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Jordan f-Derivations on Prime and Semiprime Rings

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Abstract Let R be a ring and $f : R \to R$ be a mapping. A mapping $D : R \to R$ is said to be a *Jordan* $f-derivation$ on R if $D(a+b) = D(a)+D(b)$ for all $a, b \in R$, and $D(a^2) = D(a)f(a)+f(a)D(a)$ for all $a \in R$. In this paper, we present some conditions on prime ring or semiprime ring R that force the zero map to be the only Jordan f-derivation on R.

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1. Introduction

A ring R is said to be *prime* if for any $a, b \in R, aRb = \{0\}$ implies that $a = 0$ or $b = 0$; and *semiprime* if for any $a \in R, aRa = \{0\}$ implies that $a = 0$. A prime ring is obviously semiprime. An addition mapping $D : R \to R$, where R is an arbitrary ring, is said to be a *derivation* on R if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. The study of derivations in rings got interesting after Posner [12], who gave striking results on derivations of prime rings. Then several authors investigated the relationships between derivations and the structure of rings. The notion of derivation has also been generalized in various direction, such as Jordan derivation. An additive mapping $D : R \to R$, where R is an arbitrary ring, is said to be a $Jordan\ derivation$ if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Every derivation is obviously a Jordan derivation and the converse is in general not true. A famous result due to Herstein [10] proved that a Jordan derivation of a prime ring of characteristic not 2 must be a derivation. This result was extended to 2-torsion free semiprime rings by Cusak [9]. Several authors have studied this problem in the setting of prime and semiprime rings (see [2], [4] - [8], and [11]). Ashraf et al. [3] provided a historical overview of the study of derivations in rings.

Recently, Al-Omary and Nauman [1] studied generalized derivations on prime rings. An additive mapping $D : R \to R$ is called a *generalized derivation* associated with a derivation d if $D(xy) = D(x)y + xd(y)$ holds for all $x, y \in R$. They showed that if R is

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a prime ring satisfying certain identities involving a generalized derivation D associated with derivation d, then R becomes commutative, and in some cases out to be zero.

In this paper, we study a generalization of the Jordan derivation, namely, the *Jordan* $f-$ *derivation* D on a ring R, where $f : R \to R$ is a mapping, and investigate some results involving these derivations. Furthermore, we discuss some conditions on a prime ring or semiprime ring R that force the zero map to be the only Jordan f-derivation on R.

2. Results

In this section, we begin with the definition of Jordan f-derivation and investigate some interesting properties. Next, we provide some conditions on a prime ring or semiprime ring R that force the zero map to only Jordan f-derivation on R

Definition 2.1. Let R be a ring and $f : R \to R$ be a mapping. A mapping $D : R \to R$ is said to be *Jordan f-derivation* on R if

$$\begin{array}{llll} (i) & D(a+b) & = & D(a)+D(b) \text{ for all } a,b \in R, \\ (ii) & D(a^2) & = & D(a)f(a)+f(a)D(a) \text{ for all } a,b \in R. \end{array}$$

Example 2.2. Let $R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a,b \in C \right\}$. Then R is a ring under matrix addition and multiplication. Define mapping $f, D : R \to R$ by $f\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & b \end{bmatrix}$, and $D\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ for all $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in R$, respectively.

It can be verified that D is a Jordan f-derivation on R.

Form now on, for any x and y in a ring R the symbol $x \circ y$ stands for $xy + yx$, and the symbol $[x, y]$ stands for $xy - yx$.

Lemma 2.3. *Let R be a ring and $f : R \to R$ be an additive mapping. If $D : R \to R$ is a Jordan f-derivation on R, then $D(a \circ b) = D(a) \circ f(b) + f(a) \circ D(b)$ for all $a, b \in R$.*

Proof. Let $a, b \in R$. Then $(a + b)^2 = a^2 + a \circ b + b^2$ and

$$\begin{aligned} D(a^2 + a \circ b + b^2) &= D((a+b)^2) \\ &= D(a+b)f(a+b) + f(a+b)D(a+b) \\ &= D(a)f(a) + f(a)D(a) + D(a)f(b) + f(b)D(a) + f(a)D(b) \\ &\quad +D(b)f(a) + D(b)f(b) + f(b)D(b). \end{aligned}$$

But $D(a^2 + b^2) \;=\; D(a)f(a) + f(a)D(a) + D(b)f(b) + f(b)D(b).$

Hence

$$\begin{aligned} D(a\circ b) &= D(a)f(b)+f(b)D(a)+f(a)D(b)+D(b)f(a) \\ &= D(a)\circ f(b)+f(a)\circ D(b). \end{aligned}$$

■

Lemma 2.4. *Let R be a ring with $char(R)\neq 2$ and $f:R\to R$ be a homomorphism. If D is a Jordan f-derivation on R, then*

$$D(aba)=D(a)f(b)f(a)+f(a)D(b)f(a)+f(a)f(b)D(a)$$

for all $a,b\in R$.

Proof. Let $a,b\in R$ By Lemma 2.3, we have

$$\begin{aligned} D(a\circ(a\circ b)) &= D(a)\circ f(a\circ b)+f(a)\circ D(a\circ b) \\ &= D(a)f(a)f(b)+D(a)f(b)f(a)+f(a)f(b)D(a) \\ &\quad +f(b)f(a)D(a)+f(a)D(a)f(b)+f(a)f(b)D(a) \\ &\quad +f(a)f(a)D(b)+f(a)D(b)f(a)+D(a)f(b)f(a) \\ &\quad +f(b)D(a)f(a)+f(a)D(b)f(a)+D(b)f(a)f(a) \end{aligned}$$

$$\begin{aligned} D(a\circ(a\circ b)) &= D(a^2\circ b+2aba) \\ &= D(a)f(a)f(b)+f(a)D(a)f(b)+f(b)D(a)f(a) \\ &\quad +f(b)f(a)D(a)+f(a)f(a)D(b)+D(b)f(a)f(a)+2D(aba). \end{aligned}$$

Combining the above two equalities, we obtain

$$2D(aba)=2D(a)f(b)f(a)+2f(a)D(b)f(a)+2f(a)f(b)D(a)$$

Since $char(R)\neq 2$, $D(aba)=D(a)f(b)f(b)+f(a)D(b)f(a)+f(a)f(b)D(a)$. ■

Theorem 2.5. *Let R be nonzero semiprime ring with $char(R)\neq 2$. Suppose that $f:R\to R$ is an epimorphism and $D:R\to R$ is a Jordan f-derivation on R satisfying the following conditions:*

(*i*) $D(xy)=f(x)\circ D(y)$ *for all* $x,y\in R$ *and*

(*ii*) $D(y)f(x)=D(x)f(y)$ *for all* $x,y\in R$.

Then $D(x)=0$ for all $x\in R$.

Proof. Assume that

(2.5.1) $D(xy) = f(x) \circ D(y)$ *for all* $x, y \in R$.

(2.5.2) $D(y)f(x) = D(x)f(y)$ *for all* $x, y \in R$.

Replacing y by yx in (2.5.1) and using (2.5.1), yielding

$$D(xyx) = f(x) \circ (f(y) \circ D(x)) \text{ for all } x, y \in R.$$

By Lemma 2.4, we have

$$f(x)D(y)f(x) = f(x)D(x)f(y) + f(y)D(x)f(x) \text{ for all } x, y \in R.$$

Hence as (2.5.2), it follows that $f(y)D(x)f(x) = 0$ for all $x, y \in R$.

And therefore $D(x)f(x)f(y)D(x)f(x) = 0$ for all $x, y \in R$.

Since $f : R \to R$ is surjective, $D(x)f(x)RD(x)f(x) = \{0\}$ for all $x \in R$.

Since R is semiprime, $D(x)f(x) = 0$ for all $x \in R$.

Substituting $x + y$ for x where $y \in R$ in the previous equation, we obtain

$D(x+y)f(x+y) = 0$ for all $x, y \in R$ and thus $D(x)f(y) + D(y)f(x) = 0$ for all $x, y \in R$.

By (2.5.2), it follows that $2D(x)f(y) = 0$ for all $x, y \in R$.

Since $char(R) \neq 2$, $D(x)f(y) = 0$ for all $x, y \in R$.

Hence $D(x)f(y)D(x) = 0$ for all $x, y \in R$.

Since $f : R \to R$ is surjective, $D(x)RD(x) = \{0\}$ for all $x \in R$.

The semiprimeness of R forces $D(x) = 0$ for all $x \in R$. ■

Theorem 2.6. *Let R be nonzero semiprime ring with $char(R) \neq 2$. Suppose that $f : R \to R$ is an epimorphism and $D : R \to R$ is a Jordan f-derivation on R satisfying the following conditions:*

(i) $D(xy) = -(f(x) \circ D(y))$ *for all* $x, y \in R$ *and*

(ii) $D(y)f(x) = D(x)f(y)$ *for all* $x, y \in R$.

Then $D(x) = 0$ *for all* $x \in R$.

The proof of Theorem 2.6 is analogous to the proof of Theorem 2.5.

Theorem 2.7. *Let R be nonzero prime ring with $char(R) \neq 2$. Suppose that $f : R \to R$ is an isomorphism and $D : R \to R$ is a Jordan f-derivation on R satisfying the following conditions:*

(i) $D(xy) = [D(x), f(y)]$ *for all* $x, y \in R$ *and*

(ii) $D(y)f(x) = -D(x)f(y)$ *for all* $x, y \in R$.

Then $D(x) = 0$ *for all* $x \in R$.

Proof. Assume that

(2.7.1) $D(xy) = [D(x), f(y)]$ *for all* $x, y \in R$.

(2.7.2) $D(y)f(x) = -D(x)f(y)$ *for all* $x, y \in R$.

Replacing x by yx in (2.7.1) and using (2.7.1), yielding

$$D(y)f(x)f(y) + f(y)D(x)f(y) + f(y)f(x)D(y) = D(yx)f(y) - f(y)D(yx) \text{ for all } x, y \in R.$$

By using (2.7.1) and (2.7.2) we have $f(x)D(y)f(y) = 0$ for all $x, y \in R$.

Therefore $D(y)f(y)f(x)D(y)f(y) = 0$ for all $x, y \in R$.

Since $f : R \to R$ is surjective, $D(y)f(y)RD(y)f(y) = \{0\}$ for all $y \in R$.

The primeness of R forces

(2.7.3) $D(y)f(y) = 0$ *for all* $y \in R$.

Replacing x by yx in (2.7.2) and using (2.7.3), we have

(2.7.4) $D(y)f(x)f(y) = 0$ *for all* $x, y \in R$.

Replacing y by yz in (2.7.4) where $z \in R$ and using (2.7.4) , we have

$$D(z)f(y)f(x)f(z)f(y) = 0 \text{ for all } x, y, z \in R.$$

Since $f : R \to R$ is surjective, $D(z)f(y)Rf(z)f(y) = \{0\}$ for all $y, z \in R$

The primeness of R forces $D(z)f(y) = 0$ for all $y, z \in R$ or $f(z)f(y) = 0$ for all $y, z \in R$.

If $f(z)f(y) = 0$ for all $y, z \in R$, then $f(y)f(z)f(y) = 0$ for all $y, z \in R$.

Since $f : R \to R$ is surjective, $f(y)Rf(y) = \{0\}$ for all $y \in R$

By primeness of R, we obtain $f(y) = 0$ for all $y \in R$.

Since $f : R \to R$ is injective, $R = \{0\}$.

Therefore $D(z)f(y) = 0$ for all $y, z \in R$. Then $D(z)f(y)D(z) = 0$ for all $y, z \in R$.

Since $f : R \to R$ is surjective, $D(z)RD(z) = \{0\}$ for all $z \in R$.

The primeness of R forces $D(z) = 0$ for all $z \in R$ ∎

Theorem 2.8. *Let R be nonzero prime ring with $char(R) \neq 2$. Suppose that $f : R \to R$ is an isomorphism and $D : R \to R$ is a Jordan f-derivation on R satisfying the following conditions:*

(i) $D(xy) = -[D(x), f(y)]$ *for all* $x, y \in R$ *and*

(ii) $D(y)f(x) = -D(x)f(y)$ *for all* $x, y \in R$.

Then $D(x) = 0$ *for all* $x \in R$.

The proof of Theorem 2.8 is analogous to the proof of Theorem 2.7.

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