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New Exact Traveling Wave Solutions of the $(3+1)$-Dimensional Chiral Nonlinear Schrödinger Equation Using Two Reliable Techniques

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Abstract In this research, we study a (3+1)-dimensional chiral nonlinear Schrödinger equation (CNLSE) and find its exact traveling wave solutions via the extended simplest equation method (ESEM) and the improved generalized tanh-coth method (IGTCM). The exact solutions of the CNSLE are complex-valued functions that can be expressed in terms of exponential, hyperbolic, trigonometric, and rational functions. The magnitudes of some representative solutions are plotted as 3D and contour plots to illustrate the physical interpretations of the solutions. The findings establish that the used methods are simple, powerful, and reliable tools for obtaining new exact traveling wave solutions for complex nonlinear partial differential equations.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) have been used as models for complex phenomena occurring in many real-world problems in science and engineering. For

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example, Bera and Siti [1] used NLPDEs as models for some quantum mechanical problems. One of the NLPDEs which has become an important model in quantum physics is the chiral nonlinear Schrödinger equation (CNLSE) which has applications to the quantum Hall effect.

In 1998, Nishino et al. [2] proposed the 1D-CNLSE and showed that it had dark and bright soliton solutions. The 1D-CNLSE can be written in the form

$$i\Phi_t + b_0\Phi_{xx} + ib_1(\Phi\Phi_x^* - \Phi^*\Phi_x)\Phi = 0, \tag{1.1}$$

where $\Phi = \Phi(x,t)$ is a complex-valued function of x and t, b_0 is the coefficient of dispersion along the x-axis, and b_1 is the coefficient of the nonlinear coupling terms. The superscript * denotes the complex conjugate and $i = \sqrt{-1}$. Many different methods have now been proposed for solving 1D-CNLSEs. For example, equation (1.1) has been solved using the extended Fan sub-equation method [3], the sine-Gordon expansion method [4] and the Riccati-Bernoulli sub-ODE method [5]. In [6], equation (1.1) with time-dependent coefficients was solved by the soliton ansatz method for deriving its soliton solution. Moreover, equation (1.1) was solved by utilizing three finite difference schemes for its numerical solutions [7].

The 1D-CNLSE in (1.1) has also been generalized to a $(2+1)$-dimensional CNLSE which can be written as [8, 9]

$$i\Phi_t + b_0(\Phi_{xx} + \Phi_{yy}) + i\left\{b_1(\Phi\Phi_x^* - \Phi^*\Phi_x) + b_2(\Phi\Phi_y^* - \Phi^*\Phi_y)\right\}\Phi = 0, \tag{1.2}$$

where $\Phi = \Phi(x,y,t)$ is a complex-valued function of x, y and t, b_0 is the coefficient of dispersion along the spatial directions, and b_1, b_2 are the coefficients of the nonlinear coupling terms. The superscript * again denotes the complex conjugate and $i = \sqrt{-1}$. Because equation (1.2) fails the Painleve test of integrability, then it is not integrable by the method of Inverse Scattering Transform [9]. However, equation (1.2) has been solved to find its exact traveling wave solutions using many different methods such as the trial solution technique [8], the extended trial equation method [10], the generalized auxiliary equation method [11], the modified Jacobi elliptic expansion method [12], the enhanced modified extended tanh expansion method [13] and the modern extended direct algebraic approach [14]. In addition, the $(1+1)$-dimensional and $(2+1)$-dimensional CNLSEs have been generalized to different forms such as the stochastic form [15–17], the non-autonomous form [18, 19] and the conformable derivative form [20, 21].

In the present work, we introduce the $(3+1)$-dimensional CNLSE which can be written as:

$$\begin{aligned} i\Phi_t + b_0(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) + i\{&b_1(\Phi\Phi_x^* - \Phi^*\Phi_x) + b_2(\Phi\Phi_y^* - \Phi^*\Phi_y) \\ &+ b_3(\Phi\Phi_z^* - \Phi^*\Phi_z)\}\Phi = 0, \end{aligned} \tag{1.3}$$

where $\Phi = \Phi(x,y,z,t)$ is a complex-valued function of x, y, z and t, b_0 is the coefficient of dispersion along the spatial directions, and b_1, b_2, b_3 are the coefficients of the nonlinear coupling terms. Again, the superscript * represents complex conjugate and $i = \sqrt{-1}$.

The main purpose of this research is to derive exact traveling solutions of (1.3) using two techniques, namely, the extended simplest equation method (ESEM) and the improved generalized tanh-coth method (IGTCM). The paper is arranged as follows. In section 2, we describe the main steps of the ESEM and IGTCM. In section 3, we first apply the two methods to derive exact solutions of (1.3) and then show some graphs of the exact solutions. Finally, we present conclusions in section 4.

2. Algorithms

In this section, we describe the algorithms of the extended simplest equation method (ESEM) and the improved generalized tanh-coth method (IGTCM) for solving NLPDEs. Consider a general nonlinear PDE of the form:

$$P(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}, ...) = 0, \tag{2.1}$$

where P is a polynomial of $u = u(x, y, z, t)$ and its various partial derivatives which consists of both linear and nonlinear terms. As the first step in the two methods, Eq. (2.1) is converted into an ordinary differential equation (ODE) using the following traveling wave transformation

$$u(x, y, z, t) = u(\zeta), \quad \text{where} \quad \zeta = \alpha x + \beta y + \gamma z + \varepsilon t, \tag{2.2}$$

and where α, β, γ and ε are nonzero constants which will be determined at a later step. Then, after inserting the transformation (2.2) into equation (2.1) using the chain rule, we integrate the resulting equation with respect to ζ and obtain an ODE in $u = u(\zeta)$ which can be written in the form

$$Q(u, u', u', u', u', u'', u'', u'', u'', ...) = 0, \tag{2.3}$$

where the prime denotes the derivative with respect to ζ.

2.1. The Extended Simplest Equation Method

In this subsection, we discuss the key steps of the extended simplest equation method (ESEM) [22–24].

Step 1: We assume that the solution of Eq. (2.3) has the form

$$u(\zeta) = \sum_{N=0}^{M} \alpha_N \left(\frac{\psi'}{\psi}\right)^N + \sum_{J=0}^{M-1} \beta_J \left(\frac{\psi'}{\psi}\right)^J \left(\frac{1}{\psi}\right), \tag{2.4}$$

where α_N $(N = 0, 1, 2, 3, ..., M)$, $\beta_J (J = 0, 1, 2, 3, ..., M-1)$ are constants with $\alpha_M \beta_{M-1} \neq 0$ and ζ is defined in (2.2). Further, we assume that the function $\psi = \psi(\zeta)$ is the solution of the ESEM auxiliary equation

$$\psi'' + \eta\psi = \mu, \tag{2.5}$$

where μ and η are constants. Following [22–24], the function ψ, satisfying Eq. (2.5), can be defined depending on the values of the parameter η as follows:

$$\psi(\zeta) = \begin{cases} \sigma_1 \cosh \sqrt{-\eta}\zeta + \sigma_2 \sinh \sqrt{-\eta}\zeta + \frac{\mu}{\eta}, & \eta < 0, \\ \sigma_1 \cos \sqrt{\eta}\zeta + \sigma_2 \sin \sqrt{\eta}\zeta + \frac{\mu}{\eta}, & \eta > 0, \\ \frac{\mu}{2}\zeta^2 + \sigma_1 \zeta + \sigma_2, & \eta = 0, \end{cases} \tag{2.6}$$

and we have

$$\left(\frac{\psi'}{\psi}\right)^2 = \begin{cases} \left(\eta\sigma_1^2 - \eta\sigma_2^2 - \frac{\mu^2}{\eta}\right)\left(\frac{1}{\psi}\right)^2 - \eta + \frac{2\mu}{\psi}, & \eta < 0, \\ \left(\eta\sigma_1^2 + \eta\sigma_2^2 - \frac{\mu^2}{\eta}\right)\left(\frac{1}{\psi}\right)^2 - \eta + \frac{2\mu}{\psi}, & \eta > 0, \\ (\sigma_1^2 - 2\mu\sigma_2)\frac{1}{\psi^2} + \frac{2\mu}{\psi}, & \eta = 0, \end{cases} \tag{2.7}$$

where σ_1 and σ_2 are arbitrary constants.

Step 2: Using the homogeneous balance principle [25], we find the positive value M in Eq. (2.4) by balancing between the highest-order derivative term and the highest-order nonlinear terms in (2.3).

Step 3: Substituting Eq. (2.4) into Eq. (2.3) along with Eqs. (2.5), (2.6), (2.7) and setting the sums of all terms with the same power of $\frac{1}{\psi^i}$ and $\left(\frac{1}{\psi^i}\right)\left(\frac{\psi'}{\psi}\right)$, $(i = 1, 2, 3, ...)$ to zero, we obtain a set of nonlinear algebraic equations for α_N, $(N = 0, 1, 2, 3, ..., M)$, β_J, $(J = 0, 1, 2, 3, ..., M-1)$, α, β, γ ε, μ and η.

Step 4: Then, if the constants α_N, $(N = 0, 1, 2, 3, ..., M)$, β_J, $(J = 0, 1, 2, 3, ..., M-1)$, α, β, γ ε, μ and η in Step 3 can be evaluated using a symbolic software package such as Maple, we substitute the obtained values and ψ in (2.6) into (2.4) and obtain the exact solutions of (2.1) via the transformation (2.2).

2.2. The Improved Generalized Tanh-Coth Method

In this subsection, we describe the steps in the improved generalized tanh-coth method (IGTCM) [26–28].

Step 1: Let the solution of Eq. (2.3) be of the form:

$$u(\zeta) = \sum_{N=0}^{M} \alpha_N \phi(\zeta)^N + \sum_{N=M+1}^{2M} \alpha_N \phi(\zeta)^{M-N}, \tag{2.8}$$

where the coefficients α_N are constants such that $\alpha_M \neq 0$, $\alpha_{2M} \neq 0$ and M is a positive integer which will be determined later using the homogeneous balance principle [25]. Also, let the function $\phi(\zeta)$ be a solution of the generalized Riccati equation

$$\phi'(\zeta) = \rho + \tau\phi(\zeta) + \sigma\phi^2(\zeta), \tag{2.9}$$

where ρ, τ, σ are arbitrary constants. Following [26–28], we define the solutions for the function $\phi(\zeta)$ of (2.9), which depend upon the values of ρ, τ and σ, as follows.

Case 1: If $\rho = 0$, then

$$\phi(\zeta) = \frac{\tau}{-\sigma + \tau\exp(-\tau\zeta)}. \tag{2.10}$$

Case 2: If $\tau = 0$, then

$$\phi(\zeta) = \begin{cases} \frac{\sqrt{\rho\sigma}}{\sigma}\tan(\sqrt{\rho\sigma}\zeta) & \rho > 0, \sigma > 0, \\ \frac{\sqrt{\rho\sigma}}{\sigma}\tanh(\sqrt{\rho\sigma}\zeta) & \rho > 0, \sigma < 0, \\ \frac{\sqrt{-\rho\sigma}}{\sigma}\tanh(-\sqrt{-\rho\sigma}\zeta) & \rho < 0, \sigma > 0, \\ \frac{\sqrt{\rho\sigma}}{\sigma}\tan(-\sqrt{\rho\sigma}\zeta) & \rho < 0, \sigma < 0. \end{cases} \tag{2.11}$$

Case 3: If $\sigma = 0$, then

$$\phi(\zeta) = \frac{-\rho + \tau\exp(\tau\zeta)}{\tau}. \tag{2.12}$$

Case 4: If $\rho = \tau = 0$, then

$$\phi(\zeta) = -\frac{1}{\sigma\zeta}. \tag{2.13}$$

Case 5: if $\tau^2 = 4\rho\sigma \neq 0$, then

$$\phi(\zeta) = -\frac{2\rho(\tau\zeta + 2)}{\tau^2\zeta}. \tag{2.14}$$

Case 6: If $\tau^2 < 4\rho\sigma$ and $\sigma \neq 0$, then

$$\phi(\zeta) = \frac{\sqrt{4\rho\sigma - \tau^2}\tan\left(\frac{1}{2}\sqrt{4\rho\sigma - \tau^2}\zeta\right) - \tau}{2\sigma}. \tag{2.15}$$

Case 7: If $\tau^2 > 4\rho\sigma$ and $\sigma \neq 0$, then

$$\phi(\zeta) = \frac{\sqrt{\tau^2 - 4\rho\sigma}\tanh\left(\frac{1}{2}\sqrt{\tau^2 - 4\rho\sigma}\zeta\right) - \tau}{2\sigma}. \tag{2.16}$$

Step 2: After obtaining the value of M from the homogeneous balance principle [25], we substitute Eq. (2.8) into Eq. (2.3) along with Eq. (2.9) and obtain a polynomial in $\phi^j(\zeta)$, $(j = ..., -2, -1, 0, 1, 2, ...)$. Setting all coefficients of $\phi^j(\zeta)$ to zero, we obtain a system of algebraic equations which can be solved by a symbolic software package such as Maple to obtain values for $\alpha_N(N = 0, ..., 2M)$, α, β, γ, ε, ρ, τ and σ.

Step 3: The exact solutions of Eq. (2.1) can then be obtained by inserting the values of $\alpha_N(N = 0, ..., 2M)$, α, β, γ, ε, ρ, τ and σ into the solution (2.8) along with the appropriate case for the solution of $\phi(\zeta)$ in Eqs. (2.10)-(2.16) and for ζ in (2.2).

3. Exact Traveling Wave Solutions of (1.3) Using the ESEM and IGTCM

In this section, we use the ESEM and IGTCM to obtain exact symbolic solutions of the $(3+1)$-dimensional CNLSE in Eq. (1.3). Firstly, we assume that the solution of Eq. (1.3) can be written in the form:

$$\Phi(x, y, z, t) = u(\zeta)\, e^{i\Theta}, \tag{3.1}$$

where ζ and Θ are wave transformations defined by

$$\zeta = cx + ky + mz - vt \qquad \text{and} \qquad \Theta = px + qy + rt + \omega t, \tag{3.2}$$

where c, k, m, v, p, q, r and ω are real constants. Then, substituting the assumed solution (3.1) into Eq. (1.3) and separating the real and imaginary parts of the resulting equation, we obtain the following equations:

$$\text{Re:} \quad b_0\left(c^2 + k^2 + m^2\right)u'' + 2\left(pb_1 + qb_2 + rb_3\right)u^3 - \left(b_0\left(p^2 + q^2 + r^2\right) + \omega\right)u = 0, \tag{3.3}$$

$$\text{Im:} \quad \left(2\, b_0(cp + kq + mr) - v\right)u' = 0. \tag{3.4}$$

Since u' in Eq. (3.4) is not zero, we have the relationship

$$v = 2b_0\left(cp + kq + mr\right). \tag{3.5}$$

Secondly, we balance the highest order derivative u'' and the nonlinear term u^3 in Eq. (3.3) using the homogeneous balance principle to obtain the value of M as $M = 1$. The remainder of the work to obtain an exact solution is then to solve Eq. (3.3) for $u(\zeta)$, ζ and Θ using either the ESEM or IGTCM.

3.1. Application of the ESEM

Inserting $M = 1$ into the solution form (2.4), we obtain

$$u(\zeta) = \alpha_0 + \alpha_1\left(\frac{\psi'}{\psi}\right) + \beta_0\left(\frac{1}{\psi}\right), \tag{3.6}$$

where α_0, α_1 and β_0 are constants which will be determined at a later step and where the function $\psi = \psi(\zeta)$ is a solution of Eq. (2.5). Substituting Eq. (3.6) into the ODE (3.3), and then using Eqs. (2.6) and (2.7) with the appropriate value of η, Eq. (3.3) becomes a polynomial of the terms $\frac{1}{\psi^i}$ and $\left(\frac{1}{\psi^i}\right)\left(\frac{\psi'}{\psi}\right)$. Then, setting each coefficient of this polynomial to zero, we have the following three systems of algebraic equations.

Result 1: For $\eta < 0$, we obtain the following set of algebraic equations:

$$\begin{aligned}
\frac{1}{\psi^0} &: -6\,\eta\,p\alpha_0\alpha_1^2b_1 - 6\,\eta\,q\alpha_0\alpha_1^2b_2 - 6\,\eta\,r\alpha_0\alpha_1^2b_3 + 2\,b_1p\alpha_0^3 + 2\,b_2q\alpha_0^3 + 2\,b_3r\alpha_0^3 \\
&\quad - b_0p^2\alpha_0 - b_0q^2\alpha_0 - b_0r^2\alpha_0 - \omega\,\alpha_0 = 0, \\
\frac{1}{\psi^1} &: -6\,\eta\,p\alpha_1^2b_1\beta_0 - 6\,\eta\,q\alpha_1^2b_2\beta_0 - 6\,\eta\,r\alpha_1^2b_3\beta_0 + 12\,\mu\,p\alpha_0\alpha_1^2b_1 - \omega\,\beta_0 \\
&\quad + 12\,\mu\,r\alpha_0\alpha_1^2b_3 - c^2\eta\,b_0\beta_0 - \eta\,k^2b_0\beta_0 - \eta\,m^2b_0\beta_0 + 6\,p\alpha_0^2b_1\beta_0 - r^2b_0\beta_0 \\
&\quad + 6\,r\alpha_0^2b_3\beta_0 - p^2b_0\beta_0 - q^2b_0\beta_0 + 12\,\mu\,q\alpha_0\alpha_1^2b_2 + 6\,q\alpha_0^2b_2\beta_0 = 0, \\
\frac{1}{\psi^2} &: 12\,\mu\,p\alpha_1^2b_1\beta_0 + 12\,\mu\,q\alpha_1^2b_2\beta_0 + 12\,\mu\,r\alpha_1^2b_3\beta_0 + 6\,p\alpha_0b_1\beta_0^2 + 3\,m^2\mu\,b_0\beta_0 \\
&\quad + 3\,k^2\mu\,b_0\beta_0 + 6\,q\alpha_0b_2\beta_0^2 + 6\,r\alpha_0b_3\beta_0^2 + 6\,\eta\,\sigma_1^2p\alpha_0\alpha_1^2b_1 + 6\,\eta\,\sigma_1^2q\alpha_0\alpha_1^2b_2 \\
&\quad + 6\,\eta\,\sigma_1^2r\alpha_0\alpha_1^2b_3 - 6\,\eta\,\sigma_2^2p\alpha_0\alpha_1^2b_1 - 6\,\eta\,\sigma_2^2q\alpha_0\alpha_1^2b_2 - 6\,\eta\,\sigma_2^2r\alpha_0\alpha_1^2b_3 \\
&\quad - 6\,\frac{\mu^2p\alpha_0\alpha_1^2b_1}{\eta} - 6\,\frac{\mu^2q\alpha_0\alpha_1^2b_2}{\eta} - 6\,\frac{\mu^2r\alpha_0\alpha_1^2b_3}{\eta} + 3\,c^2\mu\,b_0\beta_0 = 0, \\
\frac{1}{\psi^3} &: 2\,pb_1\beta_0^3 + 2\,qb_2\beta_0^3 + 2\,rb_3\beta_0^3 + 2\,\eta\,\sigma_1^2c^2b_0\beta_0 + 2\,\eta\,\sigma_1^2k^2b_0\beta_0 \\
&\quad - 2\,\eta\,\sigma_2^2c^2b_0\beta_0 - 2\,\eta\,\sigma_2^2k^2b_0\beta_0 - 2\,\eta\,\sigma_2^2m^2b_0\beta_0 - 2\,\frac{\mu^2c^2b_0\beta_0}{\eta} \\
&\quad - 2\,\frac{\mu^2m^2b_0\beta_0}{\eta} - 6\,\eta\,\sigma_2^2r\alpha_1^2b_3\beta_0 - 6\,\frac{\mu^2p\alpha_1^2b_1\beta_0}{\eta} - 6\,\frac{\mu^2q\alpha_1^2b_2\beta_0}{\eta} \\
&\quad + 6\,\eta\,\sigma_1^2p\alpha_1^2b_1\beta_0 + 6\,\eta\,{\sigma_1}^2q\alpha_1^2b_2\beta_0 + 6\,\eta\,{\sigma_1}^2r\alpha_1^2b_3\beta_0 - 6\,\eta\,{\sigma_2}^2p\alpha_1^2b_1\beta_0 \\
&\quad - 6\,\eta\,\sigma_2^2q\alpha_1^2b_2\beta_0 - 6\,\frac{\mu^2r\alpha_1^2b_3\beta_0}{\eta} - 2\,\frac{\mu^2k^2b_0\beta_0}{\eta} + 2\,\eta\,\sigma_1^2m^2b_0\beta_0 = 0, \\
\frac{1}{\psi^0}\left(\frac{\psi'}{\psi}\right) &: -2\,\eta\,p\alpha_1^3b_1 - 2\,\eta\,q\alpha_1^3b_2 - 2\,\eta\,r\alpha_1^3b_3 + 6\,p\alpha_0^2\alpha_1b_1 + 6\,q\alpha_0^2\alpha_1b_2 \\
&\quad - p^2\alpha_1b_0 - q^2\alpha_1b_0 - r^2\alpha_1b_0 - \omega\,\alpha_1 + 6\,r\alpha_0^2\alpha_1b_3 = 0, \\
\frac{1}{\psi^1}\left(\frac{\psi'}{\psi}\right) &: 4\,\mu\,p\alpha_1^3b_1 + 4\,\mu\,q\alpha_1^3b_2 + 4\,\mu\,r\alpha_1^3b_3 + c^2\mu\,\alpha_1b_0 + k^2\mu\,\alpha_1b_0 + m^2\mu\,\alpha_1b_0 \\
&\quad + 12\,p\alpha_0\alpha_1b_1\beta_0 + 12\,q\alpha_0\alpha_1b_2\beta_0 + 12\,r\alpha_0\alpha_1b_3\beta_0 = 0, \\
\frac{1}{\psi^2}\left(\frac{\psi'}{\psi}\right) &: 6\,b_1p\alpha_1\beta_0^2 + 6\,b_2q\alpha_1\beta_0^2 + 6\,b_3r\alpha_1\beta_0^2 - 2\,\eta\,\sigma_2^2k^2\alpha_1b_0 - 2\,\eta\,{\sigma_2}^2m^2\alpha_1b_0 \\
&\quad - 2\,\frac{\mu^2p\alpha_1^3b_1}{\eta} - 2\,\frac{\mu^2q\alpha_1^3b_2}{\eta} - 2\,\frac{\mu^2r\alpha_1^3b_3}{\eta} - 2\,\frac{\mu^2c^2\alpha_1b_0}{\eta} - 2\,\frac{\mu^2k^2\alpha_1b_0}{\eta} \\
&\quad - 2\,\frac{\mu^2m^2\alpha_1b_0}{\eta} + 2\,\eta\,\sigma_1^2p\alpha_1^3b_1 + 2\,\eta\,\sigma_1^2q\alpha_1^3b_2 + 2\,\eta\,\sigma_1^2r\alpha_1^3b_3
\end{aligned} \tag{3.7}$$

$$+\,2\,\eta\,\sigma_1^2k^2\alpha_1b_0+2\,\eta\,\sigma_1^2m^2\alpha_1b_0-2\,\eta\,\sigma_2^2p\alpha_1^3b_1-2\,\eta\,\sigma_2^2q\alpha_1^3b_2$$
$$-\,2\,\eta\,\sigma_2^2r\alpha_1^3b_3-2\,\eta\,\sigma_2^2c^2\alpha_1b_0+2\,\eta\,\sigma_1^2c^2\alpha_1b_0=0.$$

Solving system (3.7) using the Maple package program, we obtain

$$\begin{aligned}
&\alpha_0=0,\ \alpha_1=\pm\frac{1}{2}\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}},\\
&\beta_0=\alpha_1\sqrt{\frac{-\left(\mu^2+\left(\sigma_2^2-\sigma_1^2\right)\eta^2\right)}{\eta}},\\
&\omega=\frac{b_0}{2}\left(\left(c^2+k^2+m^2\right)\eta-2(p^2+q^2+r^2)\right),
\end{aligned}\tag{3.8}$$

where $\eta(<0)$, μ, σ_1, σ_2, c, k, m, p, q, r, b_0, b_1, b_2, and b_3 are arbitrary constants such that $\alpha_1,\ \beta_0\in\mathbb{R}$.

Substituting Eq. (3.8) into Eq. (3.6) along with Eq. (2.6), we obtain the exact solution of Eq. (1.3) by inserting the resulting equation, Eq. (3.2) and the relation (3.5) into Eq. (3.1) as follows:

$$\begin{aligned}
\Phi(x,y,z,t)=\frac{1}{2}&\left\{\pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\right.\\
&\times\left(\frac{\sigma_1\sinh\left(\sqrt{-\eta}\zeta\right)\sqrt{-\eta}+\sigma_2\cosh\left(\sqrt{-\eta}\zeta\right)\sqrt{-\eta}}{\sigma_1\cosh\left(\sqrt{-\eta}\zeta\right)+\sigma_2\sinh\left(\sqrt{-\eta}\zeta\right)+\frac{\mu}{\eta}}\right)\\
&\pm\sqrt{\frac{b_0\left(\left(\sigma_2^2-\sigma_1^2\right)\eta^2+\mu^2\right)\left(c^2+k^2+m^2\right)}{\eta\ \left(pb_1+qb_2+rb_3\right)}}\\
&\left.\times\left(\frac{1}{\sigma_1\cosh\left(\sqrt{-\eta}\zeta\right)+\sigma_2\sinh\left(\sqrt{-\eta}\zeta\right)+\frac{\mu}{\eta}}\right)\right\}\\
&\times\exp\left(i\left(px+qy+rz+\omega t\right)\right),
\end{aligned}\tag{3.9}$$

where $\zeta=cx+ky+mz-2b_0\left(cp+kq+mr\right)t$.

Result 2: For $\eta>0$, we obtain the following system:

$$\begin{aligned}
\frac{1}{\psi^0}\ :\ &-6\,\eta\,p\alpha_0\alpha_1^2b_1-6\,\eta\,q\alpha_0\alpha_1^2b_2-6\,\eta\,r\alpha_0\alpha_1^2b_3+2\,b_1p\alpha_0^3+2\,b_2q\alpha_0^3\\
&-b_0p^2\alpha_0-b_0q^2\alpha_0-b_0r^2\alpha_0-\omega\,\alpha_0+2\,b_3r\alpha_0^3=0,\\
\frac{1}{\psi^1}\ :\ &-6\,\eta\,p\alpha_1^2b_1\beta_0-6\,\eta\,q\alpha_1^2b_2\beta_0-6\,\eta\,r\alpha_1^2b_3\beta_0+12\,\mu\,p\alpha_0\alpha_1^2b_1-\omega\,\beta_0\\
&+12\,\mu\,r\alpha_0\alpha_1^2b_3-b_0c^2\eta\,\beta_0-b_0\eta\,k^2\beta_0-b_0\eta\,m^2\beta_0+6\,p\alpha_0^2b_1\beta_0-b_0r^2\beta_0\\
&+6\,r\alpha_0^2b_3\beta_0-ap^2\beta_0-b_0q^2\beta_0+12\,\mu\,q\alpha_0\alpha_1^2b_2+6\,q\alpha_0^2b_2\beta_0=0,
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{\psi^2} &: 12\,\mu\, p\alpha_1^2 b_1\beta_0 + 12\,\mu\, q\alpha_1^2 b_2\beta_0 + 12\,\mu\, r\alpha_1^2 b_3\beta_0 + 3\,b_0 m^2\mu\,\beta_0 + 3\,b_0 c^2\mu\,\beta_0 \\
&\quad + 6\,p\alpha_0 b_1\beta_0^2 + 6\,q\alpha_0 b_2\beta_0^2 + 6\,r\alpha_0 b_3\beta_0^2 - 6\,\frac{\mu^2 q\alpha_0\alpha_1^2 b_2}{\eta} - 6\,\frac{\mu^2 r\alpha_0\alpha_1^2 b_3}{\eta} \\
&\quad + 6\,\eta\,\sigma_1^2 p\alpha_0\alpha_1^2 b_1 + 6\,\eta\,\sigma_1^2 q\alpha_0\alpha_1^2 b_2 + 6\,\eta\,\sigma_1^2 r\alpha_0\alpha_1^2 b_3 + 6\,\eta\,\sigma_2^2 p\alpha_0\alpha_1^2 b_1 \\
&\quad + 6\,\eta\,\sigma_2^2 q\alpha_0\alpha_1^2 b_2 + 6\,\eta\,\sigma_2^2 r\alpha_0\alpha_1^2 b_3 - 6\,\frac{\mu^2 p\alpha_0\alpha_1^2 b_1}{\eta} + 3\,b_0 k^2\mu\,\beta_0 = 0, \\
\frac{1}{\psi^3} &: 2\,p b_1\beta_0^3 + 2\,q b_2\beta_0^3 + 2\,r b_3\beta_0^3 + 2\,\eta\,\sigma_1^2 b_0 c^2\beta_0 + 2\,\eta\,\sigma_1^2 b_0 k^2\beta_0 \\
&\quad + 2\,\eta\,\sigma_2^2 b_0 c^2\beta_0 + 2\,\eta\,\sigma_2^2 b_0 k^2\beta_0 + 2\,\eta\,\sigma_2^2 b_0 m^2\beta_0 - 2\,\frac{\mu^2 b_0 c^2\beta_0}{\eta} - 2\,\frac{\mu^2 b_0 k^2\beta_0}{\eta} \\
&\quad - 2\,\frac{\mu^2 b_0 m^2\beta_0}{\eta} + 6\,\eta\,{\sigma_1}^2 p\alpha_1^2 b_1\beta_0 + 6\,\eta\,\sigma_1^2 q\alpha_1^2 b_2\beta_0 + 6\,\eta\,\sigma_1^2 r\alpha_1^2 b_3\beta_0 \\
&\quad + 6\,\eta\,\sigma_2^2 p\alpha_1^2 b_1\beta_0 + 6\,\eta\,\sigma_2^2 q\alpha_1^2 b_2\beta_0 + 6\,\eta\,\sigma_2^2 r\alpha_1^2 b_3\beta_0 - 6\,\frac{\mu^2 p\alpha_1^2 b_1\beta_0}{\eta} \\
&\quad - 6\,\frac{\mu^2 q\alpha_1^2 b_2\beta_0}{\eta} - 6\,\frac{\mu^2 r\alpha_1^2 b_3\beta_0}{\eta} + 2\,\eta\,\sigma_1^2 b_0 m^2\beta_0 = 0, \qquad (3.10)\\
\frac{1}{\psi^0}\left(\frac{\psi'}{\psi}\right) &: -2\,\eta\, p\alpha_1^3 b_1 - 2\,\eta\, q\alpha_1^3 b_2 - 2\,\eta\, r\alpha_1^3 b_3 + 6\,p\alpha_0^2\alpha_1 b_1 + 6\,q\alpha_0^2\alpha_1 b_2 \\
&\quad + 6\,r\alpha_0^2\alpha_1 b_3 - b_0 p^2\alpha_1 - b_0 q^2\alpha_1 - b_0 r^2\alpha_1 - \omega\,\alpha_1 = 0, \\
\frac{1}{\psi^1}\left(\frac{\psi'}{\psi}\right) &: 4\,\mu\, p\alpha_1^3 b_1 + 4\,\mu\, q\alpha_1^3 b_2 + 4\,\mu\, r\alpha_1^3 b_3 + b_0 c^2\mu\,\alpha_1 + b_0 k^2\mu\,\alpha_1 + b_0 m^2\mu\,\alpha_1 \\
&\quad + 12\,p\alpha_0\alpha_1 b_1\beta_0 + 12\,q\alpha_0\alpha_1 b_2\beta_0 + 12\,r\alpha_0\alpha_1 b_3\beta_0 = 0, \\
\frac{1}{\psi^2}\left(\frac{\psi'}{\psi}\right) &: 2\,\eta\,\sigma_1^2 p{\alpha_1}^3 b_1 + 2\,\eta\,\sigma_1^2 q\alpha_1^3 b_2 + 2\,\eta\,\sigma_1^2 r\alpha_1^3 b_3 + 2\,\eta\,\sigma_1^2 b_0 c^2\alpha_1 \\
&\quad + 2\,\eta\,\sigma_1^2 a k^2\alpha_1 + 2\,\eta\,\sigma_1^2 b_0 m^2\alpha_1 + 2\,\eta\,\sigma_2^2 p{\alpha_1}^3 b_1 + 2\,\eta\,\sigma_2^2 q\alpha_1^3 b_2 \\
&\quad + 2\,\eta\,\sigma_2^2 r\alpha_1^3 b_3 + 2\,\eta\,\sigma_2^2 b_0 c^2\alpha_1 + 2\,\eta\,\sigma_2^2 b_0 k^2\alpha_1 + 2\,\eta\,\sigma_2^2 b_0 m^2\alpha_1 \\
&\quad + 6\,p\alpha_1 b_1\beta_0^2 + 6\,q\alpha_1 b_2\beta_0^2 + 6\,r\alpha_1 b_3\beta_0^2 - 2\,\frac{\mu^2 p\alpha_1^3 b_1}{\eta} - 2\,\frac{\mu^2 q\alpha_1^3 b_2}{\eta} \\
&\quad - 2\,\frac{\mu^2 r\alpha_1^3 b_3}{\eta} - 2\,\frac{\mu^2 b_0 c^2\alpha_1}{\eta} - 2\,\frac{\mu^2 b_0 k^2\alpha_1}{\eta} - 2\,\frac{\mu^2 b_0 m^2\alpha_1}{\eta} = 0.
\end{aligned}
$$

Solving system (3.10) using the Maple package program, we have the following results

$$
\begin{aligned}
&\alpha_0 = 0,\ \alpha_1 = \pm\frac{1}{2}\sqrt{-\frac{b_0(c^2+k^2+m^2)}{pb_1+qb_2+rb_3}}, \\
&\beta_0 = \alpha_1\sqrt{\frac{-\left(\mu^2-(\sigma_1^2+\sigma_2^2)\,\eta^2\right)}{\eta}}, \qquad (3.11)\\
&\omega = \frac{b_0}{2}\left(\left(c^2+k^2+m^2\right)\eta - 2(p^2+q^2+r^2)\right),
\end{aligned}
$$

where $\eta(>0)$, μ, σ_1, σ_2, c, k, m, p, q, r, b_0, b_1, b_2, and b_3 are arbitrary constants such that α_1, $\beta_0 \in \mathbb{R}$.

Substituting Eq. (3.11) into Eq. (3.6) along with Eq. (2.6), we obtain the exact solution of Eq. (1.3) by inserting the resulting equation, Eq. (3.2) and the relation (3.5) into Eq. (3.1) as follows:

$$\begin{aligned}\Phi(x,y,z,t) = \frac{1}{2}\Bigg\{ &\pm\sqrt{-\frac{b_0(c^2+k^2+m^2)}{pb_1+qb_2+rb_3}} \\ &\times\left(\frac{-\sigma_1\sin\left(\sqrt{\eta}\zeta\right)\sqrt{\eta}+\sigma_2\cos\left(\sqrt{\eta}\zeta\right)\sqrt{\eta}}{\sigma_1\cos\left(\sqrt{\eta}\zeta\right)+\sigma_2\sin\left(\sqrt{\eta}\zeta\right)+\frac{\mu}{\eta}}\right) \\ &\pm\sqrt{\frac{b_0\left(c^2+k^2+m^2\right)\left(\left(-\sigma_1^2-\sigma_2^2\right)\eta^2+\mu^2\right)}{\eta\,\left(pb_1+qb_2+rb_3\right)}} \\ &\times\left(\frac{1}{\sigma_1\cos\left(\sqrt{\eta}\zeta\right)+\sigma_2\sin\left(\sqrt{\eta}\zeta\right)+\frac{\mu}{\eta}}\right)\Bigg\} \\ &\times\exp\left(i\left(px+qy+rz+\omega t\right)\right),\end{aligned} \tag{3.12}$$

where $\zeta = cx+ky+mz-2b_0\left(cp+kq+mr\right)t$.

Result 3: For $\eta = 0$, we have

$$\begin{aligned}
\frac{1}{\psi^0}\ &: 2\,b_1p{\alpha_0}^3+2\,b_2q\alpha_0^3+2\,b_3r\alpha_0^3-b_0p^2\alpha_0-b_0q^2\alpha_0-b_0r^2\alpha_0-\omega\,\alpha_0=0,\\
\frac{1}{\psi^1}\ &: 12\,\mu\,p\alpha_0\alpha_1^2b_1+12\,\mu\,q\alpha_0\alpha_1^2b_2+12\,\mu\,r\alpha_0\alpha_1^2b_3+6\,p\alpha_0^2b_1\beta_0\\
&\quad+6\,q\alpha_0^2b_2\beta_0+6\,r{\alpha_0}^2b_3\beta_0-b_0p^2\beta_0-b_0q^2\beta_0-b_0r^2\beta_0-\omega\,\beta_0=0,\\
\frac{1}{\psi^2}\ &: -12\,\mu\,p\sigma_2\alpha_0\alpha_1^2b_1-12\,\mu\,q\sigma_2\alpha_0\alpha_1^2b_2-12\,\mu\,r\sigma_2\alpha_0\alpha_1^2b_3+6\,p\sigma_1^2\alpha_0\alpha_1^2b_1\\
&\quad+6\,q\sigma_1^2\alpha_0{\alpha_1}^2b_2+6\,r\sigma_1^2\alpha_0\alpha_1^2b_3+12\,\mu\,p\alpha_1^2b_1\beta_0+12\,\mu\,q\alpha_1^2b_2\beta_0\\
&\quad+12\,\mu\,r\alpha_1^2b_3\beta_0+3\,b_0c^2\mu\,\beta_0+3\,b_0k^2\mu\,\beta_0+3\,b_0m^2\mu\,\beta_0+6\,p\alpha_0b_1\beta_0^2\\
&\quad+6\,q\alpha_0b_2\beta_0^2+6\,r\alpha_0b_3\beta_0^2=0,\\
\frac{1}{\psi^3}\ &: -12\,\mu\,p\sigma_2\alpha_1^2b_1\beta_0-12\,\mu\,q\sigma_2\alpha_1^2b_2\beta_0-12\,\mu\,r\sigma_2\alpha_1^2b_3\beta_0+6\,p\sigma_1^2\alpha_1^2b_1\beta_0\\
&\quad+6\,q\sigma_1^2\alpha_1^2b_2\beta_0+6\,r\sigma_1^2\alpha_1^2b_3\beta_0-4\,b_0c^2\mu\,\sigma_2\beta_0+2\,b_0c^2\sigma_1^2\beta_0+2\,rb_3\beta_0^3\\
&\quad+2\,ak^2\sigma_1^2\beta_0-4\,am^2\mu\,\sigma_2\beta_0+2\,am^2\sigma_1^2\beta_0+2\,pb_1\beta_0^3+2\,qb_2\beta_0^3\\
&\quad-4\,b_0k^2\mu\,\sigma_2\beta_0=0,\\
\frac{1}{\psi^0}\left(\frac{\psi'}{\psi}\right)\ &: 6p\alpha_0^2\alpha_1b_1+6q\alpha_0^2\alpha_1b_2+6r\alpha_0^2\alpha_1b_3-ap^2\alpha_1-b_0q^2\alpha_1-b_0r^2\alpha_1-\omega\alpha_1=0,\\
\frac{1}{\psi^1}\left(\frac{\psi'}{\psi}\right)\ &: 4\,\mu\,p\alpha_1^3b_1+4\,\mu\,q\alpha_1^3b_2+4\,\mu\,r\alpha_1^3b_3+ac^2\mu\,\alpha_1+b_0k^2\mu\,\alpha_1+b_0m^2\mu\,\alpha_1\\
&\quad+12\,p\alpha_0\alpha_1b_1\beta_0+12\,q\alpha_0\alpha_1b_2\beta_0+12\,r\alpha_0\alpha_1b_3\beta_0=0,
\end{aligned} \tag{3.13}$$

$$\frac{1}{\psi^2}\left(\frac{\psi'}{\psi}\right) : -4\,\mu\, p\sigma_2\alpha_1^3 b_1 - 4\,\mu\, q\sigma_2\alpha_1^3 b_2 - 4\,\mu\, r\sigma_2\alpha_1^3 b_3 + 2\,p\sigma_1^2\sigma\alpha_1^3 b_1 + 2\,q\sigma_1^2\alpha_1^3 b_2$$
$$+ 2\,r\sigma_1^2\alpha_1^3 b_3 - 4\,b_0 c^2\mu\,\sigma_2\alpha_1 + 2\,b_0 c^2\sigma_1^2\alpha_1 - 4\,b_0 k^2\mu\,\sigma_2\alpha_1 + 2\,b_0 k^2\sigma_1^2\alpha_1$$
$$- 4\,b_0 m^2\mu\,\sigma_2\alpha_1 + 2\,am^2\sigma_1^2\alpha_1 + 6\,p\alpha_1 b_1\beta_0^2 + 6\,q\alpha_1 b_2\beta_0^2 + 6\,r\alpha_1 b_3\beta_0^2 = 0.$$

By solving system (3.13) using the Maple package program, we have the following results

$$\begin{aligned}&\alpha_0 = 0,\ \alpha_1 = \pm\frac{1}{2}\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}},\\ &\beta_0 = \alpha_1\sqrt{\sigma_1^2 - 2\,\mu\,\sigma_2},\ \omega = -b_0\left(p^2+q^2+r^2\right),\end{aligned} \tag{3.14}$$

where μ, σ_1, σ_2, c, k, m, p, q, r, b_0, b_1, b_2, and b_3 are arbitrary constants such that α_1, β_0 $\in \mathbb{R}$.

Substituting (3.14) into Eq. (3.6) along with Eq. (2.6), we obtain the exact solution of Eq. (1.3) by replacing the resulting equation, Eq. (3.2) and the relation (3.5) into Eq. (3.1) as follows:

$$\begin{aligned}\Phi(x,y,z,t) =& \frac{1}{2}\left\{\pm\sqrt{-\frac{a\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\left(\frac{\mu\,\zeta+\sigma_1}{\frac{1}{2}\mu\,\zeta^2+\sigma_1\zeta+\sigma_2}\right)\right.\\ &\left.\pm\sqrt{\frac{a\left(c^2+k^2+m^2\right)\left(2\,\mu\,\sigma_2-\sigma_1^2\right)}{pb_1+qb_2+rb_3}}\left(\frac{1}{\frac{1}{2}\mu\,\zeta^2+\sigma_1\zeta+\sigma_2}\right)\right\}\\ &\times \exp\left(ix+qy+rz-b_0\left(p^2+q^2+r^3\right)t\right),\end{aligned} \tag{3.15}$$

where $\zeta = cx+ky+mz-2b_0\left(cp+kq+mr\right)t$.

3.2. Application of the IGTCM

Using $M=1$ and the solution form (2.8), we can write the solution of Eq. (3.3) as

$$u(\zeta) = \alpha_0 + \alpha_1\phi(\zeta) + \alpha_2(\phi(\zeta))^{-1}, \tag{3.16}$$

where α_0, α_1 and α_2 are constants which will be determined at a later step and where the function $\phi(\zeta)$ satisfies Eq. (2.9). Next, we substitute Eqs. (3.16) and (2.9) into Eq. (3.3) to obtain a polynomial in $\phi^j(\zeta)$ $(j=-3,-2,-1,0,1,2,3)$. After setting all coefficients of $\phi^j(\zeta)$ to zero, we obtain the following algebraic equations.

$$\begin{aligned}\phi^{-3}(\zeta)\ :\ & 2b_0c^2\rho^2\alpha_2 + 2b_0k^2\rho^2\alpha_2 + 2b_0m^2\rho^2\alpha_2 + 2p\alpha_2^3b_1 + 2q\alpha_2^3b_2 + 2r\alpha_2^3b_3 = 0,\\ \phi^{-2}(\zeta)\ :\ & 3b_0c^2\rho\tau\alpha_2 + 3b_0k^2\rho\tau\alpha_{-1} + 3b_0m^2\rho\tau\alpha_{-1} + 6p\alpha_2^2\alpha_0b_1 + 6q\alpha_2^2\alpha_0b_2\\ & + 6r\alpha_2^2\alpha_0b_3 = 0,\\ \phi^{-1}(\zeta)\ :\ & 2b_0c^2\rho\sigma\alpha_2 + b_0c^2\tau^2\alpha_2 + 2b_0k^2\rho\sigma\alpha_2 + b_0k^2\tau^2\alpha_{-1} + 2b_0m^2\rho\sigma\alpha_2 - \omega\alpha_2\\ & + b_0m^2\tau^2\alpha_{-1} + 6p\alpha_2^2\alpha_1b_1 + 6p\alpha_2\alpha_0^2b_1 + 6q\alpha_2^2\alpha_1b_2 + 6q\alpha_2\alpha_0^2b_2\\ & + 6r\alpha_2^2\alpha_1b_3 + 6r\alpha_2\alpha_0^2b_3 - b_0p^2\alpha_2 - b_0q^2\alpha_2 - b_0r^2\alpha_2 = 0,\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\phi^0(\zeta) \ :\ & b_0c^2\rho\tau\alpha_1 + b_0c^2\sigma\tau\alpha_2 + b_0k^2\rho\tau\alpha_1 + b_0k^2\sigma\tau\alpha_2 + b_0m^2\rho\tau\alpha_1 + b_0m^2\sigma\tau\alpha_2 \\
& + 12p\alpha_2\alpha_0\alpha_1b_1 + 2p\alpha_0^3b_1 + 12q\alpha_2\alpha_0\alpha_1b_2 + 2q\alpha_0^3b_2 + 12r\alpha_2\alpha_0\alpha_1b_3 \\
& + 2r\alpha_0^3b_3 - b_0p^2\alpha_0 - aq^2\alpha_0 - b_0r^2\alpha_0 - \omega\alpha_0 = 0, \\
\phi^1(\zeta) \ :\ & 2b_0c^2\rho\sigma\alpha_1 + b_0c^2\tau^2\alpha_1 + 2b_0k^2\rho\sigma\alpha_1 + b_0k^2\tau^2\alpha_1 + 2b_0m^2\rho\sigma\alpha_1 + b_0m^2\tau^2\alpha_1 \\
& + 6p\alpha_2\alpha_1^2b_1 + 6p\alpha_0^2\alpha_1b_1 + 6q\alpha_2\alpha_1^2b_2 + 6q{\alpha_0}^2\alpha_1b_2 + 6r\alpha_2\alpha_1^2b_3 + 6r\alpha_0^2\alpha_1b_3 \\
& - b_0p^2\alpha_1 - b_0q^2\alpha_1 - b_0r^2\alpha_1 - \omega\alpha_1 = 0, \\
\phi^2(\zeta) \ :\ & 3b_0c^2\sigma\tau\alpha_1 + 3b_0k^2\sigma\tau\alpha_1 + 3b_0m^2\sigma\tau\alpha_1 + 6p\alpha_0\alpha_1^2b_1 + 6q\alpha_0\alpha_1^2b_2 + 6r\alpha_0\alpha_1^2b_3 = 0, \\
\phi^3(\zeta) \ :\ & 2b_0c^2\sigma^2\alpha_1 + 2b_0k^2\sigma^2\alpha_1 + 2b_0m^2\sigma^2\alpha_1 + 2p\alpha_1^3b_1 + 2q\alpha_1^3b_2 + 2r\alpha_1^3b_3 = 0.
\end{aligned}$$

Then, solving system (3.17) using the symbolic package Maple and the equation $\zeta = cx + ky - 2a\,(cp + kq + mr)\,t$, we obtain the following two results.

Result 1:

$$\begin{aligned}
&\alpha_0 = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^3\right)}{4(pb_1+qb_2+rb_3)}}\tau,\ \alpha_1 = 0,\ \alpha_2 = \frac{2\alpha_0\rho}{\tau}, \\
&\omega = 2b_0\rho\left(c^2+k^2+m^2\right)\sigma - \frac{b_0}{2}\left(\left(c^2+k^2+m^2\right)\tau^2 + 2\left(p^2+q^2+r^2\right)\right),
\end{aligned} \tag{3.18}$$

where ρ, τ, σ, c, k, m, p, q, r, b_0, b_1, b_2, and b_3 are arbitrary constants such that $\alpha_0 \in \mathbb{R}$.

Case 1: If $\rho = 0$, then the exact solutions of Eq. (1.3) are

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^3\right)}{4(pb_1+qb_2+rb_3)}}\tau\exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.19}$$

Case 2: If $\tau = 0$, then the exact solutions of Eq. (1.3) are

$$\begin{aligned}
\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\sqrt{\rho\,\sigma}\cot\left(\sqrt{\rho\,\sigma}\zeta\right) \\
\times\exp\left(i\left(px+qy+mz+\omega t\right)\right),
\end{aligned} \tag{3.20}$$

where $\rho\sigma > 0$ and

$$\begin{aligned}
\Phi(x,y,z,t) = \pm\sqrt{\frac{-b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\sqrt{-\rho\,\sigma}\coth\left(\sqrt{-\rho\,\sigma}\zeta\right) \\
\times\exp\left(i\left(px+qy+mz+\omega t\right)\right),
\end{aligned} \tag{3.21}$$

where $\rho\sigma < 0$.

Case 3: If $\sigma = 0$, then the exact solutions of Eq. (1.3) are

$$\begin{aligned}
\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\tau\left(\frac{\rho}{\tau\exp(\tau\zeta)-\rho}+\frac{1}{2}\right) \\
\times\exp\left(i\left(px+qy+mz+\omega t\right)\right).
\end{aligned} \tag{3.22}$$

Case 4: If $\rho = \tau = 0$, then we obtain a trivial solution $\Phi(x,y,z,t) = 0$.

Case 5: When $\tau^2 = 4\sigma\rho \neq 0$, then we obtain the exact solutions of Eq. (1.3) as

$$\Phi(x,y,z,t) = \pm \frac{1}{2}\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}\tau}\left(\frac{\tau\zeta}{\tau\zeta+2}-1\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.23}$$

Case 6: If $\tau^2 < 4\rho\sigma$ and $\sigma \neq 0$, then we obtain the exact solutions of Eq. (1.3) as

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\left(\frac{2\rho\sigma}{\sqrt{4\rho\sigma-\tau^2}\tan\left(\frac{1}{2}\sqrt{4\rho\sigma-\tau^2}\zeta\right)-\tau}+\frac{\tau}{2}\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.24}$$

Case 7: If $\tau^2 > 4\rho\sigma$ and $\sigma \neq 0$, then the exact solutions of Eq. (1.3) are

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\left(\frac{2\rho\sigma}{\sqrt{\tau^2-4\rho\sigma}\tanh\left(\frac{1}{2}\sqrt{\tau^2-4\rho\sigma}\zeta\right)-\tau}+\frac{\tau}{2}\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.25}$$

Result 2:

$$\begin{aligned} \alpha_0 &= \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{4pb_1+4qb_2+4rb_3}}\tau,\ \alpha_1 = \frac{2\alpha_0\sigma}{\tau},\ \alpha_2 = 0, \\ \omega &= \left(2\rho\left(c^2+k^2+m^2\right)\sigma - \frac{1}{2}\left(c^2+k^2+m^2\right)\tau^2 - \left(p^2+q^2+r^2\right)\right)b_0, \end{aligned} \tag{3.26}$$

where ρ, τ, σ, c, k, m, p, q, r, b_0, b_1, b_2, and b_3 are arbitrary constants such that $\alpha_0 \in \mathbb{R}$.

Case 1: If $\rho = 0$, then the exact solutions of Eq. (1.3) are

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\tau\left(\frac{1}{2}+\frac{\sigma}{\left(-\sigma+\tau e^{-\tau\zeta}\right)}\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.27}$$

Case 2: If $\tau = 0$, then the exact solutions of Eq. (1.3) are

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\sqrt{\rho\sigma}\tan\left(\sqrt{\rho\sigma}\zeta\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right), \tag{3.28}$$

where $\rho\sigma > 0$ and

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{pb_1+qb_2+rb_3}}\sqrt{-\rho\sigma}\tanh\left(\sqrt{-\rho\sigma}\zeta\right) \times \exp\left(i\left(px+qy+mz+\omega t\right)\right), \tag{3.29}$$

where $\rho\sigma < 0$.

Case 3: If $\sigma = 0$, then we obtain the exact solutions of Eq. (1.3) as

$$\Phi(x,y,z,t) = \pm\sqrt{-\frac{b_0\left(c^2+k^2+m^2\right)}{4pb_1+4qb_2+4rb_3}}\tau\exp\left(i\left(px+qy+mz+\omega t\right)\right). \tag{3.30}$$

Case 4: If $\rho = \tau = 0$, then we obtain the exact solutions of Eq. (1.3) as

$$\Phi(x, y, z, t) = \mp \sqrt{-\frac{b_0 \left(c^2 + k^2 + m^2\right)}{pb_1 + qb_2 + rb_3}} \left(\frac{1}{\zeta}\right) \exp\left(i\left(px + qy + mz + \omega t\right)\right). \tag{3.31}$$

Case 5: If $\tau^2 = \sigma\rho \neq 0$, then the exact solutions of Eq. (1.3) are

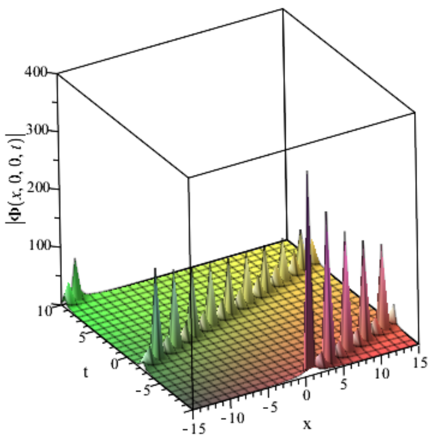
$$\Phi(x, y, z, t) = \pm \sqrt{-\frac{b_0 \left(c^2 + k^2 + m^2\right)}{4(pb_1 + qb_2 + rb_3)}} \left(\tau - \frac{\tau\zeta + 2}{\zeta}\right) \times \exp\left(i\left(px + qy + mz + \omega t\right)\right). \tag{3.32}$$

Case 6: If $\tau^2 < 4\rho\sigma$ and $\sigma \neq 0$, then the exact solutions of Eq. (1.3) are

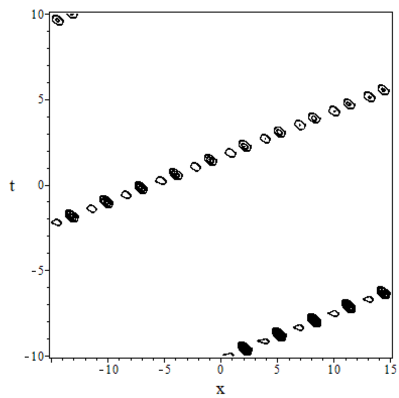
$$\Phi(x, y, z, t) = \pm \sqrt{-\frac{b_0 \left(c^2 + k^2 + m^2\right)}{4(pb_1 + qb_2 + rb_3)}} \left(\sqrt{4\rho\sigma - \tau^2} \tan\left(\frac{1}{2}\sqrt{4\rho\sigma - \tau^2}\zeta\right)\right) \times \exp\left(i\left(px + qy + mz + \omega t\right)\right). \tag{3.33}$$

Case 7: If $\tau^2 > 4\rho\sigma$ and $\sigma \neq 0$, then the exact solutions of Eq. (1.3) are

$$\Phi(x, y, z, t) = \pm \sqrt{-\frac{b_0 \left(c^2 + k^2 + m^2\right)}{4(pb_1 + qb_2 + rb_3)}} \left(\sqrt{\tau^2 - 4\rho\sigma} \tanh\left(\frac{1}{2}\sqrt{\tau^2 - 4\rho\sigma}\zeta\right)\right) \times \exp\left(i\left(px + qy + mz + \omega t\right)\right). \tag{3.34}$$



(a)



(b)

FIGURE 1. Graphs of $|\Phi(x, 0, 0, t)|$ for $\Phi(x, y, z, t)$ in (3.12): (a) 3D plot. (b) Contour plot.

Next, we demonstrate graphs of some exact solutions of Eq. (1.3) obtained using the two methods. In particular, fixing $y = 0$, $z = 0$ and using the domain $-15 \leq x \leq 15$ and $-10 \leq t \leq 10$, we obtain the graphs in Figures 1 (a) and (b) which show the 3D and contour graphs of the magnitude of $\Phi(x, y, z, t)$ in (3.12), respectively, when $\eta = 2$, $\mu = 0.25$, $\sigma_1 = 1$, $\sigma_2 = 1$, $c = 0.1$, $k = 0.5$, $m = 0.3$, $p = 0.1$, $q = 0.15$, $r = 0.5$, $b_0 = 0.8$, $b_1 = 0.5$, $b_2 = 0.5$, $b_3 = 0.5$ are used for the computation. The 3D solution graph in Figures 1

(a) displays the behavior of a singularly periodic traveling wave solution. The effects of varying the parameter values of b_0, b_1, b_2, b_3 on the behavior of the solutions of (3.12) for the remaining parameter values mentioned above are shown in Figure 2. In particular, the effect of varying the value of b_0 on $|\Phi(0,0,0,t)|$ for $-10 \leq t \leq 10$ can be seen in Figure 2 (a) for $b_0 = 0.8,\ 0.5,\ 0.2$. The most important effect of changing the values of b_0 is that the singular point of the solution is moved. The effects of b_1, b_2, b_3 on $|\Phi(0,0,0,t)|$ as t is varied can be seen in Figures 2 (b), (c) and (d) in which the values of b_1, b_2, b_3 are $\{0.5,\ 0.3,\ 0.1\}$. It is worth noting that, unlike b_0, varying the values of b_1, b_2, b_3 does not affect the position of the singular point of the solution (3.12). For example, Figure 2 (b) shows the 2D relationship between $|\Phi(0,0,0,t)|$ and t for $b_1 = 0.5,\ 0.3,\ 0.1$. It can be seen from the figure that for all values of b_1, all the curves have the same structure but the values of $|\Phi(0,0,0,t)|$ at a specific time t are very slightly different for each curve whereas the position of the singular point is the same for each curve. Similar behavior can be seen in Figures 2 (c) and (d) when the parameters b_2 and b_3 are varied, respectively.

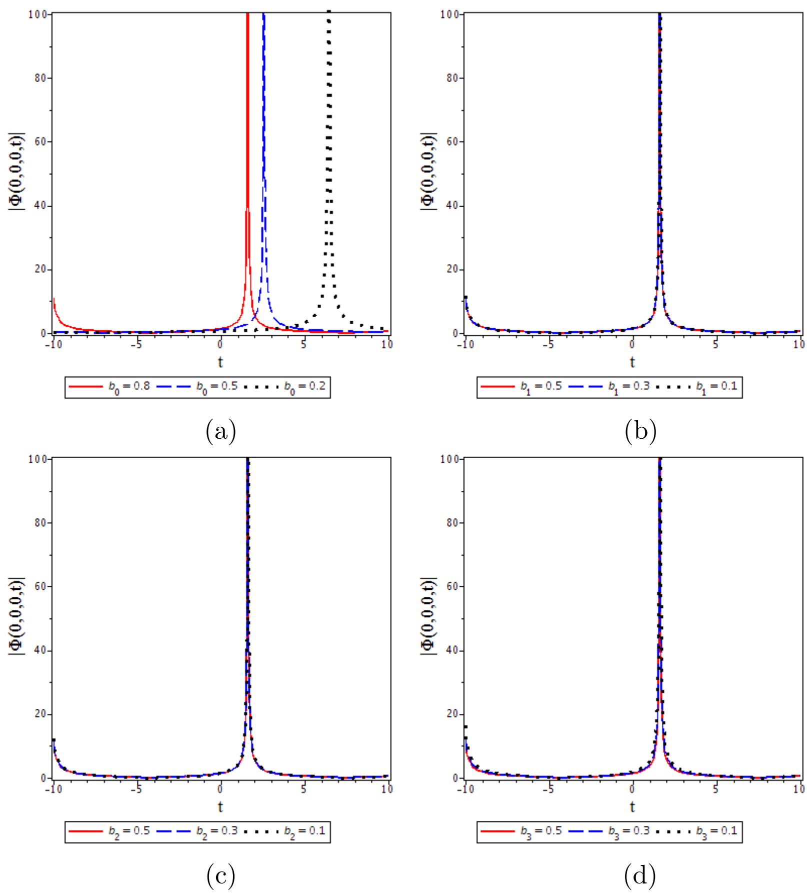
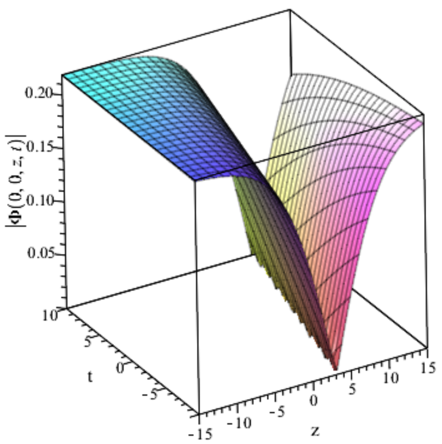
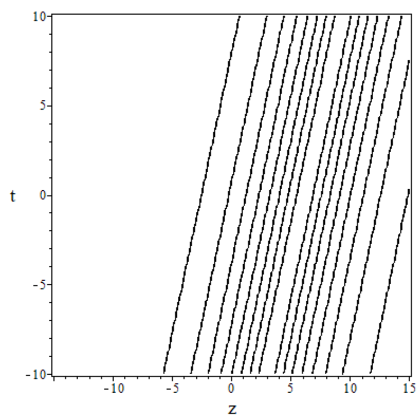


FIGURE 2. Graphs of $|\Phi(0,0,0,t)|$ for $\Phi(x,y,z,t)$ in (3.12): (a) $b_0 = 0.8,\ 0.5,\ 0.2$. (b) $b_1 = 0.5,\ 0.3,\ 0.1$. (c) $b_2 = 0.5,\ 0.3,\ 0.1$. (d) $b_3 = 0.5,\ 0.3,\ 0.1$.

Figures 3 (a) and (b) show the 3D and contour graphs of the magnitude of $\Phi(x, y, z, t)$ in (3.25), respectively, when $x = 0$, $y = 0$, $\rho = 0.75$, $\tau = 0.85$, $\sigma = 0.1$, $c = 0.1$, $k = 0.5$, $m = 0.5$, $p = 0.25$, $q = 0.25$, $r = 0.5$, $b_0 = 0.2$, $b_1 = 0.1$, $b_2 = 0.3$, $b_3 = 0.25$ are used for the simulation on the domain $-15 \leq z \leq 15$ and $-10 \leq t \leq 10$. Since solution (3.25) can be expressed in terms of a hyperbolic cotangent function, then its magnitude portrayed as the 3D graph in Figures 3 (a) shows the behavior of a singular soliton solution. The effects of varying the parameter values of b_0, b_1, b_2, b_3 on the behavior of the solutions of (3.25) for the values of the remaining parameters given above are shown in Figure 4. In particular, the effect of changing the values of b_0 on $|\Phi(0, 0, 0, t)|$ when $-10 \leq t \leq 10$ can be seen in Figure 4 (a) for $b_0 = 0.2, 0.5, 0.8$. The most important effect of changing the value of b_0 on the solutions is that the cusp point of the solution graph is translated. Figures 4 (b), (c) and (d) show the effects of varying the values of b_1, b_2, b_3 on the relationship between $|\Phi(0, 0, 0, t)|$ and t. The values of b_1, b_2 and b_3 used are $\{0.1, 0.4, 0.7\}$, $\{0.3, 0.6, 0.9\}$ and $\{0.25, 0.55, 0.85\}$ for Figures 4 (b), (c) and (d), respectively. It is worth noting that, unlike b_0, the solutions (3.25) for b_1, b_2, b_3 do not have cusp points. Figure 4 (b) shows the 2D relationship between $|\Phi(0, 0, 0, t)|$ and t for $b_1 = 0.1, 0.4, 0.7$. In this case, varying the values of b_1 only makes a change in the value of $|\Phi(0, 0, 0, t)|$ at each value of t. Similar behavior can be seen in Figures 4 (c) and (d) when the parameters b_2 and b_3 are varied, respectively.



(a)



(b)

FIGURE 3. Graphs of $|\Phi(0, 0, z, t)|$ for $\Phi(x, y, z, t)$ in (3.25): (a) 3D plot. (b) Contour plot.

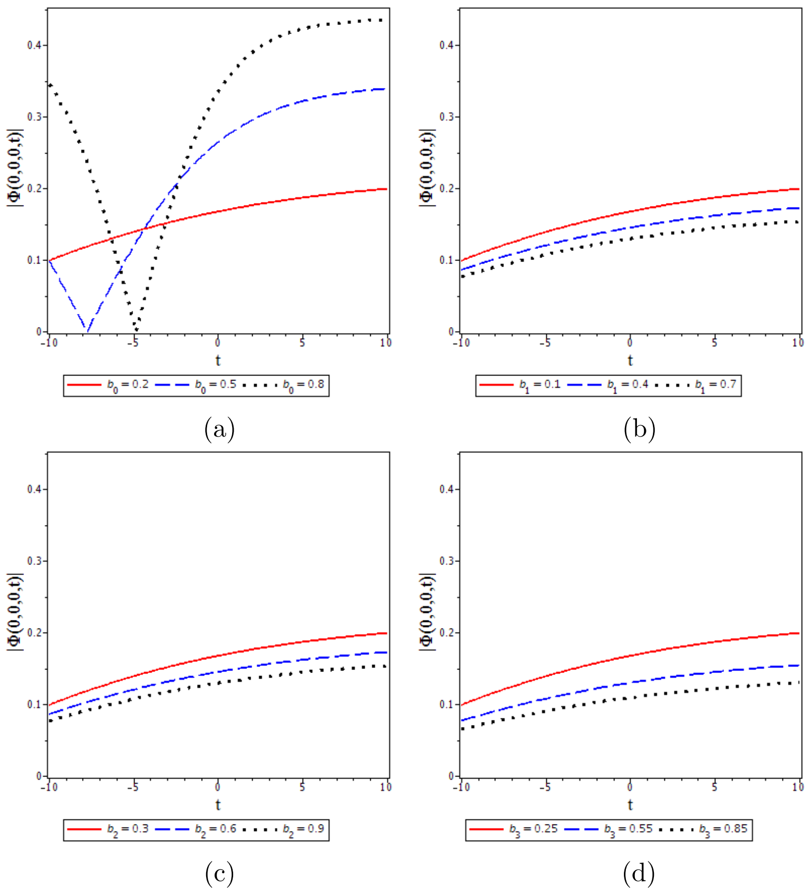


FIGURE 4. Graphs of $|\Phi(0,0,0,t)|$ for $\Phi(x,y,z,t)$ in (3.25): (a) $b_0 = 0.2, 0.5, 0.8$. (b) $b_1 = 0.1, 0.4, 0.7$. (c) $b_2 = 0.3, 0.6, 0.9$. (d) $b_3 = 0.25, 0.55, 0.85$.

4. Conclusions

In this article, the $(3+1)$-dimensional chiral nonlinear Schrödinger equation (1.3) has been analytically solved to get exact traveling wave solutions using the extended simplest equation method (ESEM) and the improved generalized tanh-coth method (IGTCM). Because the equation has complex-valued solutions, we have expressed the exact solutions in the form shown in (3.1) as a product of a real function $u(\zeta)$ and $e^{i\Theta}$. Using the ESEM, we obtained the exact solutions for (3.1) as traveling wave solutions in terms of hyperbolic, trigonometric, and rational functions. Similarly, using the IGTCM, we obtained exact solutions for (3.1) in terms of hyperbolic, trigonometric, and rational functions. For both methods, the algebraic manipulations required to obtain the exact solutions were carried out using the Maple software package. The 3D and contour plots of magnitudes of some solutions have been plotted using the Maple package to show their physical behavior.

In [7], the three different finite difference schemes including a nonlinear implicit scheme and two linearly implicit finite difference schemes were used to numerically solve equation (1.1) whose exact solution $\Phi(x,t)$ can be either a bright soliton solution expressed in terms of $\operatorname{sech}(\Omega_1(x,t))\exp(i\Omega_2(x,t))$ or a dark soliton solution written in terms of $\tanh(\Omega_1(x,t))\exp(i\Omega_2(x,t))$, where $\Omega_1(x,t)$ and $\Omega_2(x,t)$ are some traveling wave transformations. If equation (1.3) is reduced to the 1D-CNLSE, then our results in (3.29) and (3.34) can be reduced to the 1D solutions written as $\tanh(\Omega_1(x,t))\exp(i\Omega_2(x,t))$.

In [8], the author used the trial solution technique to find exact solutions of equation (1.2). The soliton solutions expressed in terms of the hyperbolic secant and cosecant functions and the singular periodic solutions written in terms of the secant and cosecant functions were found. Roughly comparing our results to the obtained solutions in [8], it can be easily done by setting $b_3 = 0$ and also ignoring the independent variable z in our solutions. Consequently, the solutions (3.29), (3.34) and (3.21), (3.25) can be converted into the hyperbolic secant function and the hyperbolic cosecant function, respectively. In addition, the solutions (3.28), (3.33) and (3.20), (3.24) can be rewritten as the secant function and the cosecant function, respectively. In [14], the authors employed the modern extended direct algebraic method to investigate exact solutions of equation (1.2). The solitary waves solutions of the equation were found such as semi-dark solitons, singular dark-pitch solitons, single solitons and intermixed hyperbolically, trigonometrically and rational solitons. When $b_3 = 0$ and the variable z disappears in (1.2), our solutions, namely, (3.9), (3.12), (3.20), (3.24), (3.28) and (3.33) are significantly similar to those in [14].

As far as the authors know, the present paper is the first time that the (3 + 1)-dimensional chiral nonlinear Schrödinger equation has been discussed in the literature and exact solutions obtained for it. The authors also believe that the ESEM and the IGTCM are powerful, straightforward and trustworthy approaches to generate solution of nonlinear partial differential equatons.

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