] = \{0\}$ and $\mathrm{Der}(\mathbf{A})^2 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})^1] = \mathrm{span}\{d_2, d_4\} = \mathrm{Der}(\mathbf{A})^1$. Therefore, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that in this case, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$ because $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 5: $[x, y] = z,\ [y, x] = \alpha z, \alpha \in \mathbb{F} \setminus \{1, -1\}$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4\}$ where

$$\begin{array}{lll} d_1(x) = x, & d_1(y) = 0, & d_1(z) = z, \\ d_2(x) = z, & d_2(y) = 0, & d_2(z) = 0, \\ d_3(x) = 0, & d_3(y) = y, & d_3(z) = z, \\ d_4(x) = 0, & d_4(y) = z, & d_4(z) = 0. \end{array}$$

Since $\mathrm{im}(d_2 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_4 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$, d_2 and d_4 are inner. There is no element $w \in \mathbf{A}$ such that $\mathrm{im}(d_1 - L_w) \subseteq \mathrm{Leib}(\mathbf{A})$ which implies d_1 is not inner. Similarly, d_3 is not inner. We observe that

$$\begin{array}{lll} [d_1, d_2] = 0, & [d_1, d_3] = 0, & [d_1, d_4] = d_4, \\ [d_2, d_3] = -d_2, & [d_2, d_4] = 0, & [d_3, d_4] = 0. \end{array}$$

Thus, $\mathrm{Der}(\mathbf{A})^{(1)} = \mathrm{Der}(\mathbf{A})^1 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})] = \mathrm{span}\{d_2, d_4\}$, $\mathrm{Der}(\mathbf{A})^{(2)} = [\mathrm{Der}(\mathbf{A})^{(1)}, \mathrm{Der}(\mathbf{A})^{(1)}] = \{0\}$ and $\mathrm{Der}(\mathbf{A})^2 = [\mathrm{Der}(\mathbf{A}), \mathrm{Der}(\mathbf{A})^1] = \mathrm{span}\{d_2, d_4\} = \mathrm{Der}(\mathbf{A})^1$. Therefore, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that in this case, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$ because $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

By straightforward computations above, we obtain the following theorems.

Theorem 3.4. *Let* $\mathbf{A}$ *be a non-Lie nilpotent Leibniz algebra and* $\dim(\mathbf{A}) = 3$. *Then* $\mathrm{Der}(\mathbf{A})$ *is solvable but not nilpotent.*

Theorem 3.5. *Let* $\mathbf{A}$ *be a non-Lie nilpotent Leibniz algebra and* $\dim(\mathbf{A}) = 3$. *Then* $\mathbf{A}$ *admits an outer derivation.*

The following is immediate from the above theorem. Note that we rely on the second condition of Definition 2.13 to obtain this result. It is shown in [3] that nilpotent Leibniz algebras are not complete relying on the first condition of 2.13.

Corollary 3.6. *Let* $\mathbf{A}$ *be a non-Lie nilpotent Leibniz algebra and* $\dim(\mathbf{A}) = 3$. *Then* $\mathbf{A}$ *is not complete.*

3.3. Three-Dimensional Solvable Leibniz Algebras

Let $\mathbf{A}$ be a non-Lie non-nilpotent solvable Leibniz algebra and $\dim(\mathbf{A}) = 3$. By [4], $\mathbf{A}$ is isomorphic to one of the following algebras spanned by the ordered basis $\mathcal{B} = \{x, y, z\}$ defined by the given nonzero multiplications.

Case 1: $[x, z] = z$,
Case 2: $[x, z] = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}, [x, y] = y, [y, x] = -y$,
Case 3: $[x, y] = y, [y, x] = -y, [x, x] = z$,
Case 4: $[x, z] = 2z, [y, y] = z, [x, y] = y, [y, x] = -y, [x, x] = z$,
Case 5: $[x, y] = y, [x, z] = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}$,
Case 6: $[x, z] = z + y, [x, y] = y$,
Case 7: $[x, z] = y, [x, y] = y, [x, x] = z$.

To find a basis for $\mathrm{Der}(\mathbf{A})$, let $d \in \mathrm{Der}(\mathbf{A})$ and define the action of d on the basis vectors as follows: $d(x) = \alpha_1 x + \alpha_2 y + \alpha_3 z$, $d(y) = \beta_1 x + \beta_2 y + \beta_3 z$ and $d(z) = \gamma_1 x + \gamma_2 y + \gamma_3 z$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}, i = 1, 2, 3$.

Case 1: $[x,z]=z$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2,d_3\}$ where

$$
\begin{array}{lll}
d_1(x)=y, & d_1(y)=0, & d_1(z)=0,\\
d_2(x)=0, & d_2(y)=y, & d_2(z)=0,\\
d_3(x)=0, & d_3(y)=0, & d_3(z)=z.
\end{array}
$$

We have that d_1 and d_2 are not inner but d_3 is inner, since $\mathrm{im}(d_3-L_x)\subseteq \mathrm{Leib}(\mathbf{A})$. Since $[d_1,d_2]=-d_1,\ [d_1,d_3]=0$ and $[d_2,d_3]=0$, we have that $\mathrm{Der}(\mathbf{A})^{(1)}=\mathrm{Der}(\mathbf{A})^1=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})]=\mathrm{span}\{d_1\}$, $\mathrm{Der}(\mathbf{A})^{(2)}=[\mathrm{Der}(\mathbf{A})^{(1)},\mathrm{Der}(\mathbf{A})^{(1)}]=\{0\}$ and $\mathrm{Der}(\mathbf{A})^2=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})^1]=\mathrm{span}\{d_1\}=\mathrm{Der}(\mathbf{A})^1$. Hence, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A}),y+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))\neq\{0\}$ because $x+\mathrm{Leib}(\mathbf{A})\in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 2: $[x,z]=\alpha z,\ \alpha\in\mathbb{F}\setminus\{0\},\ [x,y]=y,\ [y,x]=-y$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2,d_3\}$ where

$$
\begin{array}{lll}
d_1(x)=y, & d_1(y)=0, & d_1(z)=0,\\
d_2(x)=0, & d_2(y)=y, & d_2(z)=0,\\
d_3(x)=0, & d_3(y)=0, & d_3(z)=z.
\end{array}
$$

We have that d_1,d_2 and d_3 are inner, since $\mathrm{im}(d_1-L_y)\subseteq\mathrm{Leib}(\mathbf{A}),\ \mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_3-L_z)\subseteq\mathrm{Leib}(\mathbf{A})$. Since $[d_1,d_2]=-d_1,\ [d_1,d_3]=0$ and $[d_2,d_3]=0$, we have that $\mathrm{Der}(\mathbf{A})^{(1)}=\mathrm{Der}(\mathbf{A})^1=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})]=\mathrm{span}\{d_1\}$, $\mathrm{Der}(\mathbf{A})^{(2)}=[\mathrm{Der}(\mathbf{A})^{(1)},\mathrm{Der}(\mathbf{A})^{(1)}]=\{0\}$ and $\mathrm{Der}(\mathbf{A})^2=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})^1]=\mathrm{span}\{d_1\}=\mathrm{Der}(\mathbf{A})^1$. Hence, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A}),y+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))=\{0\}$.

Case 3: $[x,y]=y,\ [y,x]=-y,\ [x,x]=z$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2,d_3\}$ where

$$
\begin{array}{lll}
d_1(x)=y, & d_1(y)=0, & d_1(z)=0,\\
d_2(x)=z, & d_2(y)=0, & d_2(z)=0,\\
d_3(x)=0, & d_3(y)=y, & d_3(z)=0.
\end{array}
$$

We have that d_1,d_2 and d_3 are inner, since $\mathrm{im}(d_1-L_y)\subseteq\mathrm{Leib}(\mathbf{A}),\ \mathrm{im}(d_2-L_z)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_3-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$. Since $[d_1,d_2]=0,\ [d_1,d_3]=-d_1$ and $[d_2,d_3]=0$, we have that $\mathrm{Der}(\mathbf{A})^{(1)}=\mathrm{Der}(\mathbf{A})^1=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})]=\mathrm{span}\{d_1\}$, $\mathrm{Der}(\mathbf{A})^{(2)}=[\mathrm{Der}(\mathbf{A})^{(1)},\mathrm{Der}(\mathbf{A})^{(1)}]=\{0\}$ and $\mathrm{Der}(\mathbf{A})^2=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})^1]=\mathrm{span}\{d_1\}=\mathrm{Der}(\mathbf{A})^1$. Hence, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A}),y+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))=\{0\}$.

Case 4: $[x,z]=2z,\ [y,y]=z,\ [x,y]=y,\ [y,x]=-y,\ [x,x]=z$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2\}$ where

$$
\begin{array}{lll}
d_1(x)=y, & d_1(y)=-z, & d_1(z)=0,\\
d_2(x)=z, & d_2(y)=y, & d_2(z)=2z.
\end{array}
$$

We have that d_1 and d_2 are inner, since $\mathrm{im}(d_1-L_y)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$. Since $[d_1,d_2]=-d_1$, we have that $\mathrm{Der}(\mathbf{A})^{(1)}=\mathrm{Der}(\mathbf{A})^1=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})]=\mathrm{span}\{d_1\}$, $\mathrm{Der}(\mathbf{A})^{(2)}=[\mathrm{Der}(\mathbf{A})^{(1)},\mathrm{Der}(\mathbf{A})^{(1)}]=\{0\}$ and $\mathrm{Der}(\mathbf{A})^2=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})^1]$ $=\mathrm{span}\{d_1\}=\mathrm{Der}(\mathbf{A})^1$. Hence, $\mathrm{Der}(\mathbf{A})$ is solvable but not nilpotent. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A}),y+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))=\{0\}$.

Case 5: $[x,y]=y,\ [x,z]=\alpha z,\ \alpha\in\mathbb{F}\setminus\{0\}$.

Case 5.1: $\alpha=1$. We have that $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2,d_3,d_4\}$ where

$$\begin{aligned}
&d_1(x)=0, && d_1(y)=y, && d_1(z)=0,\\
&d_2(x)=0, && d_2(y)=z, && d_2(z)=0,\\
&d_3(x)=0, && d_3(y)=0, && d_3(z)=y,\\
&d_4(x)=0, && d_4(y)=0, && d_4(z)=z.
\end{aligned}$$

Since $\mathrm{im}(d_1-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, $\mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, $\mathrm{im}(d_3-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_4-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, d_1,d_2,d_3 and d_4 are inner. We observe that

$$\begin{aligned}
&[d_1,d_2]=-d_2, && [d_1,d_3]=d_3, && [d_1,d_4]=0,\\
&[d_2,d_3]=-d_1+d_4, && [d_2,d_4]=-d_2, && [d_3,d_4]=d_3.
\end{aligned}$$

Thus, $\mathrm{Der}(\mathbf{A})^{(1)}=\mathrm{Der}(\mathbf{A})^1=[\mathrm{Der}(\mathbf{A}),\mathrm{Der}(\mathbf{A})]=\mathrm{span}\{d_1,d_2,d_3,d_4\}$ which implies that $\mathrm{Der}(\mathbf{A})$ is neither solvable nor nilpotent.

Case 5.2: $\alpha\neq 1$. We have that $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2\}$ where

$$\begin{aligned}
&d_1(x)=0, && d_1(y)=y, && d_1(z)=0,\\
&d_2(x)=0, && d_2(y)=0, && d_2(z)=z.
\end{aligned}$$

Since $\mathrm{im}(d_1-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, d_1 and d_2 are inner. Since $[d_1,d_2]=0$, we have that $\mathrm{Der}(\mathbf{A})$ is abelian.

For both Case 5.1 and 5.2, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{y,z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))\neq\{0\}$ because $x+\mathrm{Leib}(\mathbf{A})\in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 6: $[x,z]=z+y,\ [x,y]=y$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2\}$ where

$$\begin{aligned}
&d_1(x)=0, && d_1(y)=y, && d_1(z)=z,\\
&d_2(x)=0, && d_2(y)=0, && d_2(z)=y,
\end{aligned}$$

Since $\mathrm{im}(d_1-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, d_1 and d_2 are inner. Since $[d_1,d_2]=0$, we have that $\mathrm{Der}(\mathbf{A})$ is abelian. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{y,z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))\neq\{0\}$ because $x+\mathrm{Leib}(\mathbf{A})\in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

Case 7: $[x,z]=y,\ [x,y]=y,\ [x,x]=z$. Then, $\mathrm{Der}(\mathbf{A})=\mathrm{span}\{d_1,d_2\}$ where

$$\begin{aligned}
&d_1(x)=y, && d_1(y)=y, && d_1(z)=y,\\
&d_2(x)=z, && d_2(y)=y, && d_2(z)=y.
\end{aligned}$$

Since $\mathrm{im}(d_1-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$ and $\mathrm{im}(d_2-L_x)\subseteq\mathrm{Leib}(\mathbf{A})$, d_1 and d_2 are inner. Since $[d_1,d_2]=0$, we have that $\mathrm{Der}(\mathbf{A})$ is abelian. Also, we see that $\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{y,z\}$, $\mathbf{A}/\mathrm{Leib}(\mathbf{A})=\mathrm{span}\{x+\mathrm{Leib}(\mathbf{A})\}$ and $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))\neq\{0\}$ because $x+\mathrm{Leib}(\mathbf{A})\in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$.

By straightforward computations above, we obtain the following theorems.

Theorem 3.7. *Let $\mathbf{A}$ be a non-Lie non-nilpotent solvable Leibniz algebra and* $\dim(\mathbf{A})=3$. *Then every derivation of $\mathbf{A}$ is inner if and only if $\mathbf{A}$ is not isomorphic to a Leibniz algebra spanned by $\{x,y,z\}$ with the nonzero product given by $[x,z]=z$.*

Theorem 3.8. *Let* $\mathbf{A}$ *be a non-Lie non-nilpotent solvable Leibniz algebra and* $\dim(\mathbf{A}) = 3$*. Then* $\mathbf{A}$ *is complete if and only if* $\mathbf{A}$ *is isomorphic to a Leibniz algebra spanned by* $\{x, y, z\}$ *with the nonzero products given by one of the following:*

(1) $[x, z] = \alpha z,\ \alpha \in \mathbb{F} \setminus \{0\},\ [x, y] = y,\ [y, x] = -y,$

(2) $[x, y] = y,\ [y, x] = -y,\ [x, x] = z,$

(3) $[x, z] = 2z,\ [y, y] = z,\ [x, y] = y,\ [y, x] = -y,\ [x, x] = z.$

Remark 3.9. (1) Note that some of our results may look similar to those in [9]. However, by comparing our classification of derivation algebras to the classification given in [9], one Leibniz algebra isomorphism class was missed in their list, i.e., our Case 2 in Section 3.2. This implies that the dimension of the derivation algebra of three-dimensional Leibniz algebras can be up to five and hence their conclusion [9, page 81] may not cover all cases.

(2) For a given Leibniz algebra $\mathbf{A} = \operatorname{span}\{x, y, z\}$, we observe that $\operatorname{Der}(\mathbf{A}) \subseteq \operatorname{Der}^{\mathrm{Lie}}(\mathbf{A})$ in [10] as the derivations defined in our work are also Lie-derivations, except the case in [10, Proposition 3.4]. They use the Leibniz algebra class $[x, y] = z$ and $[y, z] = z$ in [10, Proposition 3.4] whereas we use $[x, x] = z$ and $[y, y] = z$ in Case 3 in Section 3.2 which corresponds to [4, Theorem 6.4 (3)]. Also, throughout their work, each $\operatorname{Leib}(\mathbf{A})$ is spanned by $[x, x], [y, y]$ and $[z, z]$ instead of $[a, a]$ for any $a \in \mathbf{A}$ so all inner and outer derivations for each case defined in [10] are different from ours.

Observe that if a Leibniz algebra $\mathbf{A}$ is isomorphic to a Leibniz algebra in Theorem 3.8 (3), then $\mathbf{A}$ is also complete by Definition in [7]. However, if a Leibniz algebra $\mathbf{A}$ is isomorphic to a Leibniz algebra in Theorem 3.8 (1) or (2), then $\mathbf{A}$ will not be complete as it admits outer derivations in the sense of [7]. (Note that in [7], the completeness definition is defined for right Leibniz algebras.)

It is known that for a Leibniz algebra $\mathbf{A}$ if the Lie algebra $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$ is complete, then $\mathbf{A}$ is a complete Leibniz algebra [3]. As shown below the converse holds for complete solvable Leibniz algebras of dimension 3.

Theorem 3.10. *Let* $\mathbf{A}$ *be a complete solvable Leibniz algebra of dimension 3. Then the Lie algebra* $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$ *is complete. Furthermore,* $\operatorname{Der}(\mathbf{A}/\operatorname{Leib}(\mathbf{A}))$ *is a complete Lie algebra.*

Proof. Suppose $\mathbf{A}$ is a complete solvable Leibniz algebra of dimension 3. By Theorem 3.8, $\operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{z\}$ and hence $\mathbf{A}/\operatorname{Leib}(\mathbf{A}) = \operatorname{span}\{x + \operatorname{Leib}(\mathbf{A}), y + \operatorname{Leib}(\mathbf{A})\}$ with $[x + \operatorname{Leib}(\mathbf{A}), y + \operatorname{Leib}(\mathbf{A})] = y + \operatorname{Leib}(\mathbf{A})$. Thus, $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$ is a non-abelian Lie algebra of dimension 2. By Theorem 2.6, we have that $\mathbf{A}/\operatorname{Leib}(\mathbf{A})$ is complete. To find a basis for $\operatorname{Der}(\mathbf{A}/\operatorname{Leib}(\mathbf{A}))$, let $d \in \operatorname{Der}(\mathbf{A}/\operatorname{Leib}(\mathbf{A}))$ and define

$$d(x + \operatorname{Leib}(\mathbf{A})) = \alpha_1(x + \operatorname{Leib}(\mathbf{A})) + \alpha_2(y + \operatorname{Leib}(\mathbf{A}))$$
$$d(y + \operatorname{Leib}(\mathbf{A})) = \beta_1(x + \operatorname{Leib}(\mathbf{A})) + \beta_2(y + \operatorname{Leib}(\mathbf{A}))$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$. Then

$$
\begin{aligned}
&d([x+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})]) \\
&= [d(x+\mathrm{Leib}(\mathbf{A})), y+\mathrm{Leib}(\mathbf{A})] + [x+\mathrm{Leib}(\mathbf{A}), d(y+\mathrm{Leib}(\mathbf{A}))] \\
&= [\alpha_1(x+\mathrm{Leib}(\mathbf{A})) + \alpha_2(y+\mathrm{Leib}(\mathbf{A})), y+\mathrm{Leib}(\mathbf{A})] \\
&\quad + [x+\mathrm{Leib}(\mathbf{A}), \beta_1(x+\mathrm{Leib}(\mathbf{A})) + \beta_2(y+\mathrm{Leib}(\mathbf{A}))] \\
&= \alpha_1[x+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})] + \alpha_2[y+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})] \\
&\quad + \beta_1[x+\mathrm{Leib}(\mathbf{A}), x+\mathrm{Leib}(\mathbf{A})] + \beta_2[x+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})] \\
&= (\alpha_1+\beta_2)[x+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})] \\
&= (\alpha_1+\beta_2)([x,y]+\mathrm{Leib}(\mathbf{A})) \\
&= (\alpha_1+\beta_2)(y+\mathrm{Leib}(\mathbf{A}))
\end{aligned}
$$

and

$$
\begin{aligned}
d([x+\mathrm{Leib}(\mathbf{A}), y+\mathrm{Leib}(\mathbf{A})]) &= d([x,y]+\mathrm{Leib}(\mathbf{A})) \\
&= d(y+\mathrm{Leib}(\mathbf{A})) \\
&= \beta_1(x+\mathrm{Leib}(\mathbf{A})) + \beta_2(y+\mathrm{Leib}(\mathbf{A})).
\end{aligned}
$$

Thus, by the linear independence of the basis vectors, $\alpha_1 = \beta_1 = 0$. This implies that

$$[d]_{\{x+\mathrm{Leib}(A), y+\mathrm{Leib}(A)\}} = \begin{pmatrix} 0 & 0 \\ \alpha_2 & \beta_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $\mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \mathrm{span}\{d_1, d_2\}$, where

$$
\begin{aligned}
d_1(x+\mathrm{Leib}(\mathbf{A})) &= y+\mathrm{Leib}(\mathbf{A}), & d_1(y+\mathrm{Leib}(\mathbf{A})) &= \mathrm{Leib}(\mathbf{A}), \\
d_2(x+\mathrm{Leib}(\mathbf{A})) &= \mathrm{Leib}(\mathbf{A}), & d_2(y+\mathrm{Leib}(\mathbf{A})) &= y+\mathrm{Leib}(\mathbf{A}).
\end{aligned}
$$

Since $[d_1, d_2] = -d_1$, we have that $\mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ is a non-abelian Lie algebra of dimension 2. By Theorem 2.6, this proves that $\mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ is complete. ∎

In [5], Meng proved that for a Lie algebra $\mathbf{L}$ such that $Z(\mathbf{L}) = \{0\}$ and $d(ad(\mathbf{L})) \subseteq ad(\mathbf{L})$ for all $d \in \mathrm{Der}(\mathrm{Der}(\mathbf{L}))$, $\mathrm{Der}(\mathbf{L})$ is a complete Lie algebra. As a result, for a complete Lie algebra $\mathbf{L}$, $\mathrm{Der}(\mathbf{L})$ is complete. However, as the following examples show there exist complete Leibniz algebras $\mathbf{A}$ such that $\mathrm{Der}(\mathbf{A})$ is not complete.

Example 3.11. Consider the complete Leibniz algebra $\mathbf{A} = \mathrm{span}\{x, y, z\}$ with nonzero multiplications $[x, z] = \alpha z,\ \alpha \in \mathbb{F} \setminus \{0\},\ [x, y] = y,\ [y, x] = -y$. We know that $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\}$ where

$$
\begin{aligned}
d_1(x) &= y, & d_1(y) &= 0, & d_1(z) &= 0, \\
d_2(x) &= 0, & d_2(y) &= y, & d_2(z) &= 0, \\
d_3(x) &= 0, & d_3(y) &= 0, & d_3(z) &= z.
\end{aligned}
$$

Since $[d_1, d_2] = -d_1,\ [d_1, d_3] = 0$ and $[d_2, d_3] = 0$, we have that $d_3 \in Z(\mathrm{Der}(\mathbf{A}))$ which implies that $Z(\mathrm{Der}(\mathbf{A})) \neq \{0\}$. Therefore, $\mathrm{Der}(\mathbf{A})$ is not complete.

Example 3.12. Consider the complete Leibniz algebra $\mathbf{A} = \mathrm{span}\{x, y, z\}$ with nonzero multiplications $[x, y] = y,\ [y, x] = -y,\ [x, x] = z$. We know that $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\}$

where

$$d_1(x) = y, \qquad d_1(y) = 0, \qquad d_1(z) = 0,$$
$$d_2(x) = z, \qquad d_2(y) = 0, \qquad d_2(z) = 0,$$
$$d_3(x) = 0, \qquad d_3(y) = y, \qquad d_3(z) = 0.$$

Since $[d_1, d_2] = 0$, $[d_1, d_3] = -d_1$ and $[d_2, d_3] = 0$, we have that $d_2 \in Z(\mathrm{Der}(\mathbf{A}))$ which implies that $Z(\mathrm{Der}(\mathbf{A})) \neq \{0\}$. Therefore, $\mathrm{Der}(\mathbf{A})$ is not complete.

We observe that $d_3 \in Z(\mathrm{Der}(\mathbf{A}))$ when $\mathrm{im}(d_3) = \mathrm{span}\{z\} \subseteq \mathrm{Leib}(\mathbf{A})$ in Example 3.11 and $d_2 \in Z(\mathrm{Der}(\mathbf{A}))$ when $\mathrm{im}(d_2) = \mathrm{span}\{z\} \subseteq \mathrm{Leib}(\mathbf{A})$ in Example 3.12. As a result, we formulate the following conjecture for complete Leibniz algebras.

Conjecture. *For a complete Leibniz algebra $\mathbf{A}$, $\mathrm{Der}(\mathbf{A})/I$ is a complete Lie algebra where $I = \{d \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$.*

In the following theorem, we show that the conjecture is true for complete solvable Leibniz algebras of dimension 3.

Theorem 3.13. *If A is a complete solvable Leibniz algebra of dimension 3 and $I = \{d \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$, then $\mathrm{Der}(\mathbf{A})/I$ is a complete Lie algebra.*

Proof. Clearly, $I \neq \emptyset$ as $0 \in I$. To show that I is a subspace of $\mathrm{Der}(\mathbf{A})$, let $d_1, d_2 \in I$ and $\alpha, \beta \in \mathbb{F}$. Then $\mathrm{im}(d_1), \mathrm{im}(d_2) \subseteq \mathrm{Leib}(\mathbf{A})$. Let $a \in \mathbf{A}$. Then we have that $(\alpha d_1 + \beta d_2)(a) = \alpha d_1(a) + \beta d_2(a) \in \mathrm{Leib}(\mathbf{A})$. Thus, $\mathrm{im}(\alpha d_1 + \beta d_2) \subseteq \mathrm{Leib}(\mathbf{A})$ and hence $\alpha d_1 + \beta d_2 \in I$ which implies that I is a subspace of $\mathrm{Der}(\mathbf{A})$. To show that I is an ideal of $\mathrm{Der}(\mathbf{A})$, let $d_1 \in I$, $d_2 \in \mathrm{Der}(\mathbf{A})$. Then $\mathrm{im}(d_1) \subseteq \mathrm{Leib}(\mathbf{A})$. Let $a \in \mathbf{A}$. Then we have that $[d_1, d_2](a) = d_1 d_2(a) - d_2 d_1(a)$. By Proposition 2.11, $d_1 d_2(a) - d_2 d_1(a) \in \mathrm{Leib}(\mathbf{A})$. Thus, $\mathrm{im}([d_1, d_2]) \subseteq \mathrm{Leib}(\mathbf{A})$ and hence $[d_1, d_2] \in I$. This proves that I is an ideal of $\mathrm{Der}(\mathbf{A})$.

Assume $\mathbf{A}$ is a complete solvable Leibniz algebra of dimension 3. By Theorem 3.8, $\mathbf{A}$ is isomorphic to one of the following algebras defined by the given nonzero multiplications. We will find I and $\mathrm{Der}(\mathbf{A})/I$ of each algebra to show that $\mathrm{Der}(\mathbf{A})/I$ of 3-dimensional complete solvable Leibniz algebras are complete.

(1) $[x, z] = \alpha z$, $\alpha \in \mathbb{F}\backslash\{0\}$, $[x, y] = y$, $[y, x] = -y$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\}$ where

$$d_1(x) = y, \qquad d_1(y) = 0, \qquad d_1(z) = 0,$$
$$d_2(x) = 0, \qquad d_2(y) = y, \qquad d_2(z) = 0,$$
$$d_3(x) = 0, \qquad d_3(y) = 0, \qquad d_3(z) = z,$$

and $[d_1, d_2] = -d_1$, $[d_1, d_3] = 0$ and $[d_2, d_3] = 0$. Thus, $I = \mathrm{span}\{d_3\}$ and $\mathrm{Der}(\mathbf{A})/I = \mathrm{span}\{d_1 + I, d_2 + I\}$ with $[d_1 + I, d_2 + I] = -d_1 + I$. Therefore, $\mathrm{Der}(\mathbf{A})/I$ is a non-abelian Lie algebra of dimension 2. By Theorem 2.6, this proves that $\mathrm{Der}(\mathbf{A})/I$ is complete.

(2) $[x, y] = y$, $[y, x] = -y$, $[x, x] = z$. Then, $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{d_1, d_2, d_3\}$ where

$$d_1(x) = y, \qquad d_1(y) = 0, \qquad d_1(z) = 0,$$
$$d_2(x) = z, \qquad d_2(y) = 0, \qquad d_2(z) = 0,$$
$$d_3(x) = 0, \qquad d_3(y) = y, \qquad d_3(z) = 0,$$

and $[d_1, d_2] = 0$, $[d_1, d_3] = -d_1$ and $[d_2, d_3] = 0$. Thus, $I = \text{span}\{d_2\}$ and $\text{Der}(\mathbf{A})/I = \text{span}\{d_1 + I, d_3 + I\}$ with $[d_1 + I, d_3 + I] = -d_1 + I$. Therefore, $\text{Der}(\mathbf{A})/I$ is a non-abelian Lie algebra of dimension 2. By Theorem 2.6, this proves that $\text{Der}(\mathbf{A})/I$ is complete.

(3) $[x, z] = 2z$, $[y, y] = z$, $[x, y] = y$, $[y, x] = -y$, $[x, x] = z$. Then $\text{Der}(\mathbf{A}) = \text{span}\{d_1, d_2\}$ where

$$d_1(x) = y, \qquad d_1(y) = -z, \qquad d_1(z) = 0,$$
$$d_2(x) = z, \qquad d_2(y) = y, \qquad d_2(z) = 2z,$$

and $[d_1, d_2] = -d_1$. Thus, $I = \{0\}$ and $\text{Der}(\mathbf{A})/I = \text{Der}(\mathbf{A})$ which is complete by Theorem 2.6. ∎

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