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On (f, g)-Semi-Derivations of Hyperrings

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Abstract In this paper, we introduce a generalization of semi-derivation on the Krasner hyperrings R. Specifically, we propose a new type of semi-derivation called (f, g)-semi-derivation, where f and g are mappings from R into itself. Our aim is to explore the properties of this new type of semi-derivation. Additionally, we conduct investigations some results on (f, g)-semi-derivations either on 2-torsion free prime hyperrings or on prime hyperrings.

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1. Introduction

Hyperstructures represent a natural extension of classical algebraic structures and the concept of hyperstructure was first introduced in 1934 by Marty [10]. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Krasner [9] introduced the notion of hyperrings. A well-known type of hyperring is called the Krasner hyperring. Krasner hyperring is an essential ring with approximately modified axioms in which addition is a hyperoperation and multiplication is a binary operation. Asokkumar [1] studied the idempotent elements of Krasner hyperrings.

Derivations is an interesting research area in the theory of algebraic structure in mathematics. Posner [11] initiated the study about derivations in rings and proved that in a prime ring of characteristic different from 2, if the iterate of two derivations is a derivation, then one of them must be zero. Based on this concept, Bell and Kappea [3] studied that rings in which derivations satisfy certain algebraic conditions. Moreover, several researchers have further studied this notion in rings and near rings. Xin et al.[14] studied the notion of a derivation, previously studied for rings, near rings and C*- algebras, for lattices and discussed some related properties. In [2], Asokkumar initiated the study of

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derivations on Krasner hyperrings. Then several authors investigated the relationships between derivations and the structure of hyperrings (see [7], [8], [10], [12], and [13]).

The concept of a semi-derivation on ring was introduced by Bergen [4]. Let f be a mapping of a ring R into R. An additive mapping D of a ring R into R is called a semi-derivation of R if $D(xy) = D(x) f(y) + xD(y) = D(x) y + f(x) D(y)$ and $D(f(x)) = f(D(x))$ for all $x, y \in R$. In [5], Bresar obtained the structure of semi-derivations of prime rings. Recently, Yilmaz and Yazarli [15] introduced a special type of derivation on hyperring, called semi-derivation and proved that the semi-derivation of prime Krasner hyperring is derivations.

In this paper, we introduce a generalization of the semi-derivation on a Krasner hyperring R, namely, the (f, g)-semi-derivation, where f and g are mappings from R into itself, and investigate some results involving these derivations. Moreover, we extend some results of Yilmaz and Yazarli [15] for the (f, g)-semi-derivation.

2. Preliminaries

Let us recall some definitions and concepts of hyperstructures which are used in the sequel. For details, we refer to Davvaz and Leoreanu-Fotea [6] and Asokummar [2]. For a set H, let $P(H)$ denote the power set of H, and $P^*(H) = P(H) - \{\phi\}$.

Definition 2.1. [6] A *hyperoperation* on a nonempty set H is a mapping $\circ : H \times H \to P^*(H)$. An algebraic system $(H, \circ)$ is called a *hypergroupoid.*

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets A and B of H, and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

and $A \circ x = A \circ \{x\}, x \circ B = \{x\} \circ B$.
A hypergroupoid $(H, \circ)$ is said to be *commutative* if $a \circ b = b \circ a$ for all $a, b \in H$.
A *semihypergroupoid* is a hypergroupoid $(H, \circ)$ such that $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c\ .$$

A *hypergroup* is a semihypergroupoid $(H, \circ)$ such that $a \circ H = H = H \circ a$ for all $a \in H$.

Definition 2.2. [6] A hypergroup $(H, \circ)$ is called a *canonical hypergroup* if

(i) $(H, \circ)$ is commutative,
(ii) $(H, \circ)$ has a scalar identity, which means that there is an element $e \in H$ such that $e \circ x = \{x\}$ for all $x \in H$,
(iii) every element x of H has a unique inverse, which means that for all $x \in H$, there exists a unique x^{-1} in H such that $e \in x \circ x^{-1}$,
(iv) if $x \in y \circ z$, then there exist the inverse y^{-1} of y and z^{-1} of z, such that $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$

Note that a scalar identity is unique since if e and e' are scalar identities of a hypergroupoid $(H, \circ)$, then $\{e\} = e \circ e' = \{e'\}$, so that $e = e'$.

Definition 2.3. [6] A *Krasner hyperring* is an algebraic structure $(R,+,\cdot)$ which satisfies the following axioms:

(1) $(R,+)$ is a canonical hypergroup, that is,
  (i) $(x+y)+z = x+(y+z)$ for every $x,y,z \in R$,
  (ii) $x+y = y+x$ for every $x,y \in R$,
  (iii) there exists $0 \in R$ such that $0+x = x$, for every $x \in R$,
  (iv) for every $x \in R$, there exists a unique element, denoted by $-x \in R$ such that $0 \in x+(-x)$,
  (v) $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$, for every $x,y,z \in R$.
(2) $(R,\cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is,
  (i) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ for every $x,y,z \in R$,
  (ii) $x\cdot 0 = 0\cdot x = 0$ for every $x \in R$.
(3) The multiplication is distributive with respect to the hyperoperation $+$, that is, $x\cdot(y+z) = x\cdot y + x\cdot z$ and $(x+y)\cdot z = x\cdot z + y\cdot z$ for every $x,y,z \in R$.

Let $(R,+,\cdot)$ be a Krasner hyperring. For nonempty subset A of R, let $-A = \{-a|\ a \in A\}$. The following elementary facts follow easily from the axioms:

(i) $-(-x) = x$ for all $x \in R$,
(ii) $-(x+y) = -x-y$ for all $x,y \in R$,
(iii) $-(x\cdot y) = (-x)\cdot y = x\cdot(-y)$ for all $x,y \in R$,
(iv) $(a+b)\cdot(c+d) = a\cdot c + b\cdot c + a\cdot d + b\cdot d$ for all $a,b,c,d \in R$.

In Definition 2.3, for simplicity of notations we write sometimes xy instead of $x\cdot y$ and in (1, iii), $0+x=x$ instead of $0+x=\{x\}$.

Throughout this paper, by a hyperring we mean a Krasner hyperring.
Let A and B be nonempty subsets of a hyperring R and $a,b \in R$. Let

$$A+B = \{x — x = a+b \text{ for some } a \in A, b \in B\},$$

$$AB = \{x — x = \sum_{i=1}^{n} a_i b_i \text{ for some } a_i \in A, b_i \in B \text{ and } n \in \mathbb{Z}^+\},$$

and

$$aRb = \{x — x = ary, \text{ for all } r \in R\}.$$

Definition 2.4. [15] A hyperring R is called a *prime hyperring* if for any $a,b \in R$, $aRb = \{0\}$ implies $a=0$ or $b=0$.
A hyperring R is said to be 2-torsion free if for any $x \in R$, $0 \in x+x$ implies $x=0$.
The *center* of a hyperring R is the set $Z(R) = \{z \in R — zx = xz \text{ for all } x \in R\}$.

Lemma 2.5. [15] *Let R be a hyperring. For any $r,s \in R$, the symbol $[r,s]$ represents for the commutator $rs-sr$ and the symbol (r,s) represents for the skew commutator $rs+sr$. The following conditions hold: for all $r,s,t \in R$,*

(i) $[r+s,t] = [r,t]+[s,t]$,
(ii) $[rs,t] \subseteq [r,t]s + r[s,t] = r[s,t]+[r,t]s$,
(iii) $(r+s,t) = (r,t)+(s,t)$,
(iv) $(rs,t) \subseteq (r,t)s + r[s,t] = r(s,t) - r,ts$.

Definition 2.6. [15] Let R be a hyperring. A mapping $D : R \to R$ is said to be a *semi-derivation* of R associated with a function $f : R \to R$ if for all $x,y \in R$,

(i) $D(x+y) \subseteq D(x) + D(y)$,
(ii) $D(xy) \in D(x) f(y) + xD(y) = D(x) y + f(x) D(y)$,
(iii) $D(f(x)) \subseteq f(D(x))$.

3. Main Results

We introduce a generalization of the semi-derivation on a hyperring as follows:

Definition 3.1. Let R be a hyperring and let $f, g : R \to R$ be mappings. A mapping $D : R \to R$ is said to be an (f, g)-*semi-derivation* of R if for all $x, y \in R$,

(i) $D(x+y) \subseteq D(x) + D(y)$,
(ii) $D(xy) \in D(x) f(y) + g(x) D(y) = D(x) g(y) + f(x) D(y)$,
(iii) $D(f(x)) \subseteq f(D(x))$ and $D(g(x)) \subseteq g(D(x))$.

Obviously, every semi-derivation is a (f, g)-semi-derivation, where $g : R \to R$ is the identity mapping.

Example 3.2. Let $R = \{0, u, v\}$ with the hyperoperation $(+)$ and the multiplication $(\cdot)$ given in the following tables:

$+$	0	u	v
0	0	u	v
u	u	$\{u, v\}$	R
v	v	R	$\{u, v\}$

$\cdot$	0	u	v
0	0	0	0
u	0	v	u
v	0	u	v

Then, R is a hyperring.
Define a mapping $D : R \to R$ by

$$D(x) = \begin{cases} 0 & ; \quad x = 0 \\ u & ; \quad x = v \\ v & ; \quad x = u \end{cases}$$

Let $f, g : R \to R$ be mappings such that $f = g = D$.
Then it can be easily verified that D is an (f, g)-semi-derivation of the hyperring R.

From now on, let R denote a hyperring and $f, g : R \to R$ be mappings.

Lemma 3.3. *Let R be a hyperring and D be an (f, g)-semi-derivation of R. If $f(0) = g(0) = 0$ then the following conditions hold:*

(i) $D(0) = 0$,
(ii) $D(-x) = -D(x)$ *for all* $x \in R$.

Proof. Let $f(0) = g(0) = 0$,

(i) $D(0) = D(0 \cdot 0) \in D(0) f(0) + g(0) D(0) = 0 + 0 = 0$. Then $D(0) = 0$.
(ii) Let $x \in R$. Then there exists $-x \in R$ such that $0 \in x + (-x)$.
Hence $0 = D(0) \in D(x + (-x)) \subseteq D(x) + D(-x)$. But $0 \in D(x) - D(x)$.
Since the inverse of an element is unique in a canonical hypergroup,
$-D(x) = D(-x)$. ∎

Lemma 3.4. *Let R be a prime hyperring and let D be an (f, g)-semi-derivation of R. Suppose that g is surjective and $r \in R$. If $rD(x) = 0$ (or $D(x)r = 0$) for all $x \in R$ then $r = 0$ or $D(x) = 0$ for all $x \in R$.*

Proof. Let $rD(x) = 0$ for all $x \in R$ and let $s, t \in R$. Then

$$\begin{aligned} 0 &= rD(st) &&\in r(D(s)f(t) + g(s)D(t)) \\ &&&= rD(s)f(t) + rg(s)D(t) \\ &&&= rg(s)D(t). \end{aligned}$$

Since g is surjective, $rRD(t) = \{0\}$ for all $t \in R$. By the primeness of R, we have $r = 0$ or $D(t) = 0$ for all $t \in R$. When $D(x)r = 0$ for all $x \in R$, the proof is similar. ∎

Theorem 3.5. *Let R be a 2-torsion free prime hyperring. Suppose that $f, g : R \to R$ are surjective mappings and D is an (f, g)-semi-derivation of R. If $D^2(x) = 0$ for all $x \in R$, then $D(x) = 0$ for all $x \in R$.*

Proof. Let $D^2(x) = 0$ for all $x \in R$. Suppose that $D(x) \neq 0$ for some $x \in R$. Then there exists an element $a \in R$ such that $D(a) \neq 0$. Since g is surjective, there exists $r \in R$ such that $g(r) = a$.
Then for any $y \in R$,

$$\begin{aligned} 0 &= D^2(ry) = D(D(ry)) \\ &\in D(D(r)f(y) + g(r)D(y)) \\ &\subseteq D(D(r)f(y)) + D(g(r)D(y)) \\ &\in 0 + D(g(r))D(f(y)) + D(g(r))D(f(y)) + 0 \\ &= D(a)D(f(y)) + D(a)D(f(y)). \end{aligned}$$

Since R is 2-torsion free, $D(a)D(f(y)) = 0$ for all $y \in R$.
Since f is surjective, $D(a)D(x) = 0$ for all $x \in R$.
Since $D(a) \neq 0$ and by Lemma 3.4, $D(x) = 0$ for all $x \in R$. This completes the proof. ∎

Theorem 3.6. *Let R be a 2-torsion free prime hyperring. For $i = 1, 2$, let $f_i, g_i : R \to R$ be surjective mappings and g_1 be an injective mapping such that $g_1(0) = 0$. Suppose that $D_i : R \to R$ is an (f_i, g_i)-semi-derivation of R, for $i = 1, 2$. If $D_1(D_2(x)) = 0$ for all $x \in R$ then $D_1(x) = 0$ for all $x \in R$ or $D_2(x) = 0$ for all $x \in R$.*

Proof. Let $D_1(D_2(x)) = 0$ for all $x \in R$ and let $x, y \in R$. Then

$$\begin{aligned} 0 &= D_1(D_2(xy)) \\ &\in D_1(D_2(x)f_2(y) + g_2(x)D_2(y)) \\ &\subseteq D_1(D_2(x)f_2(y)) + D_1(g_2(x)D_2(y)) \\ &\subseteq g_1(D_2(x))D_1(f_2(y)) + D_1(g_2(x))f_1(D_2(y)). \end{aligned}$$

Therefore,

$$0 \in g_1(D_2(x))D_1(f_2(y)) + D_1(g_2(x))f_1(D_2(y)) \text{ for all } x, y \in R.$$

Replacing x by $D_2(x)$, we get $0 \in g_1\left(D_2^2(x)\right)D_1(f_2(y))$ for all $x, y \in R$.
Hence $g_1\left(D_2^2(x)\right)D_1(f_2(y)) = 0$ for all $x, y \in R$.
Since f_2 is surjective, $g_1\left(D_2^2(x)\right)D_1(y) = 0$ for all $x, y \in R$.
By Lemma 3.4, we get $g_1\left(D_2^2(x)\right) = 0$ for all $x \in R$ or $D_1(y) = 0$ for all $y \in R$.
If $g_1\left(D_2^2(x)\right) = 0$ for all $x \in R$, then, by g_1 is injective, we get $D_2^2(x) = 0$ for all $x \in R$.
By Theorem 3.5, we have $D_2(x) = 0$ for all $x \in R$. This completes the proof. ∎

Corollary 3.7. *Let R be a 2-torsion free prime hyperring. Suppose that $f, g : R \to R$ are surjective mappings and g is an injective mapping such that $g_1(0) = 0$. If D_1, D_2 are (f, g)-semi-derivations of R such that $D_1(D_2(x)) = 0$ for all $x \in R$ then $D_1(x) = 0$ for all $x \in R$ or $D_2(x) = 0$ for all $x \in R$.*

Theorem 3.8. *Let R be a 2-torsion free prime hyperring. Let f and g be surjective mappings and $D : R \to R$ be an (f, g)-semi-derivation of R. Suppose that $r \in R - Z(R)$ and $[D(x), r] = \{0\}$ for all $x \in R$. Then $D(x) = 0$ for all $x \in R$.*

Proof. Let $r \in R - Z(R)$ be such that $[D(x), r] = \{0\}$ for all $x \in R$. Then, for $x, y \in R$, we have

$$\begin{aligned} \{0\} &= [D(yD(x)), r] \subseteq \left[D(y) f(D(x)) + g(y) D^2(x),\ r\right] \\ &= [D(y) f(D(x)), r] + \left[g(y) D^2(x), r\right] \\ &\subseteq [g(y), r] D^2(x)\ . \end{aligned}$$

Since g is surjective, $0 \in [z, r] D^2(x)$ for all $x, z \in R$. This implies $0 = sD^2(x)$ for some $s \in [r, z]$. By Lemma 3.4, we have $s = 0$ or $D^2(x) = 0$ for all $x \in R$. If $s = 0$, then $0 \in [r, z]$ for all $z \in R$. This implies $r \in Z(R)$, a contradiction. Therefore, $D^2(x) = 0$ for all $x \in R$. By Lemma 3.5, we obtain $D(x) = 0$ for all $x \in R$. ■

Theorem 3.9. *Let R be a prime hyperring and D be an (f, g)-semi-derivation of R. Let $f, g :\ R \to R$ be surjective mappings and $f(0) = 0$. Suppose there exists $r \in R$ such that $r \notin Z(R)$ and $(D(x), r) = \{0\}$ for all $x \in R$. Then $D((x, r)) = \{0\}$ for all $x \in R$.*

Proof. Suppose there exists $r \in R$ such that $r \notin Z(R)$ and $(D(x), r) = \{0\}$ for all $x \in R$. Since f is surjective, there exists $q \in R$ such that $f(q) = r$. Then, for all $x \in R$,

$$\begin{aligned} \{0\} &= (D(xq), r) \\ &\subseteq (D(x) f(q) + g(x) D(q), r) \\ &= (D(x) f(q), r) + (g(x) D(q), r) \\ &\subseteq (D(x), r) f(q) + D(x) [f(q), r] + g(x) (D(q), r) - [g(x), r] D(q)\ . \end{aligned}$$

Thus, $0 \in [g(x), r] D(q)$ for all $x \in R$.
Since g is surjective, $0 \in [t, r] D(q)$ for all $t \in R$. For each $y \in R$, replacing t by ty and we get $0 \in [ty, r] D(q) \subseteq [t, r] yD(q) + t[y, r] D(q)$.
Thus, $0 \in [t, r] yD(q)$ for all $t, y \in R$.
This implies that, for any $t \in R$, $0 = sRD(q)$ for some $s \in [t, r]$.
The primeness of R implies that $s = 0$ or $D(q) = 0$. If $s = 0$, then $r \in Z(R)$ which is a contradiction. Hence $D(q) = 0$. Then, we have $D(r) = D(f(q)) = f(D(q)) = f(0) = 0$. Thus, for all $x \in R$,

$$\begin{aligned} D((x, r)) = D(xr + rx) &\subseteq D(xr) + D(rx) \\ &\in D(x) f(r) + g(x) D(r) + D(r) g(x) + f(r) D(x) \\ &= D(x) f(r) + f(r) D(x) \\ &= (D(x), f(r)) \\ &= \{0\}. \end{aligned}$$

Therefore, $D((x, r)) = \{0\}$ for all $x \in R$. ■

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