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Two-parameter Taxicab Trigonometric Identities

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Abstract The metric on $\mathbb{R}^2$ defined by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ is known as the ℓ^1 or the taxicab metric. Delp and Filipski define and provide explicit formulas for sine and cosine functions for the taxicab space. Their version agrees with the right-triangle definition of the standard trigonometric functions. In particular, the sine (cosine) of an acute angle in a right triangle is equal to the ratio of the length of its opposite (adjacent) side and the length of the hypotenuse. These functions must have two parameters because a general rotation is not an isometry in the taxicab metric. We derive new identities for the taxicab sine and cosine functions. Specifically, we derive the Pythagorean, angle sum, double-angle, half-angle, and negative-angle identities. Additionally, we derive derivative identities for the taxicab tangent, secant, cotangent, and cosecant functions. We find that the derivatives of these functions behave similarly to their Euclidean counterparts.

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1. Introduction

The metric on $\mathbb{R}^2$ defined by $d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ is known as the ℓ^1 or the taxicab metric, and $(\mathbb{R}^2, d_T)$ is known as the taxicab space. It is also the metric generated by the ℓ^1 norm, $\|(x_1, x_2)\| = |x_1| + |x_2|$, on $\mathbb{R}^2$ as a normed vector space.

In the taxicab space, if Γ is the graph of a monotone increasing or decreasing function f over an interval $[a, b]$, then the length of Γ is equal to $(b-a)+|f(b)-f(a)|$ [6, Theorem 2.1]. In other words, the length of path from $(a, f(a))$ to $(b, f(b))$ is independent of the function f under the above conditions. It follows that in the taxicab space, shortest paths are not unique. In fact, there are infinitely many shortest paths between any two points unless they have the same x- or y-coordinates. Therefore, in this article, we shall define *lines* using vector space properties of $\mathbb{R}^2$ instead. We define a *line* as a 1-dimensional affine subspace of $\mathbb{R}^2$, i.e., a line is the set $\{t\vec{v} + \vec{w} : t \in \mathbb{R}\}$ for fixed vectors $\vec{v}$ and $\vec{w}$. Related geometric terms such as *rays*, *angles*, *line segments*, and *triangles* are defined

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using this definition of "lines." The length of a line segment is equal to the distance between the endpoints.

We can measure an angle in the taxicab space in terms of a subtended arc of a circle of radius r centered at the origin, which is given by the equation $|x| + |y| = r$. We call this angle measure *t-radians*. This notion has been defined and used in previous works including [2] and [5].

Definition 1.1. Let A and B be points on the unit circle of radius r centered at the origin. Let s be the length of the subtended arc of the unit circle from A to B. The *t-radian* measure, θ, of the taxicab angle $\angle AOB$ is given by

$$\theta = \frac{s}{r}.$$

It follows from the above definition that a taxicab circle has 8 t-radians, and a semicircle has 4 t-radians. Thompson and Dray [5, Lemma 2.5] show that an angle has taxicab measure of 2 t-radians if, and only if, it has Euclidean measure of $\frac{\pi}{2}$. In other words, two lines are perpendicular in the taxicab space whenever they are perpendicular in the Euclidean space.

Angles of other measure do not behave as nicely. For example, in Figure 1, angles θ and γ have the same taxicab measure of $\frac{1}{2}$ t-radians, but they have different Euclidean measure. This is because a general rotation is not an isometry on the taxicab space. The only isometric rotations are 0°, 90°, 180°, and 270° rotations. The full group of isometries, the semidirect product $D_4 \rtimes \mathbb{R}^2$, is generated by translations and the symmetries of a square [3, 4].

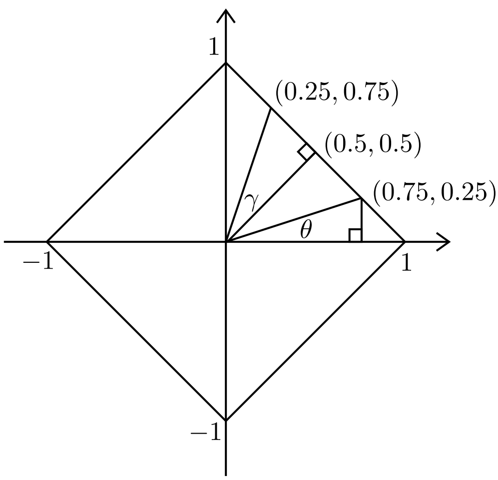

FIGURE 1. Angles θ and γ have the same taxicab measure of $\frac{1}{2}$ t-radians, but they have different Euclidean measure.

Delp and Filipski [2] define sine and cosine functions for the taxicab space as analogous to the right-triangle definitions of the standard sine and cosine functions. In particular, the (absolute value of) sine of an acute angle in a right triangle is defined as the ratio of the length of its opposite side and the length of the hypotenuse, and the (absolute value of) cosine is the ratio of the length of the adjacent side and the length of the hypotenuse.

However, as seen in Figure 1, the side ratios of the two right triangles are different even though angles θ and γ have the same taxicab measure. In the θ-triangle, the ratio of the length of the adjacent side to the length of the hypotenuse is less than 1, but in the γ-triangle, the side ratio is equal to 1. Delp and Filipski [2] show that such functions, therefore, must have two parameters. They also provide the explicit formulas and basic properties for the sine and cosine functions. Aka and Kaya [1] and Thompson and Dray [5] define and derive some identities for similar sine and cosine functions on the taxicab space. However, they have one parameter and only cover triangles where the bases are parallel to the x-axis.

In this article, we derive trigonometric identities for the two-parameter taxicab sine and cosine functions. Specifically, we derive the Pythagorean identity, angle sum identities, double-angle identities, half-angle identities, and negative-angle identities. Additionally, we derive derivative identities for the taxicab tangent, secant, cotangent, and cosecant functions. We find that the derivatives of these functions behave similarly to their Euclidean counterparts.

2. Two-parameter Taxicab Sine and Cosine Functions

In this section, we review the definition and basic facts of the taxicab sine and cosine functions from [2].

2.1. Definition and Explicit Formulas

Let O denote the origin and let $\triangle PRO$ be a right triangle where $\angle R$ is the right angle. We need two angle parameters to define sine and cosine in $\triangle PRO$. The first parameter, θ, measures $\angle POR$. The second parameter, ϕ, measures the angle from the x-axis to the base $\overline{OR}$. We shall call ϕ the *reference angle*. Figure 2 illustrates this construction.

Since two lines are perpendicular in the taxicab space whenever they are perpendicular in the Euclidean space, we shall use the same notion of *orthogonal projection* in the taxicab space.

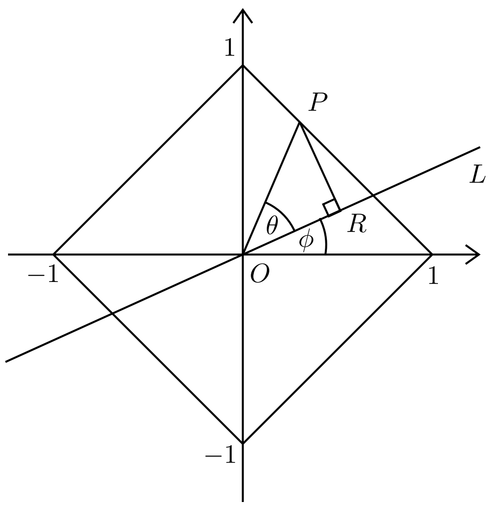

FIGURE 2. Right triangle $\triangle PRO$ in the unit circle with reference angle ϕ

Definition 2.1 (Definition 5 in [2]). Let L be the line through the origin O which makes reference angle ϕ with the x-axis, where $0 \leq \phi < 2$, and let $P = (p_1, p_2)$ be a point on the unit circle so that segment $\overline{OP}$ makes angle θ with L. Let $R = (r_1, r_2)$ be the orthogonal projection of P onto L. We define the *taxicab cosine and sine of angle θ at reference angle ϕ* as

$$\operatorname{tcos}_\phi \theta = r_1 + r_2 \qquad \text{and} \qquad \operatorname{tsin}_\phi \theta = (r_1 - p_1) + (p_2 - r_2).$$

Given a right triangle $\triangle PRO$ with hypotenuse $\overline{OP}$ of length 1 and $\theta = \angle POR$, Definition 2.1 implies that $\operatorname{tcos}_\phi \theta = OR$, which is the length of the side adjacent to θ, and $\operatorname{tsin}_\phi \theta = PR$, which is the length of the side opposite to θ (see Figure 2). Since any right triangle with hypotenuse of length 1 can be mapped with a taxicab isometry into a right triangle of this form, we may view Definition 2.1 as an analogy to the right-triangle definition of the Euclidean sine and cosine functions.

Let $L^\perp$ be the line perpendicular to L passing through the origin. These two lines divide the plane into four quadrants. Figure 3 shows the signs of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ in each L-$L^\perp$ quadrant.

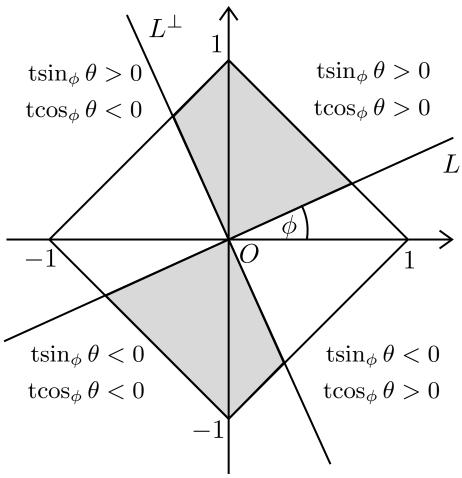

FIGURE 3. The signs of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ in each L-$L^\perp$ quadrant

We remark that the parameter θ in $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ is measured from the line L which makes a fixed angle ϕ to the x-axis (see Figure 2). Hence $\theta = 0$ corresponds to the positive direction of line L. Since a right angle has taxicab measure of 2 t-radians, $\theta = -\phi, 2-\phi, 4-\phi$, and $6-\phi$ correspond to the positive x-axis, positive y-axis, negative x-axis, and negative y-axis, respectively. As a point P travels one round on the unit circle from $(1, 0)$ counterclockwise, the parameter θ increases from $-\phi$ to $8-\phi$.

The following theorem gives explicit formulas of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ for $-\phi \leq \theta < 8-\phi$. The explicit formulas are piecewise functions with breakpoints at $\theta = -\phi, 2-\phi, 4-\phi, 6-\phi$, and $8-\phi$, which correspond to the corners of the unit circle. Figure 4 shows examples of the graphs of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$.

Theorem 2.2 (Theorem 9 in [2]). *Let ϕ be a taxicab reference angle such that $0 \le \phi < 2$ and let θ be a taxicab angle measured relative to ϕ. Let*

$$\alpha = \frac{1}{\phi^2 - 2\phi + 2},$$

which is well defined for all ϕ since $\phi^2 - 2\phi + 2 > 0$. The taxicab sine and cosine of θ with reference angle ϕ are given by

$$\operatorname{tsin}_\phi \theta = \begin{cases} \alpha\theta & ; -\phi \le \theta < 2-\phi, \\ 1 + \alpha(\theta - 2)(\phi - 1) & ; 2-\phi \le \theta < 4-\phi, \\ \alpha(4-\theta) & ; 4-\phi \le \theta < 6-\phi, \\ -1 + \alpha(6-\theta)(\phi-1) & ; 6-\phi \le \theta < 8-\phi, \end{cases}$$

and

$$\operatorname{tcos}_\phi \theta = \begin{cases} 1 + \alpha\theta(\phi - 1) & ; -\phi \le \theta < 2-\phi, \\ \alpha(2-\theta) & ; 2-\phi \le \theta < 4-\phi, \\ -1 + \alpha(4-\theta)(\phi-1) & ; 4-\phi \le \theta < 6-\phi, \\ \alpha(\theta-6) & ; 6-\phi \le \theta < 8-\phi. \end{cases}$$

Remark 2.3. From now on in this article, given a taxicab reference angle ϕ, we shall let α denote $\dfrac{1}{\phi^2 - 2\phi + 2}$ as defined in Theorem 2.2.

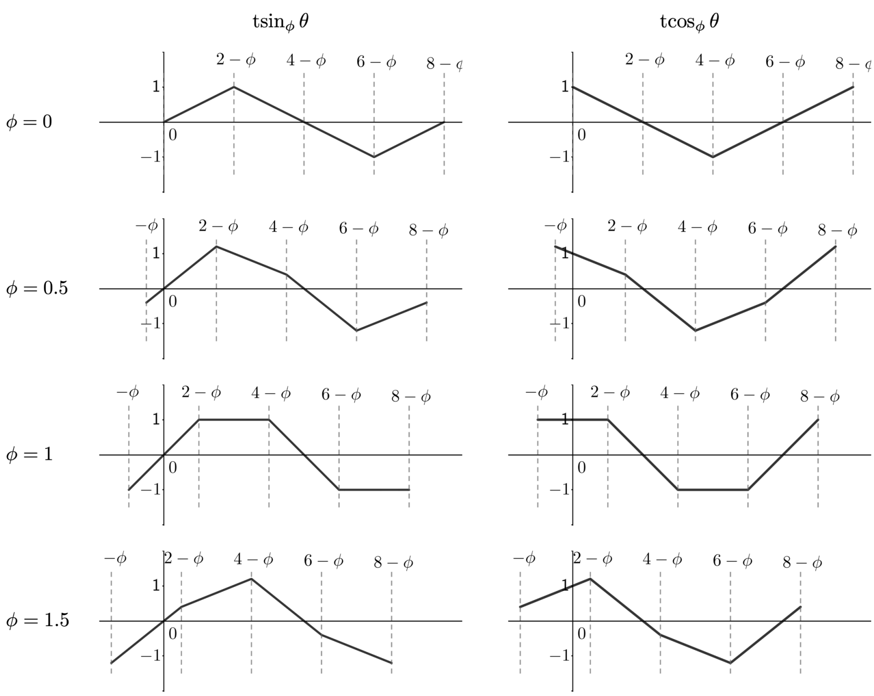

FIGURE 4. The graphs of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ for some values of ϕ

2.2. Basic Properties

From Theorem 2.2, $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ are continuous on $[-\phi, 8-\phi)$. They can be continuously extended to all real values of θ and ϕ.

Proposition 2.4 (Section 4.1 in [2]). *The extensions*

$$\begin{aligned}\operatorname{tsin}_\phi(\theta+8k) &= \operatorname{tsin}_\phi \theta,\\ \operatorname{tcos}_\phi(\theta+8k) &= \operatorname{tcos}_\phi \theta,\\ \operatorname{tsin}_{\phi+2k} \theta &= \operatorname{tsin}_\phi \theta,\\ \operatorname{tcos}_{\phi+2k} \theta &= \operatorname{tcos}_\phi \theta,\end{aligned}$$

for $k \in \mathbb{Z}$, continuously extend $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ to all real values of θ and ϕ.

Proposition 2.5 (Proposition 8 and 11 in [2]). *The following identities hold.*

(1) Half-periodic identities:

$$\begin{aligned}\operatorname{tsin}_\phi(\theta-4) &= -\operatorname{tsin}_\phi \theta,\\ \operatorname{tcos}_\phi(\theta-4) &= -\operatorname{tcos}_\phi \theta.\end{aligned}$$

(2) Cofunction identity:

$$\operatorname{tsin}_\phi(\theta+2) = \operatorname{tcos}_\phi \theta.$$

3. Taxicab Trigonometric Identities

In this section, we derive new identities for the taxicab sine and cosine functions. From now on, we will assume that ϕ is a taxicab reference angle such that $0 \le \phi < 2$, and θ and γ are taxicab angles measured relative to ϕ. We also recall that α denotes $\frac{1}{\phi^2-2\phi+2}$.

3.1. Pythagorean Identity

For the Euclidean space, the Pythagorean identity states that $\sin^2\theta + \cos^2\theta = 1$ as $(\cos\theta, \sin\theta)$ is a point on the unit circle, which is given by $x^2+y^2=1$. If $\phi = 0$, we also have that $(\operatorname{tcos}_0 \theta, \operatorname{tsin}_0 \theta)$ is a point on the taxicab unit circle, which is given by $|x|+|y| = 1$. Hence we have the identity

$$|\operatorname{tsin}_0 \theta| + |\operatorname{tcos}_0 \theta| = 1. \tag{3.1}$$

However, this does not hold for other values of ϕ because $(\operatorname{tcos}_\phi \theta, \operatorname{tsin}_\phi \theta)$ is not a point on the taxicab unit circle. We construct a step function which will be useful for deriving an analogous identity.

Definition 3.1. We define a step function f_ϕ on the interval $[-\phi, 8-\phi)$ as follows.

$$f_\phi(\theta) = \begin{cases} -(\phi-1) & ; -\phi \le \theta < 2-\phi,\\ 1 & ; 2-\phi \le \theta < 4-\phi,\\ \phi-1 & ; 4-\phi \le \theta < 6-\phi,\\ -1 & ; 6-\phi \le \theta < 8-\phi.\end{cases}$$

We extend f_ϕ to $\mathbb{R}$ by $f_\phi(\theta+8k) = f_\phi(\theta)$ for $k \in \mathbb{Z}$.

Theorem 3.2.

$$f_\phi(\theta)\operatorname{tsin}_\phi \theta + f_\phi(\theta+2)\operatorname{tcos}_\phi \theta = 1. \tag{3.2}$$

We note that when $\phi = 0$, the above identity becomes identity (3.1). It can be verified by direct substitution using the explicit formulas in Theorem 2.2. Here, we provide a geometric proof instead. Our proof makes use of taxicab circles and orthogonality. This is necessary because, in general, similar triangles theorems and familiar congruent triangles conditions such as ASA (angle-side-angle) and SAS do not hold in the taxicab space. In fact, the only congruent triangles condition for taxicab triangles is SASAS [5, Section 4].

Proof. We will give the proof for the case $-\phi \leq \theta < 2 - \phi$. The proofs for other cases are similar. We also assume that $0 \leq \phi < 1$ and $0 < \theta < 2 - \phi$. The statement in the case $\phi > 1$ or $-\phi < \theta < 0$ follows from the same argument. Identity (3.2) clearly holds when $\phi = 1$ or when $\theta = 0$ or $-\phi$.

Let L be the line through the origin O which makes reference angle ϕ with the x-axis, where $0 \leq \phi < 1$, and intersects the unit circle at a point Q. Let P be a point on the unit circle so that segment $\overline{OP}$ makes angle $0 < \theta < 2 - \phi$ with L, and let R be the orthogonal projection of P onto L. From the definition of taxicab sine and cosine (Definition 2.1), $OR = \operatorname{tcos}_\phi \theta$ and $PR = \operatorname{tsin}_\phi \theta$.

Then, we construct a point H on the unit circle so that segment $\overline{OH}$ is perpendicular to the unit circle in the first quadrant. It follows that $\overline{OH}$ makes angle of taxicab measure 1 with the x-axis, and angle $\angle QOH$ has taxicab measure $1 - \phi$. Let G be the orthogonal projection of point R onto the unit circle in the first quadrant. Figure 5 (left) illustrates this construction.

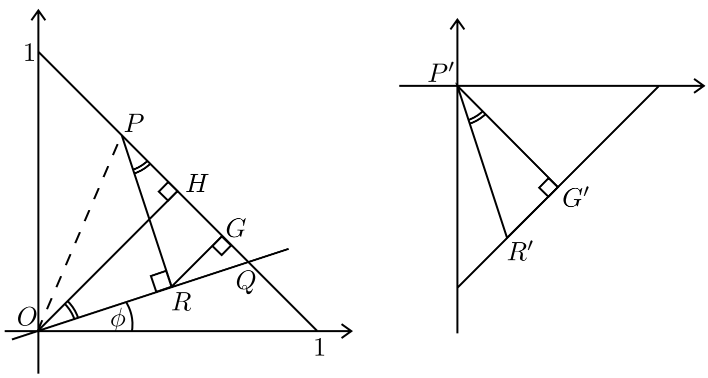

FIGURE 5. Construction in the proof of Theorem 3.2

Next, we consider triangles $\triangle OHQ$ and $\triangle PRQ$. Angles $\angle OQH$ and $\angle PQR$ are the same, and both $\angle OHQ$ and $\angle PRQ$ have taxicab measure of 2 t-radians. Since the angle sum of a taxicab triangle is 4 t-radians [5, Theorem 4.2], we have that angles $\angle HOQ$ and $\angle RPQ$ have the same taxicab measure of $1 - \phi$ t-radians. Because point G is on segment $\overline{PQ}$, angle $\angle RPG$ also has taxicab measure of $1 - \phi$ t-radians.

We recall that a translation is a taxicab isometry. We translate triangle $\triangle PRG$ to the triangle $\triangle P'R'G'$ where the point P' is at the origin. As a result, points G' and R' are on a taxicab circle where both $\overline{P'G'}$ and $\overline{P'R'}$ are radii (see Figure 5 (right)). Since $\angle RPG = \angle R'P'G' = 1 - \phi$, we have that

$$\frac{RG}{PR} = \frac{R'G'}{P'R'} = 1 - \phi.$$

Hence

$$RG = (1-\phi)PR = (1-\phi)\,\mathrm{tsin}_\phi\,\theta.$$

As segment $\overline{RG}$ is perpendicular to the unit circle, there exists a taxicab circle centered at point R where both $\overline{RG}$ and $\overline{RQ}$ are radii. In particular,

$$RQ = RG = (1-\phi)\,\mathrm{tsin}_\phi\,\theta.$$

Because segment $\overline{OQ}$ is a radius of the unit circle and $R \in \overline{OQ}$, we have that

$$1 = OQ = OR + RQ = \mathrm{tcos}_\phi\,\theta + (1-\phi)\,\mathrm{tsin}_\phi\,\theta,$$

which is equivalent to (3.2). ∎

Other taxicab trigonometric functions can be defined in the usual ways. We also get the Pythagorean identities for these functions directly from Theorem 3.2.

Definition 3.3. The taxicab tangent, cotangent, secant, and cosecant functions are defined as follows:

$$\mathrm{ttan}_\phi\,\theta = \frac{\mathrm{tsin}_\phi\,\theta}{\mathrm{tcos}_\phi\,\theta}, \quad \mathrm{tcot}_\phi\,\theta = \frac{\mathrm{tcos}_\phi\,\theta}{\mathrm{tsin}_\phi\,\theta}, \quad \mathrm{tsec}_\phi\,\theta = \frac{1}{\mathrm{tcos}_\phi\,\theta}, \quad \mathrm{tcsc}_\phi\,\theta = \frac{1}{\mathrm{tsin}_\phi\,\theta}.$$

Corollary 3.4.

$$f_\phi(\theta)\,\mathrm{ttan}_\phi\,\theta + f_\phi(\theta+2) = \mathrm{tsec}_\phi\,\theta;$$
$$f_\phi(\theta) + f_\phi(\theta+2)\,\mathrm{tcot}_\phi\,\theta = \mathrm{tcsc}_\phi\,\theta.$$

3.2. Angle Sum Identities

We want to derive identities that express $\mathrm{tsin}_\phi(\theta+\gamma)$ and $\mathrm{tcos}_\phi(\theta+\gamma)$ in terms of sine and cosine of θ and γ. We shall first derive angle sum identities for $-\phi \le \theta < 2-\phi$ and $-\phi \le \gamma < 2-\phi$. For other values of θ and γ, angle sum formulas can be calculated by using the derived identities together with periodic, half-periodic, and cofunction properties of taxicab sine and cosine. We define primitive regions for the angle sum identities as follows. Figure 6 shows these regions in the (θ,γ)-parameter plane.

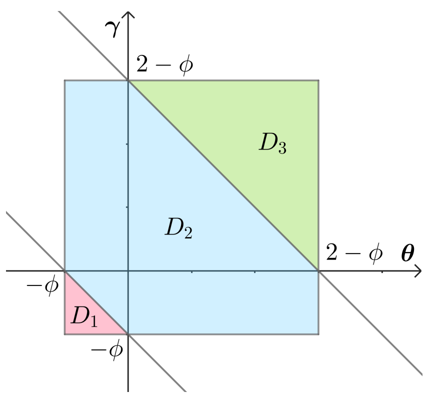

FIGURE 6. Primitive regions for the angle sum identities in the parameter plane

$$D_1 = \{(\theta,\gamma) \in \mathbb{R}^2 : \theta \geq -\phi, \gamma \geq -\phi \text{ and } \theta+\gamma < -\phi\};$$
$$D_2 = \{(\theta,\gamma) \in \mathbb{R}^2 : -\phi \leq \theta < 2-\phi, -\phi \leq \gamma < 2-\phi, \text{ and } -\phi \leq \theta+\gamma < 2-\phi\};$$
$$D_3 = \{(\theta,\gamma) \in \mathbb{R}^2 : \theta < 2-\phi, \gamma < 2-\phi, \text{ and } \theta+\gamma \geq 2-\phi\}.$$

Theorem 3.5. *Let θ and γ be taxicab angles measured relative to ϕ. If $-\phi \leq \theta < 2-\phi$ and $-\phi \leq \gamma < 2-\phi$, then the following identities hold whenever they are well defined.*

$$\operatorname{tsin}_\phi(\theta+\gamma) = \begin{cases} -(\phi-1)(\operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma + 2\alpha) - 1 & \text{if } (\theta,\gamma)\in D_1; \\ \operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma & \text{if } (\theta,\gamma)\in D_2; \\ (\phi-1)(\operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma - 2\alpha) + 1 & \text{if } (\theta,\gamma)\in D_3; \end{cases}$$
$$= \begin{cases} -\operatorname{tcos}_\phi\theta - \operatorname{tcos}_\phi\gamma - 2\alpha(\phi-1) + 1 & \text{if } (\theta,\gamma)\in D_1; \\ \dfrac{\operatorname{tcos}_\phi\theta + \operatorname{tcos}_\phi\gamma - 2}{\phi-1} & \text{if } (\theta,\gamma)\in D_2; \\ \operatorname{tcos}_\phi\theta + \operatorname{tcos}_\phi\gamma - 2\alpha(\phi-1) - 1 & \text{if } (\theta,\gamma)\in D_3; \end{cases}$$

and

$$\operatorname{tcos}_\phi(\theta+\gamma) = \begin{cases} \operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma + 2\alpha & \text{if } (\theta,\gamma)\in D_1; \\ (\phi-1)(\operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma) + 1 & \text{if } (\theta,\gamma)\in D_2; \\ -\operatorname{tsin}_\phi\theta - \operatorname{tsin}_\phi\gamma + 2\alpha & \text{if } (\theta,\gamma)\in D_3; \end{cases}$$
$$= \begin{cases} \dfrac{\operatorname{tcos}_\phi\theta + \operatorname{tcos}_\phi\gamma - 2}{\phi-1} + 2\alpha & \text{if } (\theta,\gamma)\in D_1; \\ \operatorname{tcos}_\phi\theta + \operatorname{tcos}_\phi\gamma - 1 & \text{if } (\theta,\gamma)\in D_2; \\ \dfrac{-\operatorname{tcos}_\phi\theta - \operatorname{tcos}_\phi\gamma + 2}{\phi-1} + 2\alpha & \text{if } (\theta,\gamma)\in D_3. \end{cases}$$

Proof. The proof is by direct calculation using explicit formulas in Theorem 2.2. Here, we present the proof for $\operatorname{tsin}_\phi(\theta+\gamma)$. The proof for $\operatorname{tcos}_\phi(\theta+\gamma)$ is similar.

Case 1 : $(\theta,\gamma)\in D_1$. In this case, $-2\phi \leq \theta+\gamma < -\phi$. From the periodic property, $\operatorname{tsin}(\theta+\gamma) = \operatorname{tsin}(\theta+\gamma+8)$. We also have that $6-\phi < 8-2\phi \leq \theta+\gamma+8 < 8-\phi$. From Theorem 2.2, $\operatorname{tsin}_\phi x = \alpha x$, for $x \in [-\phi, 2-\phi)$, and $\operatorname{tsin}_\phi x = -1 + \alpha(6-x)(\phi-1)$, for $x \in [6-\phi, 8-\phi)$. Thus,

$$\begin{aligned} \operatorname{tsin}_\phi(\theta+\gamma) = \operatorname{tsin}_\phi(\theta+\gamma+8) &= -1 + \alpha(6-(\theta+\gamma+8))(\phi-1) \\ &= -(\phi-1)\alpha\theta - (\phi-1)\alpha\gamma - 2\alpha(\phi-1) - 1 \\ &= -(\phi-1)(\operatorname{tsin}_\phi\theta + \operatorname{tsin}_\phi\gamma + 2\alpha) - 1. \end{aligned}$$

From the Pythagorean identity (Theorem 3.2), for $x \in [-\phi, 2-\phi)$,

$$-(\phi-1)\operatorname{tsin}_\phi x = 1 - \operatorname{tcos}_\phi x. \tag{3.3}$$

Therefore, we also have that

$$\operatorname{tsin}_\phi(\theta+\gamma) = -\operatorname{tcos}_\phi\theta - \operatorname{tcos}_\phi\gamma - 2\alpha(\phi-1) + 1.$$

Case 2 : $(\theta, \gamma) \in D_2$. In this case, $-\phi \leq \theta + \gamma < 2 - \phi$. From Theorem 2.2, $\operatorname{tsin}_\phi x = \alpha x$ for all $x \in [-\phi, 2 - \phi)$. This is a linear function, so

$$\operatorname{tsin}_\phi(\theta + \gamma) = \operatorname{tsin}_\phi \theta + \operatorname{tsin}_\phi \gamma.$$

Using equation (3.3), if $\phi \neq 1$, we also get that

$$\operatorname{tsin}_\phi(\theta + \gamma) = \frac{\operatorname{tcos}_\phi \theta + \operatorname{tcos}_\phi \gamma - 2}{\phi - 1}.$$

Case 3 : $(\theta, \gamma) \in D_3$. In this case, $2 - \phi \leq \theta + \gamma < 4 - \phi$. From Theorem 2.2, $\operatorname{tsin}_\phi x = 1 + \alpha(x - 2)(\phi - 1)$ for all $x \in [2 - \phi, 4 - \phi)$. Therefore,

$$\begin{aligned} \operatorname{tsin}_\phi(\theta + \gamma) &= 1 + \alpha((\theta + \gamma) - 2)(\phi - 1) \\ &= (\phi - 1)\alpha\theta + (\phi - 1)\alpha\gamma - 2\alpha(\phi - 1) + 1 \\ &= (\phi - 1)(\operatorname{tsin}_\phi \theta + \operatorname{tsin}_\phi \gamma - 2\alpha) + 1. \end{aligned}$$

Using equation (3.3), we also get that

$$\operatorname{tsin}_\phi(\theta + \gamma) = \operatorname{tcos}_\phi \theta + \operatorname{tcos}_\phi \gamma - 2\alpha(\phi - 1) - 1.$$

■

Remark 3.6. Some identities in Theorem 3.5 have $\phi - 1$ as a denominator. This means that when $\phi = 1$, an angle sum identity of that type does not exist. For example, when $\phi = 1$ and $(\theta, \gamma) \in D_3$, there is no formula expressing $\operatorname{tcos}_\phi(\theta + \gamma)$ in terms of $\operatorname{tcos}_\phi \theta$ and $\operatorname{tcos}_\phi \gamma$. This is because $\operatorname{tcos}_\phi x$ is constant on $[-\phi, 2 - \phi)$ but not a constant on $[2 - \phi, 4 - \phi)$ (see Figure 4). In this case, we can only write $\operatorname{tcos}_\phi(\theta + \gamma)$ in terms of $\operatorname{tsin}_\phi \theta$ and $\operatorname{tsin}_\phi \gamma$.

For other values of θ and γ, we can use periodic, half-periodic, and cofunction properties together with Theorem 3.5 to find angle sum identities for $\operatorname{tsin}_\phi(\theta + \gamma)$ and $\operatorname{tcos}_\phi(\theta + \gamma)$. Example 3.7 illustrates these calculations. In Table 1, we present angle sum identities for $\theta, \gamma \in [-\phi, 8 - \phi)$ such that $\theta + \gamma < 8 - \phi$. We choose combinations of taxicab sine and cosine of θ and γ so that the identities are well defined for all values of ϕ. The domains for θ and γ in Table 1 are given in Figure 7.

Example 3.7. If $\theta \in [-\phi, 2 - \phi)$, $\gamma \in [2 - \phi, 4 - \phi)$, and $\theta + \gamma \in [2 - \phi, 4 - \phi)$, then

$$\operatorname{tsin}_\phi(\theta + \gamma) = (\phi - 1) \operatorname{tsin}_\phi \theta + \operatorname{tsin}_\phi \gamma.$$

Proof. Let $\tilde{\gamma} = \gamma - 2$. Then, $\tilde{\gamma} \in [-\phi, 2 - \phi)$ and $\theta + \tilde{\gamma} \in [-\phi, 2 - \phi)$. That is $(\theta, \tilde{\gamma}) \in D_2$ in Theorem 3.5. Hence

$$\operatorname{tcos}_\phi(\theta + \gamma - 2) = \operatorname{tcos}_\phi(\theta + \tilde{\gamma}) = \operatorname{tcos}_\phi \theta + \operatorname{tcos}_\phi \tilde{\gamma} - 1.$$

From the cofunction property (Proposition 2.5), $\operatorname{tcos}_\phi(\theta + \gamma - 2) = \operatorname{tsin}_\phi(\theta + \gamma)$ and $\operatorname{tcos}_\phi \tilde{\gamma} = \operatorname{tcos}_\phi(\gamma - 2) = \operatorname{tsin}_\phi \gamma$. We have that

$$\operatorname{tsin}_\phi(\theta + \gamma) = \operatorname{tcos}_\phi \theta + \operatorname{tsin}_\phi \gamma - 1.$$

From the Pythagorean identity (Theorem 3.2), $-(\phi - 1) \operatorname{tsin}_\phi \theta + \operatorname{tcos}_\phi \theta = 1$. Therefore, the above identity becomes

$$\operatorname{tsin}_\phi(\theta + \gamma) = (\phi - 1) \operatorname{tsin}_\phi \theta + \operatorname{tsin}_\phi \gamma.$$

■

TABLE 1. Angle sum identities for $\theta, \gamma \in [-\phi, 8-\phi)$ such that $\theta+\gamma < 8-\phi$. The domains D_i are given in Figure 7. In this table, we say that an angle x is in Q_1 if $-\phi \le x < 2-\phi$, Q_2 if $2-\phi \le x < 4-\phi$, Q_3 if $4-\phi \le x < 6-\phi$, and Q_4 if $6-\phi \le x < 8-\phi$ or $-2-\phi \le x < -\phi$.

Domain	θ	γ	$\theta+\gamma$	$\operatorname{tsin}_\phi(\theta+\gamma)$	$\operatorname{tcos}_\phi(\theta+\gamma)$
D_1	Q_1	Q_1	Q_4	$-(\phi-1)(\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma+2\alpha)-1$	$\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma+2\alpha$
D_2	Q_1	Q_1	Q_1	$\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma$	$\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma-1$
D_3	Q_1	Q_1	Q_2	$(\phi-1)(\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma-2\alpha)+1$	$-\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma+2\alpha$
D_4	Q_2	Q_2	Q_2	$\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma+2\alpha(\phi-1)-1$	$\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma-2\alpha$
D_5	Q_2	Q_2	Q_3	$\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma$	$(\phi-1)(\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma)-1$
D_6	Q_2	Q_2	Q_4	$-\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma+2\alpha(\phi-1)+1$	$-\operatorname{tcos}_\phi\theta-\operatorname{tcos}_\phi\gamma-2\alpha$
D_7	Q_3	Q_3	Q_4	$(\phi-1)(\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma-2\alpha)-1$	$-\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma+2\alpha$
D_8	Q_1	Q_2	Q_1	$\operatorname{tsin}_\phi\theta-\operatorname{tcos}_\phi\gamma+2\alpha$	$\operatorname{tcos}_\phi\theta-(\phi-1)\operatorname{tcos}_\phi\gamma+2\alpha(\phi-1)$
D_9	Q_1	Q_2	Q_2	$(\phi-1)\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma$	$-\operatorname{tsin}_\phi\theta+\operatorname{tcos}_\phi\gamma$
D_{10}	Q_1	Q_2	Q_3	$-\operatorname{tsin}_\phi\theta+\operatorname{tcos}_\phi\gamma+2\alpha$	$-\operatorname{tcos}_\phi\theta+(\phi-1)\operatorname{tcos}_\phi\gamma+2\alpha(\phi-1)$
D_{11}	Q_2	Q_3	Q_3	$\operatorname{tcos}_\phi\theta+\operatorname{tsin}_\phi\gamma-2\alpha$	$(\phi-1)\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma-2\alpha(\phi-1)$
D_{12}	Q_2	Q_3	Q_4	$-\operatorname{tsin}_\phi\theta+(\phi-1)\operatorname{tsin}_\phi\gamma$	$-\operatorname{tcos}_\phi\theta-\operatorname{tsin}_\phi\gamma$
D_{13}	Q_1	Q_3	Q_2	$(\phi-1)(\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma+2\alpha)+1$	$-\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma-2\alpha$
D_{14}	Q_1	Q_3	Q_3	$-\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma$	$-\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma+1$
D_{15}	Q_1	Q_3	Q_4	$(\phi-1)(-\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma+2\alpha)-1$	$\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma+2\alpha$
D_{16}	Q_2	Q_4	Q_4	$-\operatorname{tsin}_\phi\theta+\operatorname{tsin}_\phi\gamma-2\alpha(\phi-1)+1$	$-\operatorname{tcos}_\phi\theta+\operatorname{tcos}_\phi\gamma+2\alpha$
D_{17}	Q_1	Q_4	Q_3	$-\operatorname{tsin}_\phi\theta-\operatorname{tcos}_\phi\gamma-2\alpha$	$-\operatorname{tcos}_\phi\theta-(\phi-1)\operatorname{tcos}_\phi\gamma-2\alpha(\phi-1)$
D_{18}	Q_1	Q_4	Q_4	$-(\phi-1)\operatorname{tsin}_\phi\theta-\operatorname{tsin}_\phi\gamma$	$\operatorname{tsin}_\phi\theta+\operatorname{tcos}_\phi\gamma$

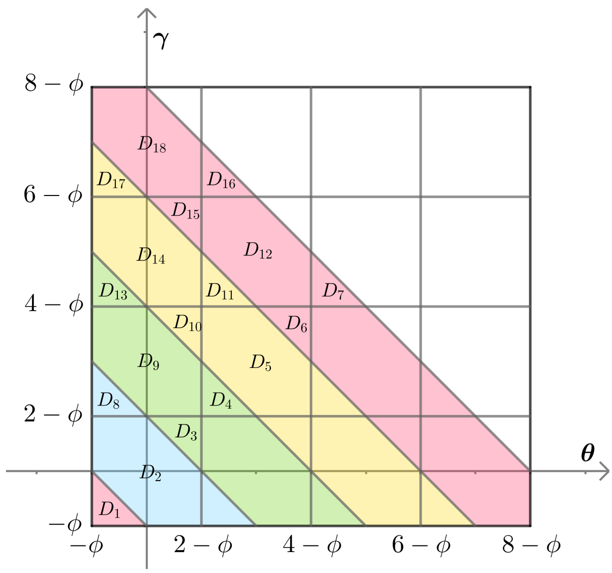

FIGURE 7. Domains for angle sum identities in Table 1

3.3. Double-angle Identities

Double-angle identities express $\operatorname{tsin}_\phi(2\theta)$ and $\operatorname{tcos}_\phi(2\theta)$ in terms of sine and cosine of θ. It suffices to derive double-angle identities for $-\phi \le \theta < 2-\phi$ as we can use periodic, half-periodic, and cofunction properties to derive double-angle identities for other values of θ. The following corollary follows directly from Theorem 3.5. Table 2 shows double-angle identities that are well defined for all values of ϕ.

Corollary 3.8. *Let θ be a taxicab angle measured relative to ϕ. If $-\phi \le \theta < 2-\phi$, then the following identities hold whenever they are well defined.*

$$\operatorname{tsin}_\phi(2\theta) = \begin{cases} -2(\phi-1)(\operatorname{tsin}_\phi\theta+\alpha)-1 & \text{if } -\phi \le \theta < -\frac{\phi}{2}; \\ 2\operatorname{tsin}_\phi\theta & \text{if } -\dfrac{\phi}{2} \le \theta < 1-\frac{\phi}{2}; \\ 2(\phi-1)(\operatorname{tsin}_\phi\theta-\alpha)+1 & \text{if } 1-\frac{\phi}{2} \le \theta < 2-\phi; \end{cases}$$

$$= \begin{cases} -2(\operatorname{tcos}_\phi\theta+\alpha(\phi-1))+1 & \text{if } -\phi \le \theta < -\frac{\phi}{2}; \\ \dfrac{2(\operatorname{tcos}_\phi\theta-1)}{\phi-1} & \text{if } -\frac{\phi}{2} \le \theta < 1-\frac{\phi}{2}; \\ 2(\operatorname{tcos}_\phi\theta-\alpha(\phi-1))-1 & \text{if } 1-\frac{\phi}{2} \le \theta < 2-\phi; \end{cases}$$

and

$$\operatorname{tcos}_\phi(2\theta) = \begin{cases} 2\operatorname{tsin}_\phi\theta+2\alpha & \text{if } -\phi \le \theta < -\frac{\phi}{2}; \\ 2(\phi-1)\operatorname{tsin}_\phi\theta+1 & \text{if } -\frac{\phi}{2} \le \theta < 1-\frac{\phi}{2}; \\ -2\operatorname{tsin}_\phi\theta+2\alpha & \text{if } 1-\frac{\phi}{2} \le \theta < 2-\phi; \end{cases}$$

$$= \begin{cases} \dfrac{2(\operatorname{tcos}_\phi\theta+\alpha(\phi-1)-1)}{\phi-1} & \text{if } -\phi \le \theta < -\frac{\phi}{2}; \\ 2\operatorname{tcos}_\phi\theta-1 & \text{if } -\frac{\phi}{2} \le \theta < 1-\frac{\phi}{2}; \\ \dfrac{-2(\operatorname{tcos}_\phi\theta-\alpha(\phi-1)-1)}{\phi-1} & \text{if } 1-\frac{\phi}{2} \le \theta < 2-\phi. \end{cases}$$

Table 2. Double angle identities for θ such that $-2\phi \le 2\theta < 8-\phi$

Domain	$\operatorname{tsin}_\phi(2\theta)$	$\operatorname{tcos}_\phi(2\theta)$
$\left[-\phi, -\frac{\phi}{2}\right)$	$-2(\phi-1)(\operatorname{tsin}_\phi\theta+\alpha)-1$	$2\operatorname{tsin}_\phi\theta+2\alpha$
$\left[-\frac{\phi}{2}, 1-\frac{\phi}{2}\right)$	$2\operatorname{tsin}_\phi\theta$	$2\operatorname{tcos}_\phi\theta-1$
$\left[1-\frac{\phi}{2}, 2-\phi\right)$	$2(\phi-1)(\operatorname{tsin}_\phi\theta-\alpha)+1$	$-2\operatorname{tsin}_\phi\theta+2\alpha$
$\left[2-\phi, 2-\frac{\phi}{2}\right)$	$2\operatorname{tsin}_\phi\theta+2\alpha(\phi-1)-1$	$2\operatorname{tcos}_\phi\theta-2\alpha$
$\left[2-\frac{\phi}{2}, 3-\frac{\phi}{2}\right)$	$2\operatorname{tcos}_\phi\theta$	$2(\phi-1)\operatorname{tcos}_\phi\theta-1$
$\left[3-\frac{\phi}{2}, 4-\phi\right)$	$-2\operatorname{tsin}_\phi\theta+2\alpha(\phi-1)+1$	$-2\operatorname{tcos}_\phi\theta-2\alpha$
$\left[4-\phi, 4-\frac{\phi}{2}\right)$	$2(\phi-1)(\operatorname{tsin}_\phi\theta-\alpha)-1$	$-2\operatorname{tsin}_\phi\theta+2\alpha$

3.4. Half-angle Identities

Half-angle identities express $\operatorname{tsin}_\phi\left(\frac{\theta}{2}\right)$ and $\operatorname{tcos}_\phi\left(\frac{\theta}{2}\right)$ in terms of sine and cosine of θ. We get the following corollary from Corollary 3.8 by setting $\theta \to 2\theta$ and $\frac{\theta}{2} \to \theta$. Table 3 shows half-angle identities that are well defined for all values of ϕ.

Corollary 3.9. *Let θ be a taxicab angle measured relative to ϕ. If $-\phi \le \theta < 4-2\phi$, then the following identities hold whenever they are well defined.*

$$\operatorname{tsin}_\phi\left(\frac{\theta}{2}\right) = \begin{cases} \frac{1}{2}\operatorname{tsin}_\phi\theta & \text{if } -\phi \le \theta < 2-\phi; \\ \dfrac{\operatorname{tsin}_\phi\theta + 2\alpha(\phi-1)-1}{2(\phi-1)} & \text{if } 2-\phi \le \theta < 4-2\phi; \end{cases}$$
$$= \begin{cases} \dfrac{\operatorname{tcos}_\phi\theta - 1}{2(\phi-1)} & \text{if } -\phi \le \theta < 2-\phi; \\ -\frac{1}{2}\operatorname{tcos}_\phi\theta + \alpha & \text{if } 2-\phi \le \theta < 4-2\phi; \end{cases}$$

and

$$\operatorname{tcos}_\phi\left(\frac{\theta}{2}\right) = \begin{cases} \frac{1}{2}(\phi-1)\operatorname{tsin}_\phi\theta + 1 & \text{if } -\phi \le \theta < 2-\phi; \\ \frac{1}{2}\operatorname{tsin}_\phi\theta + \alpha(\phi-1) + \frac{1}{2} & \text{if } 2-\phi \le \theta < 4-2\phi; \end{cases}$$
$$= \begin{cases} \frac{1}{2}\operatorname{tcos}_\phi\theta + \frac{1}{2} & \text{if } -\phi \le \theta < 2-\phi; \\ -\frac{1}{2}(\phi-1)\operatorname{tcos}_\phi\theta + \alpha(\phi-1) + 1 & \text{if } 2-\phi \le \theta < 4-2\phi. \end{cases}$$

TABLE 3. Half-angle identities for $-\phi \le \theta < 8-\phi$

Domain	$\operatorname{tsin}_\phi\left(\frac{\theta}{2}\right)$	$\operatorname{tcos}_\phi\left(\frac{\theta}{2}\right)$
$[-\phi, 2-\phi)$	$\frac{1}{2}\operatorname{tsin}_\phi\theta$	$\frac{1}{2}\operatorname{tcos}_\phi\theta + \frac{1}{2}$
$[2-\phi, 4-2\phi)$	$-\frac{1}{2}\operatorname{tcos}_\phi\theta + \alpha$	$-\frac{1}{2}(\phi-1)\operatorname{tcos}_\phi\theta + \alpha(\phi-1) + 1$
$[4-2\phi, 4-\phi)$	$\frac{1}{2}\operatorname{tsin}_\phi\theta - \alpha(\phi-1) + \frac{1}{2}$	$\frac{1}{2}\operatorname{tcos}_\phi\theta + \alpha$
$[4-\phi, 6-\phi)$	$-\frac{1}{2}(\phi-1)\operatorname{tsin}_\phi\theta + 1$	$\frac{1}{2}\operatorname{tsin}_\phi\theta$
$[6-\phi, 8-2\phi)$	$-\frac{1}{2}\operatorname{tsin}_\phi\theta + \alpha(\phi-1) + \frac{1}{2}$	$-\frac{1}{2}\operatorname{tcos}_\phi\theta - \alpha$
$[8-2\phi, 8-\phi)$	$-\frac{1}{2}\operatorname{tcos}_\phi\theta + \alpha$	$-\frac{1}{2}(\phi-1)\operatorname{tcos}_\phi\theta + \alpha(\phi-1) - 1$

3.5. Negative-angle Identities

Negative-angle identities express $\operatorname{tsin}_\phi(-\theta)$ and $\operatorname{tcos}_\phi(-\theta)$ in terms of $\operatorname{tsin}_\phi\theta$ or $\operatorname{tcos}_\phi\theta$. It suffices to derive the negative-angle identities for $\theta \in (0,4]$. For other values of θ, we can use the half-periodic property to derive negative-angle identities. Unlike the previous identities, negative-angle identities also depend on the values of ϕ. In the following theorem, we choose to provide versions of negative-angle identities that are well defined for all values of ϕ.

Theorem 3.10. *Let $0 < \theta \le 4$ be a taxicab angle measured relative to ϕ. If $0 \le \phi < 1$, then*

$$\operatorname{tsin}_\phi(-\theta) = \begin{cases} -\operatorname{tsin}_\phi\theta & \text{if } 0 < \theta \le \phi; \\ (\phi-1)(\operatorname{tsin}_\phi\theta - 2\alpha) - 1 & \text{if } \phi < \theta \le 2-\phi; \\ \operatorname{tsin}_\phi\theta - 2 & \text{if } 2-\phi < \theta \le 2+\phi; \\ -\operatorname{tcos}_\phi\theta - 2\alpha & \text{if } 2+\phi < \theta \le 4-\phi; \\ -\operatorname{tsin}_\phi\theta & \text{if } 4-\phi < \theta \le 4; \end{cases}$$

and

$$\operatorname{tcos}_\phi(-\theta) = \begin{cases} -\operatorname{tcos}_\phi \theta + 2 & \text{if } 0 < \theta \le \phi; \\ -\operatorname{tsin}_\phi \theta + 2\alpha & \text{if } \phi < \theta \le 2-\phi; \\ \operatorname{tcos}_\phi \theta & \text{if } 2-\phi < \theta \le 2+\phi; \\ -(\phi-1)(\operatorname{tcos}_\phi \theta + 2\alpha) - 1 & \text{if } 2+\phi < \theta \le 4-\phi; \\ -\operatorname{tcos}_\phi \theta - 2 & \text{if } 4-\phi < \theta \le 4. \end{cases}$$

If $1 \le \phi < 2$, *then*

$$\operatorname{tsin}_\phi(-\theta) = \begin{cases} -\operatorname{tsin}_\phi \theta & \text{if } 0 < \theta \le 2-\phi; \\ \operatorname{tcos}_\phi \theta - 2\alpha & \text{if } 2-\phi < \theta \le \phi; \\ \operatorname{tsin}_\phi \theta - 2 & \text{if } \phi < \theta \le 4-\phi; \\ -(\phi-1)(\operatorname{tsin}_\phi \theta - 2\alpha) - 1 & \text{if } 4-\phi < \theta \le 2+\phi; \\ -\operatorname{tsin}_\phi \theta & \text{if } 2+\phi < \theta \le 4; \end{cases}$$

and

$$\operatorname{tcos}_\phi(-\theta) = \begin{cases} -\operatorname{tcos}_\phi \theta + 2 & \text{if } 0 < \theta \le 2-\phi; \\ (\phi-1)(\operatorname{tcos}_\phi \theta - 2\alpha) + 1 & \text{if } 2-\phi < \theta \le \phi; \\ \operatorname{tcos}_\phi \theta & \text{if } \phi < \theta \le 4-\phi; \\ \operatorname{tsin}_\phi \theta - 2\alpha & \text{if } 4-\phi < \theta \le 2+\phi; \\ -\operatorname{tcos}_\phi \theta - 2 & \text{if } 2+\phi < \theta \le 4. \end{cases}$$

Proof. The proof is by direct calculation using explicit formulas in Theorem 2.2. Here, we demonstrate the proof for one case. The proofs for other cases are similar.

Case $\phi < 1$ **and** $2+\phi < \theta \le 4-\phi$**:** In this case, we have that $-4+\phi \le -\theta < -2-\phi$. So $4-\phi \le 4+\phi \le 8-\theta < 6-\phi$. From Theorem 2.2, $\operatorname{tcos}_\phi x = \alpha(2-x)$, for $2-\phi \le x < 4-\phi$, and $\operatorname{tcos}_\phi x = -1 + \alpha(4-x)(\phi-1)$, for $4-\phi \le x < 6-\phi$. Then,

$$\begin{aligned} \operatorname{tcos}_\phi(-\theta) = \operatorname{tcos}_\phi(8-\theta) &= -1 + \alpha(4-(8-\theta))(\phi-1) \\ &= -1 + \alpha(2-\theta)(\phi-1) - 2\alpha(\phi-1) \\ &= -(\phi-1)(\operatorname{tcos}_\phi \theta + 2\alpha) - 1. \end{aligned}$$

■

4. Derivative Identities

For the Euclidean space, $\frac{d}{d\theta}[\sin\theta] = \cos\theta$ and $\frac{d}{d\theta}[\cos\theta] = -\sin\theta$. Because the taxicab sine and cosine are piecewise linear, their derivatives do not have similar relation. However, it turns out that the derivatives of other taxicab trigonometric functions behave similarly to their Euclidean counterparts. For example, the derivative of taxicab tangent is proportional to the square of taxicab secant.

The following theorem collects new derivative identities for the taxicab tangent, secant, cotangent, and cosecant functions, which extend the results for the case $\phi = 0$ in [7].

Theorem 4.1. *The following identities hold whenever the derivatives are well defined.*

$$\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] = \alpha \operatorname{tsec}^2_\phi \theta;$$

$$\frac{d}{d\theta}[\operatorname{tsec}_\phi \theta] = \alpha f_\phi(\theta) \operatorname{tsec}^2_\phi \theta$$
$$= \begin{cases} -\alpha(\phi-1)\operatorname{tsec}^2_\phi \theta & \textit{if } 8k-\phi < \theta < (8k+2)-\phi; \\ \alpha \operatorname{tsec}^2_\phi \theta & \textit{if } (8k+2)-\phi < \theta < (8k+4)-\phi; \\ \alpha(\phi-1)\operatorname{tsec}^2_\phi \theta & \textit{if } (8k+4)-\phi < \theta < (8k+6)-\phi; \\ -\alpha \operatorname{tsec}^2_\phi \theta & \textit{if } (8k+6)-\phi < \theta < (8k+8)-\phi; \end{cases}$$
$$\frac{d}{d\theta}[\operatorname{tcot}_\phi \theta] = -\alpha \operatorname{tcsc}^2_\phi \theta;$$
$$\frac{d}{d\theta}[\operatorname{tcsc}_\phi \theta] = -\alpha f_\phi(\theta+2) \operatorname{tcsc}^2_\phi \theta$$
$$= \begin{cases} -\alpha \operatorname{tcsc}^2_\phi \theta & \textit{if } 8k-\phi < \theta < (8k+2)-\phi; \\ -\alpha(\phi-1)\operatorname{tcsc}^2_\phi \theta & \textit{if } (8k+2)-\phi < \theta < (8k+4)-\phi; \\ \alpha \operatorname{tcsc}^2_\phi \theta & \textit{if } (8k+4)-\phi < \theta < (8k+6)-\phi; \\ \alpha(\phi-1)\operatorname{tcsc}^2_\phi \theta & \textit{if } (8k+6)-\phi < \theta < (8k+8)-\phi. \end{cases}$$

Here, k is an arbitrary integer.

Proof. The derivative identities can be proved by direct calculation using explicit formulas in Theorem 2.2. We show the calculation for the derivative of $\operatorname{ttan}_\phi \theta$ here. The proofs for other derivatives are similar.

We note that the derivative of $\operatorname{ttan}_\phi \theta$ does not exist at $\theta \in \{2k-\phi : k \in \mathbb{Z}\}$, where the graphs of taxicab sine and cosine have corners, and whenever $\operatorname{tcos}_\phi \theta = 0$.

Case 1 : $-\phi < \theta < 2-\phi$. From Theorem 2.2, $\operatorname{tsin}_\phi \theta = \alpha\theta$ and $\operatorname{tcos}_\phi \theta = 1+\alpha\theta(\phi-1)$. Then,

$$\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] = \frac{d}{d\theta}\left[\frac{\alpha\theta}{1+\alpha\theta(\phi-1)}\right] = \frac{(1+\alpha\theta(\phi-1))\alpha - (\alpha\theta)(\alpha(\phi-1))}{(1+\alpha\theta(\phi-1))^2}$$
$$= \frac{\alpha}{(1+\alpha\theta(\phi-1))^2}$$
$$= \frac{\alpha}{\operatorname{tcos}^2_\phi \theta}$$
$$= \alpha \operatorname{tsec}^2_\phi \theta.$$

Case 2 : $2-\phi < \theta < 4-\phi$. In this case, $\operatorname{tsin}_\phi \theta = 1+\alpha(\theta-2)(\phi-1)$ and $\operatorname{tcos}_\phi \theta = \alpha(2-\theta)$. Then,

$$\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] = \frac{d}{d\theta}\left[\frac{1+\alpha(\theta-2)(\phi-1)}{\alpha(2-\theta)}\right] = \frac{1}{\alpha(2-\theta)^2} = \frac{\alpha}{(\alpha(2-\theta))^2} = \alpha \operatorname{tsec}^2_\phi \theta.$$

Case 3 : $4-\phi < \theta < 6-\phi$. We have that $\operatorname{tsin}_\phi \theta = \alpha(4-\theta)$ and $\operatorname{tcos}_\phi \theta = -1+\alpha(4-\theta)(\phi-1)$. Then,

$$\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] = \frac{d}{d\theta}\left[\frac{\alpha(4-\theta)}{-1+\alpha(4-\theta)(\phi-1)}\right]$$
$$= \frac{(-1+\alpha(4-\theta)(\phi-1))(-\alpha) - \alpha(4-\theta)(-\alpha(\phi-1))}{(-1+\alpha(4-\theta)(\phi-1))^2}$$
$$= \frac{\alpha}{(-1+\alpha(4-\theta)(\phi-1))^2}$$

$$= \alpha \operatorname{tsec}_\phi^2 \theta.$$

Case 4 : $6-\phi < \theta < 8-\phi$. In this case, $\operatorname{tsin}_\phi \theta = -1+\alpha(6-\theta)(\phi-1)$ and $\operatorname{tcos}_\phi \theta = \alpha(\theta-6)$. Then,

$$\begin{aligned}
\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] &= \frac{d}{d\theta}\left[\frac{-1+\alpha(6-\theta)(\phi-1)}{\alpha(\theta-6)}\right] \\
&= \frac{\alpha(\theta-6)(-\alpha(\phi-1)) - (-1+\alpha(6-\theta)(\phi-1))(\alpha)}{(\alpha(\theta-6))^2} \\
&= \frac{\alpha}{(\alpha(\theta-6))^2} \\
&= \alpha \operatorname{tsec}_\phi^2 \theta.
\end{aligned}$$

From all cases, we get that $\frac{d}{d\theta}[\operatorname{ttan}_\phi \theta] = \alpha \operatorname{tsec}_\phi^2 \theta$. ∎

Applying the Pythagorean identities (Corollary 3.4), we get the following identities.

Corollary 4.2. *The following identities hold whenever the derivatives are well defined.*

$$\frac{d}{d\theta}[\operatorname{tsec}_\phi \theta] = \alpha f_\phi^2(\theta) \operatorname{tsec}_\phi \theta \operatorname{ttan}_\phi \theta + \alpha f_\phi(\theta) f_\phi(\theta+2) \operatorname{tsec}_\phi \theta$$

$$= \begin{cases} \alpha(\phi-1)^2 \operatorname{tsec}_\phi \theta \operatorname{ttan}_\phi \theta - \alpha(\phi-1) \operatorname{tsec}_\phi \theta & \text{if } \theta \in (8k, 8k+2) \cup (8k+4, 8k+6); \\ \alpha \operatorname{tsec}_\phi \theta \operatorname{ttan}_\phi \theta + \alpha(\phi-1) \operatorname{tsec}_\phi \theta & \text{if } \theta \in (8k+2, 8k+4) \cup (8k+6, 8k+8); \end{cases}$$

$$\frac{d}{d\theta}[\operatorname{tcsc}_\phi \theta] = -\alpha f_\phi^2(\theta+2) \operatorname{tcsc}_\phi \theta \operatorname{tcot}_\phi \theta - \alpha f_\phi(\theta) f_\phi(\theta+2) \operatorname{tcsc}_\phi \theta$$

$$= \begin{cases} -\alpha \operatorname{tcsc}_\phi \theta \operatorname{tcot}_\phi \theta + \alpha(\phi-1) \operatorname{tcsc}_\phi \theta & \text{if } \theta \in (8k, 8k+2) \cup (8k+4, 8k+6); \\ -\alpha(\phi-1)^2 \operatorname{tcsc}_\phi \theta \operatorname{tcot}_\phi \theta - \alpha(\phi-1) \operatorname{tcsc}_\phi \theta & \text{if } \theta \in (8k+2, 8k+4) \cup (8k+6, 8k+8), \end{cases}$$

Here, k is an arbitrary integer.

5. Conclusion

If $D \subset \mathbb{R}^2$ is a closed bounded convex region which is centrally symmetric (i.e. $p \in D$ if and only if $-p \in D$), then D defines a norm $\|\cdot\|$ on $\mathbb{R}^2$ where D is the unit disk. The norm $\|\cdot\|$ then induces a metric d on $\mathbb{R}^2$ given by $d(x,y) = \|y-x\|$. Finite dimensional normed spaces are also known as *Minkowski spaces* [4]. Examples of two-dimensional Minkowski spaces include the taxicab space, where the unit disk is the square region $D = \{(x,y) : |x|+|y| \le 1\}$, and more generally, normed linear spaces defined by regular $2n$-gons.

In this article, we derive several algebraic and derivative identities for two-parameter trigonometric functions in the taxicab space. Our result suggests that some constants and functions that are originally constructed only as algebraic aids might have geometric meaning related to the geometry of the unit disk. This hints at a possible unified theory for two-parameter trigonometric functions for a larger class of Minkowski spaces such as two-dimensional normed linear spaces defined by regular $2n$-gons.

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