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On the Relative Rank of Orientation-preserving or Orientation-reversing Transformation Semigroups with Restricted Range

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Abstract Let G and U be subsets of a semigroup S. The rank of a semigroup S is the minimal size of a generating set of S. By the definition of rank, it gives a new idea of definition of rank which is called the relative rank of S modulo U which is the minimal size of a subset G such that $G \cup U$ is a generating set of S. A set G is called a generating set of S modulo U. Let X be a finite chain and let Y be a subchain of X. The semigroup $\mathcal{T}(X,Y)$ is so-called the full transformation semigroup on X with restricted range Y which is a subsemigroup of the semigroup $\mathcal{T}(X)$. In this work, we determine the relative rank of the semigroup $\mathcal{OPR}(X,Y)$ of all orientation-preserving or orientation-reversing transformations with restricted range modulo the semigroup $\mathcal{OD}(X,Y)$ of all order-preserving or order-reversing transformations with restricted range.

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1. Introduction and Preliminaries

Let S be a semigroup and let G and U be subsets of the semigroup S. Then a generating set G of S is denoted by $\langle G\rangle = S$. The rank of S is defined to be the minimal size of a generating set G, i.e. $rank(S) := \min\{|G| : G \subseteq S, \langle G\rangle = S\}$. The relative rank of S modulo U is the minimal size of a subset G of S such that $G \cup U$ generates S, i.e. $rank(S : U) := \min\{|G| : G \subseteq S, \langle G \cup U\rangle = S\}$. By the definition of the relative rank, we obtain immediately that $rank(S : \emptyset) = rank(S)$, $rank(S : S) = 0$, $rank(S : A) = rank(S : \langle A\rangle)$ and $rank(S : A) = 0$ if and only if $\langle A\rangle = S$. In addition, a set G with $\langle G \cup U\rangle = S$ is called a generating set of S modulo U. The relative rank

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generalizes the rank of a semigroup which was introduced by Howie, Ruškuc and Higgins [9].

Let $X = \{1 < 2 < \cdots < n\}$ be a finite chain with $|X| = n$. Denote $\mathcal{T}(X)$ by the semigroup of all full transformations on X under the usual composition of functions. Next, we will introduce some notations in the full transformation semigroup. Let $\alpha \in \mathcal{T}(X)$. The image of a transformation α which is denoted by $im(\alpha)$, i.e. $im(\alpha) := \{x\alpha : x \in X\}$ and the rank of a transformation α is the cardinality of $im(\alpha)$ which is denoted by $rank(\alpha)$, i.e. $rank(\alpha) := |im(\alpha)|$. The kernel of a transformation α is the set $\ker(\alpha) := \{(x, y) \in X \times X : x\alpha = y\alpha\}$. It is easy to verify that $\ker(\alpha)$ is an equivalence relation on X and it is called $\ker(\alpha)$-classes or $\ker(\alpha)$-blocks. Let B_1, B_2 be subsets of X. Define $B_1 < B_2$ if and only if $x_1 < x_2$ for all $x_1 \in B_1$ and for all $x_2 \in B_2$.

A transformation $\alpha \in \mathcal{T}(X)$ is called orientation-preserving (orientation-reversing, respectively) if there is a decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, $y_1\alpha \geq y_2\alpha$ ($y_1\alpha \leq y_2\alpha$, respectively) for all $y_1 \in X_1$ and $y_2 \in X_2$, and $x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$, respectively) for all $x \leq y \in X_1$ or $x \leq y \in X_2$. If $X_2 = \emptyset$ with $x\alpha \leq y\alpha$ for all $x \leq y \in X_1$ then α is called order-preserving. Moreover, if $X_1 = \emptyset$ with $x\alpha \geq y\alpha$ for all $x \leq y \in X_2$ then α is called order-reversing. We obtain that the product of two orientation-preserving transformations is an orientation-preserving and the product of two orientation-reversing transformations is also an orientation-preserving. We denote by $\mathcal{O}(X)$, $\mathcal{OD}(X)$, $\mathcal{OP}(X)$, $\mathcal{OR}(X)$ and $\mathcal{OPR}(X)$ the semigroup of all order-preserving transformations, the semigroup of all order-preserving or order-reversing transformations, the semigroup of all orientation-preserving transformations, the set of all orientation-reversing transformations and the semigroup of all orientation-preserving or orientation-reversing transformations, respectively. It is easy to verify that $\mathcal{O}(X)$, $\mathcal{OD}(X)$, $\mathcal{OP}(X)$ and $\mathcal{OPR}(X)$ are subsemigroup of $\mathcal{T}(X)$ under the usual composition of functions but $\mathcal{OR}(X)$ is not a subsemigroup of $\mathcal{T}(X)$. It is also clear that $\mathcal{O}(X)$ is a proper subsemigroup of $\mathcal{OD}(X)$, $\mathcal{OP}(X)$ and $\mathcal{OPR}(X)$. In addition, we also know that $\mathcal{OD}(X)$ is a proper subsemigroup of $\mathcal{OPR}(X)$. The semigroup $\mathcal{OP}(X)$ has been widely studied (see in [1], [2], [3], [5] and [12]). The rank of $\mathcal{OP}(X)$, $\mathcal{O}(X)$ and $\mathcal{T}(X)$ are equal $2, n$ and 3, respectively (see in [3] and [9]). Moreover, we obtain that $rank(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$, $rank(\mathcal{T}(X) : \mathcal{O}(X)) = 2$, and $rank(\mathcal{T}(X) : \mathcal{OP}(X)) = 1$ (see in [1] and [9]).

Let $Y = \{l_1 < l_2 < \cdots < l_m\}$ be a subchain of X with $|Y| = m$ and $1 < m < n$. Then we consider the following sets:

$$\begin{aligned}
\mathcal{T}(X, Y) &:= \{\alpha \in \mathcal{T}(X) : X\alpha \subseteq Y\},\\
\mathcal{O}(X, Y) &:= \{\alpha \in \mathcal{O}(X) : X\alpha \subseteq Y\},\\
\mathcal{OD}(X, Y) &:= \{\alpha \in \mathcal{OD}(X) : X\alpha \subseteq Y\},\\
\mathcal{OP}(X, Y) &:= \{\alpha \in \mathcal{OP}(X) : X\alpha \subseteq Y\},\\
\mathcal{OPR}(X, Y) &:= \{\alpha \in \mathcal{OPR}(X) : X\alpha \subseteq Y\}.
\end{aligned}$$

It is easy to see that all of them form to be subsemigroups of $\mathcal{T}(X)$ under the usual composition of functions. The semigroup $\mathcal{T}(X, Y)$ is defined by Symons and it is called the full transformation semigroup with restricted range (see in [11]). The other semigroups are introduced by Fernandes et al. in [4] and [5]. Moreover, the transformation semigroups with restricted range have been widely investigated (see in [4], [6] and [10]). The rank of $\mathcal{T}(X, Y)$ is equal to $S(n, m)$ is the stirling number of second kind [8]. In [4] and [5], the authors proved that $rank(\mathcal{O}(X,Y)) = \binom{n-1}{m-1} + |Y^\sharp|$ where $Y^\sharp$ is the set of captive elements and $rank(\mathcal{OP}(X,Y)) = rank(\mathcal{OPR}(X,Y)) = \binom{n}{m}$. In addition, the number of distinct kernels of the semigroup $\mathcal{OP}(X,Y)$ and the semigroup $\mathcal{OPR}(X,Y)$ with rank m are coincided (see in [1] and [5]). In [12], we obtained that $rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y))$ is equal to $S(n,m) - \binom{n-1}{m-1}$ or $S(n,m) - \binom{n-1}{m-1} + 1$ depends on set Y. In [13], the author determined the relative rank of the semigroup $\mathcal{T}(X,Y)$ modulo the semigroup $\mathcal{OD}(X,Y)$ and modulo the semigroup $\mathcal{OPR}(X,Y)$, respectively.

In this paper, we determine the relative rank of some subsemigroups of $\mathcal{T}(X,Y)$. In section 2.1, we describe the relative rank $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Finally, we determine the relative rank $\mathcal{OPR}(X,Y)$ modulo $\mathcal{OD}(X,Y)$ in section 2.2.

2. Main Results

In this section, we study the semigoup $\mathcal{OPR}(X,Y)$ and the subsemigroup $\mathcal{OD}(X,Y)$ in order to determine the relative rank of the semigoup $\mathcal{OPR}(X,Y)$ modulo the subsemigroup $\mathcal{OD}(X,Y)$.

2.1. Reletive Rank of $\mathcal{OP}(X,Y)$ Modulo $\mathcal{O}(X,Y)$

In this section, we study and describe the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ [2]. Define the set $\mathcal{P}$ by

$$\mathcal{P} := \{\ker(\alpha) : \alpha \in \mathcal{OP}(X,Y), rank(\alpha) = m\} \setminus \{\ker(\alpha) : \alpha \in \mathcal{O}(X,Y), rank(\alpha) = m\}.$$

Therefore, $\mathcal{P}$ is the set of all partitions of X into $m-1$ intervals and one block, which is the union of two intervals B_1 and B_n such that $1 \in B_1$ and $n \in B_n$. For each $P \in \mathcal{P}$, we fix an $\alpha_P \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y)$ with $\ker(\alpha_P) = P$. Then we can compute the cardinality of $\mathcal{P}$ as the following lemma.

Lemma 2.1. [2] $|\mathcal{P}| = \binom{n-1}{m}$.

Next, we define a transformation $\eta^* : X \to Y$ by

$$x\eta^* := \begin{cases} l_{i+1} & \text{if } l_i \le x < l_{i+1},\ 1 \le i < m \\ l_1 & \text{if } l_m \le x \text{ or } x < l_1. \end{cases}$$

It is easy to see that $\eta^* \in \mathcal{OP}(X,Y)$. Then we can state the main result as the following theorem.

Theorem 2.2. [2] $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_P : P \in \mathcal{P}\}, \eta^* \rangle$.

Therefore, we get the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ as follows:

Proposition 2.3. [2] *If $1 \notin Y$ or $n \notin Y$, then* $rank(\mathcal{OP}(X,Y) : \mathcal{O}(X,Y)) = \binom{n-1}{m}$.

Proposition 2.4. [2] *If $\{1,n\} \subseteq Y$, then* $rank(\mathcal{OP}(X,Y) : \mathcal{O}(X,Y)) = 1 + \binom{n-1}{m}$.

2.2. Reletive Rank of $\mathcal{OPR}(X,Y)$ Modulo $\mathcal{OD}(X,Y)$

In this section, we calculate the relative rank of $\mathcal{OPR}(X,Y)$ modulo $\mathcal{OD}(X,Y)$. Since we know that $\mathcal{O}(X,Y)$ is a proper subsemigroup of $\mathcal{OD}(X,Y)$, we obtain immediately that $rank(\mathcal{OPR}(X,Y):\mathcal{OD}(X,Y)) \leq rank(\mathcal{OPR}(X,Y):\mathcal{O}(X,Y))$.

For Y is a subchain of X with $|Y|=2$, we found that $\mathcal{OPR}(X,Y)=\mathcal{OP}(X,Y)$ and notice that $\{\ker(\alpha):\alpha\in\mathcal{O}(X,Y), rank(\alpha)=2\}=\{\ker(\alpha):\alpha\in\mathcal{OD}(X,Y), rank(\alpha)=2\}$. Then we obtain the following result.

Theorem 2.5. *$\mathcal{OPR}(X,Y)=\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle$.*

Proof. Since $|Y|=2$ and $\mathcal{O}(X,Y)\subseteq\mathcal{OD}(X,Y)$, we get that $\mathcal{OPR}(X,Y)=\mathcal{OP}(X,Y)=\langle\mathcal{O}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle\subseteq\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle$. Then $\mathcal{OPR}(X,Y)\subseteq\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle$. It is clear that $\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle\subseteq\mathcal{OPR}(X,Y)$. Altogether, we get that $\mathcal{OPR}(X,Y)=\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle$ ■

Remark 2.6. If $Y=\{1,n\}$, it is easy to see that $\eta^*\in\mathcal{OD}(X,Y)$.

By Proposition 2.3, Proposition 2.4, Theorem 2.5 and Remark 2.6, we obtain the following theorem.

Theorem 2.7. *If Y is a subchain of X with $|Y|=2$, then $rank(\mathcal{OPR}(X,Y):\mathcal{OD}(X,Y))=\binom{n-1}{2}$.*

Proof. Suppose that Y is a subchain of X with $|Y|=2$. If $1\notin Y$ or $n\notin Y$, then $rank(\mathcal{OPR}(X,Y):\mathcal{OD}(X,Y))=rank(\mathcal{OP}(X,Y):\mathcal{O}(X,Y))=\binom{n-1}{2}$ by Proposition 2.3 and Theorem 2.5. If $Y=\{1,n\}$, we obtain that $\eta^*\in\mathcal{OD}(X,Y)$ and $\mathcal{OPR}(X,Y)=\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\},\eta^*\rangle=\langle\mathcal{OD}(X,Y),\{\alpha_P:P\in\mathcal{P}\}\rangle$ by Remark 2.6 and Theorem 2.5, respectively. By Proposition 2.4, $rank(\mathcal{OPR}(X,Y):\mathcal{OD}(X,Y))=rank(\mathcal{OP}(X,Y):\mathcal{O}(X,Y))-1=\binom{n-1}{2}$. Altogether, we conclude that $rank(\mathcal{OPR}(X,Y):\mathcal{OD}(X,Y))=\binom{n-1}{2}$. ■

The rest of this work will consider the case that Y is a subchain of X with $|Y|\geq 3$. Notice that $\{\ker(\alpha):\alpha\in\mathcal{O}(X,Y), rank(\alpha)=m\}=\{\ker(\alpha):\alpha\in\mathcal{OD}(X,Y), rank(\alpha)=m\}$ and $\{\ker(\alpha):\alpha\in\mathcal{OP}(X,Y), rank(\alpha)=m\}=\{\ker(\alpha):\alpha\in\mathcal{OPR}(X,Y), rank(\alpha)=m\}$. Then we define the set $\mathcal{K}$ as the following

$$\mathcal{K}:=\{\ker(\alpha):\alpha\in\mathcal{OPR}(X,Y), rank(\alpha)=m\}\setminus\{\ker(\alpha):\alpha\in\mathcal{OD}(X,Y), rank(\alpha)=m\}.$$

For each $K\in\mathcal{K}$, we fix an $\alpha_K\in\mathcal{OPR}(X,Y)\setminus\mathcal{OP}(X,Y)$ with $\ker(\alpha_K)=K$. Then we obtain that $|\mathcal{K}|=|\mathcal{P}|$, i.e. $|\mathcal{K}|=\binom{n-1}{m}$. Next, we define a transformation $\beta_1^*:X\to Y$ and a transformation $\beta^*:X\to Y$ by

$$x\beta_1^*:=\begin{cases} l_1 & \text{if } l_1\leq x<l_2 \\ l_{m-i+1} & \text{if } l_{i+1}\leq x<l_{i+2} \text{ and } 1\leq i<m-1 \\ l_2 & \text{if } l_m\leq x \text{ or } x<l_1 \end{cases}$$

and

$$x\beta^* := \begin{cases} l_m & \text{if } x < l_1 \\ l_{m-i+1} & \text{if } l_i \le x < l_{i+1} \text{ and } 1 \le i < m \\ l_1 & \text{if } x \ge l_m \ . \end{cases}$$

It is easy to see that $\beta_1^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\ker(\beta_1^*) = \ker(\eta^*)$ and $\beta^* \in \mathcal{OD}(X,Y)$. We can compute that $(\beta_1^*\beta^*)^{m-1} = \eta^*$. Then the next proposition will show that $\mathcal{OD}(X,Y) \cup \{\alpha_K : K \in \mathcal{K}\} \cup \{\beta_1^*\}$ is a generating set for $\mathcal{OPR}(X,Y)$.

Theorem 2.8. $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$.

Proof. Let $\beta \in \mathcal{OPR}(X,Y)$. Then we will consider two cases.

Case 1. $\beta \in \mathcal{OP}(X,Y)$. For each $K \in \mathcal{K}$, we put $\theta_K := \alpha_K\beta_1^*$, where $\alpha_K \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. Therefore, $\theta_K \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y)$ with $rank(\theta_K) = m$ and $\ker(\theta_K) = \ker(\alpha_K)$. By the previous argument, let us put $\{\alpha_P : P \in \mathcal{P}\} = \{\theta_K : K \in \mathcal{K}\}$. By Theorem 2.2, we obtain that $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_P : P \in \mathcal{P}\}, \eta^* \rangle$. So, $\beta \in \mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_P : P \in \mathcal{P}\}, \eta^* \rangle \subseteq \langle \mathcal{OD}(X,Y), \{\theta_K : K \in \mathcal{K}\}, (\beta_1^*\beta^*)^{m-1} \rangle \subseteq \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$.

Case 2. $\beta \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. We put $\theta := \beta\beta_1^*$ and we observe that $\theta \in \mathcal{OP}(X,Y)$ as the product of two orientation-reversing transformations. From Case 1, we have $\theta \in \mathcal{OP}(X,Y) \subseteq \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$. Let $x \in X$. Then $x\theta\beta_1^* = x(\beta\beta_1^*)\beta_1^* = x\beta(\beta_1^*\beta_1^*) = x\beta(id|_Y) = x\beta$, i.e. $\beta = \theta\beta_1^*$. Hence, $\beta \in \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$.

Altogerther, we obtain that $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$. ∎

Lemma 2.9. *Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y)$ such that $\langle \mathcal{OD}(X,Y), A \rangle = \mathcal{OPR}(X,Y)$. Then there is a set $A' \subseteq A$ with $\{\ker\alpha : \alpha \in A'\} = \mathcal{K}$.*

Proof. Assume that there is $K \in \mathcal{K}$ with $K \notin \{\ker\alpha : \alpha \in A\}$. Since $\alpha_K \in \mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), A \rangle$, there are $\theta_1 \in \mathcal{OD}(X,Y) \cup A$ and $\theta_2 \in \mathcal{OPR}(X,Y)$ sucht that $\alpha_K = \theta_1\theta_2$. Since $rank(\alpha_K) = m$, we obtain that $\ker(\alpha_K) = \ker(\theta_1)$, i.e. $\ker(\theta_1) = K$. Hence, $\theta_1 \notin A$ by the assmption and $\theta_1 \notin \mathcal{OD}(X,Y)$ because $K \notin \{\ker(\alpha) : \alpha \in \mathcal{OD}(X,Y)\}$ that is a contradiction. Therefore, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{K}$. ∎

Then we contribute the main result of this section that consider two possibilities. Firstly, we consider the case that $|X \setminus Y| = 1$, i.e. $|X| = m+1$. So, $|\mathcal{K}| = \binom{m+1-1}{m} = 1$ that means $\mathcal{K} = \{K\}$. Then we obtain the following results.

Theorem 2.10. *If $|X \setminus Y| = 1$ and $1 \notin Y$ or $n \notin Y$, then we have $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = 1$.*

Proof. Since $1 \notin Y$ or $n \notin Y$ and $|\mathcal{K}| = 1$, we can assume without loss of generaltity that $\beta_1^* = \alpha_K$. Moreover, we know that $\beta^* \in \mathcal{OD}(X,Y)$. By Theorem 2.8, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \alpha_K \rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \le 1$. Since $\mathcal{OD}(X,Y)$ is a proper subsemigroup of $\mathcal{OPR}(X,Y)$, we get immediately that $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \ge 1$. Altogether, $rank(\mathcal{OP}(X,Y) : \mathcal{OD}(X,Y)) = 1$. ∎

Theorem 2.11. *If $|X \setminus Y| = 1$ and $\{1,n\} \subseteq Y$, then we have $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = 2$.*

Proof. By Theorem 2.8, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \alpha_K, \beta_1^* \rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \leq 2$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y)$ such that $\langle \mathcal{OD}(X,Y), A \rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.9, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{K}$. Therefore, $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \geq |A'| \geq |\mathcal{K}| = 1$. Assume that $\langle \mathcal{OD}(X,Y), \alpha_K \rangle = \mathcal{OPR}(X,Y)$. By the definition of η^*, we have $\eta^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y) \subseteq \mathcal{OPR}(X,Y)$ and $\ker(\eta^*) \notin \mathcal{K}$ because $(1,n) \notin \ker(\eta^*)$. Since $\eta^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y) \subseteq \mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \alpha_K \rangle$, there are $\theta_1, \ldots, \theta_l \in \mathcal{OD}(X,Y) \cup \{\alpha_K\}$ such that $\eta^* = \theta_1 \cdots \theta_l$. Since $rank(\eta^*) = m$ and $\{1,n\} \subseteq Y$, we obtain that $(1,n) \notin \ker(\theta_i)$ for all $i \in \{2,3,\ldots,l\}$ that implies $\theta_2 \cdots \theta_l \in \mathcal{OD}(X,Y)$. Since $rank(\eta^*) = m$, we get that $\ker(\eta^*) = \ker(\theta_1)$. If $\theta_1 \in \mathcal{OD}(X,Y)$, we have $\theta_1\theta_2 \cdots \theta_k \in \mathcal{OD}(X,Y)$ that is a contradiction. If $\theta_1 = \alpha_K$, we have $(1,n) \in \ker(\eta^*)$ that contradicts with $(1,n) \notin \ker(\eta^*)$. That is $\eta^* \notin \langle \mathcal{OD}(X,Y), \alpha_K \rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \geq 2$.

Altogether, we obtain that $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = 2$. ∎

From Theorem 2.10 and Theorem 2.11, we obtain immediately two corollaries as the following.

Corollary 2.12. *If* $|X \setminus Y| = 1$ *and* $1 \notin Y$ *or* $n \notin Y$*, then* $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \beta_1^* \rangle$.

Corollary 2.13. *If* $|X \setminus Y| = 1$ *and* $\{1,n\} \subseteq Y$*, then* $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \alpha_K, \beta_1^* \rangle$.

Secondly, we consider $|X \setminus Y| \geq 2$ and we can consider two cases as the case of $|X \setminus Y| = 1$.

Theorem 2.14. *If* $|X \setminus Y| \geq 2$ *and* $1 \notin Y$ *or* $n \notin Y$*, then we have* $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = \binom{n-1}{m}$.

Proof. Since $1 \notin Y$ or $n \notin Y$, we can assume without loss of generaltity that $\beta_1^* \in \{\alpha_K : K \in \mathcal{K}\}$. By Theorem 2.8, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\} \rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \leq |\{\alpha_K : K \in \mathcal{K}\}| = \binom{n-1}{m}$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y)$ such that $\langle \mathcal{OD}(X,Y), A \rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.9, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{K}$. Therefore, $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \geq |A'| \geq |\mathcal{K}| = \binom{n-1}{m}$.

Altogether, we obtain that $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = \binom{n-1}{m}$. ∎

Theorem 2.15. *If* $|X \setminus Y| \geq 2$ *and* $\{1,n\} \subseteq Y$*, then we have* $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = 1 + \binom{n-1}{m}$.

Proof. By Theorem 2.8, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \beta_1^* \rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \leq 1 + |\{\alpha_K : K \in \mathcal{K}\}| = 1 + \binom{n-1}{m}$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y)$ such that $\langle \mathcal{OD}(X,Y), A \rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.9, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{K}$. Therefore, $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \geq |A'| \geq |\mathcal{K}| = \binom{n-1}{m}$. Assume that $\langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\} \rangle =$

$\mathcal{OPR}(X,Y)$. By definition of η^*, we have $\eta^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y) \subseteq \mathcal{OPR}(X,Y)$ and $\ker(\eta^*) \notin \mathcal{K}$ because $(1,n) \notin \ker(\eta^*)$. Since $\eta^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OD}(X,Y) \subseteq \mathcal{OPR}(X,Y) = \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}\rangle$, there are $\theta_1, \ldots, \theta_l \in \mathcal{OD}(X,Y) \cup \{\alpha_K : K \in \mathcal{K}\}$ such that $\eta^* = \theta_1 \cdots \theta_l$. Since $rank(\eta^*) = m$ and $\{1,n\} \subseteq Y$, we obtain that $(1,n) \notin \ker(\theta_i)$ for all $i \in \{2,3,\ldots,l\}$ that implies $\theta_2 \cdots \theta_l \in \mathcal{OD}(X,Y)$. Since $rank(\eta^*) = m$, we get $\ker(\eta^*) = \ker(\theta_1)$. If $\theta_1 \in \mathcal{OD}(X,Y)$, we have $\theta_1\theta_2 \cdots \theta_k \in \mathcal{OD}(X,Y)$ that is a contradiction. If $\theta_1 = \alpha_K$ for some $K \in \mathcal{K}$, we have $(1,n) \in \ker(\eta^*)$ that contradicts with $(1,n) \notin \ker(\eta^*)$. That is $\eta^* \notin \langle \mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) \geq 1 + \binom{n-1}{m}$.

Consequently, we get that $rank(\mathcal{OPR}(X,Y) : \mathcal{OD}(X,Y)) = 1 + \binom{n-1}{m}$. ■

3. Conclusion

In this paper, we study and observe the transformation semigroup with restricted range $\mathcal{T}(X,Y)$ and its subsemigroups. In section 1, we introduce some notation and some definition of the transformation semigroups to use through this paper. In section 2.1, we study and describe the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. In section 2.2, we determine the relative rank of $\mathcal{OPR}(X,Y)$ modulo $\mathcal{OD}(X,Y)$. In future work, we plan to study other kind structure of the transformation semigroup with restricted range.

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