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Exact Solutions of Steady Plane Flows of an Incompressible Fluid of Variable Viscosity in The Presence of Unknown External Force Using (ε,ψ)– or (η,ψ)-Coordinates

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Abstract : It is indicated that equations governing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an external force for an arbitrary state equation admits an infinite set of solutions for the streamline pattern of the form $\frac{\eta-f(\xi)}{g(\xi)}$ = constant or $\frac{\xi-f(\eta)}{g(\eta)}$=constant defined in section (3).

Keywords : Incompressible fluid of variable viscosity; fluid flow in the presence of an external force; steady plane flows with unknown external force.
2000 Mathematics Subject Classification : 70D05 (2000 MSC)

1 Introduction

Researchers have studied equations governing the motion of an incompressible fluid of variable viscosity in the absence of external force and presented solution [1-7]. No work for an incompressible fluid of variable viscosity in the presence of external force for an arbitrary state equation to the best of our knowledge, has appeared in the literature at yet.

The objective of this paper is to present some exact solutions of equations describing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an external force for arbitrary state equation.

The plane of the paper is as follows. In section (2), we consider the non dimensional flow equation of the of

the fluid under considerations and give some results from differential geometry utilized in section (3) to transform flow equation in the streamline coordinate (ϕ,ψ). In section (3), we present flow equations into a new form in the (ϕ,ψ) system and outline the Labrapulu and Chandna approach [8]. In section (4) we determine the solutions of the flow equations when the expression/ term containing the force component are non zero.

2 Basic Flow Equations

The basic non-dimensional equations governing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an unknown external force with no heat addition are:

$$u_x + u_y = 0 \tag{2.1}$$

$$uu_x + vu_y = -p_x + \frac{1}{R_e}[(2\mu u_x)_x + (\mu(u_y + v_x))_y] + \lambda^* f_1 \tag{2.2}$$

$$uv_x + vv_y = -p_y + \frac{1}{R_e}[(2\mu v_y)_y + (\mu(u_y + v_x))_x] + \lambda^* f_2 \tag{2.3}$$

$$uT_x + vT_y = \frac{1}{R_e P_r}[T_{xx} + T_{yy}] + \frac{E_c}{R_e}[2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2] \tag{2.4}$$

$$\mu = \mu(T) \tag{2.5}$$

where are u,v the velocity components, p the pressure, μ the fluid viscosity, T the fluid temperature, R_e the Reynolds number, P_r the Prandtl number and E_c the Eckert number and f_1, f_2 are the components of the external force. In (2.2) and (2.3) λ^*, is a non-dimensional number, and in case of motion under the gravitational force, λ^* is called the Froude number (F_r). We define the following functions

$$\omega = v_x - u_y \tag{2.6}$$

$$L = p + \frac{u^2 + v^2}{2} - \frac{2\mu u_x}{R_e} \tag{2.7}$$

In term of these functions, the system of (2.1-2.5) is replaced by the following system

$$v_x + u_y = 0 \tag{2.8}$$

$$-v\omega = -L_x + \frac{[u(u_y + v_x)]_y}{R_e} + F_1 \tag{2.9}$$

$$u\omega = -L_y - \frac{4(\mu u_x)_y}{R_e} + \frac{[\mu(u_y + v_x)]_x}{R_e} + F2 \tag{2.10}$$

$$\omega = v_x - u_y \tag{2.11}$$

$$uT_x + vT_y = \frac{1}{R_eP_r}(T_{xx} + T_{yy}) + \frac{E_c}{R_e}[2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2] \quad (2.12)$$

$$\mu = \mu(T) \quad (2.13)$$

of six equations in eight unknowns $u,\ v,\ L,\ \mu, T, \omega, F_{1,} F_2$ as functions of $x,\ y$. The advantage of this system over the original system is that the order of the partial differential (2.2) and (2.3) has decreased from two to one. In (2.9) and (2.10) for convenience we have put $F_1 = \lambda^* f_{1,} F_2 = \lambda^* f_2$. Equation (2.8) implies the existence of a stream function $\psi(x,\ y)$ such that

$$u = \psi_y, \qquad v = -\psi_x \quad (2.14)$$

Let $\psi(x,y)$=constant defines the family of streamlines. Let us assume $\phi(x, y)$= constant to be some arbitrary family of curves such that it generate with $\psi(x,y)$=constant a curvilinear net (ϕ, ψ) in the physical plane. Let

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi) \quad (2.15)$$

define the curvilinear net in $(x, y)-$ plane and let the squared element of arc length along any curve be

$$ds^2 = E(\phi, \psi)d\phi^2 + 2F(\phi, \psi)d\phi d\psi + G(\phi, \psi)d\psi^2 \quad (2.16)$$

where

$$\left.\begin{array}{c} E = x_\phi^2 + y_\phi^2 \\ F = x_\phi x_\psi + y_\phi y_\psi \\ G = x_\psi^2 + y_\psi^2 \end{array}\right\} \quad (2.17)$$

Equation (2.15) can be solved to obtain

$$\phi = (x, y), \qquad \psi = (x, y) \quad (2.18)$$

such that

$$\left.\begin{array}{ll} x_\phi = J\psi_y, & x_\psi = -J\phi_y \\ y_\phi = -J\psi_x, & y_\psi = J\phi_x \end{array}\right\} \quad (2.19)$$

provided that $0 < |J| < \infty$, where J is the transformation Jacobian, and is defined as

$$J = x_\phi y_\psi - x_\psi y_\phi \quad (2.20)$$

If α is the angle of inclination of the tangent to the coordinate line $\psi =$ constant, directed in the sense of increasing ϕ, we have from differential geometry, the following results:

$$\left.\begin{array}{ll} J = \pm W, & \\ x_{\phi} = \sqrt{E}\cos\alpha, & y_{\phi} = \sqrt{E}\sin\alpha \\ x_{\psi} = \frac{F\cos\alpha - J\sin\alpha}{\sqrt{E}}, & y_{\psi} = \frac{F\sin\alpha + J\cos\alpha}{\sqrt{E}} \\ \alpha_{\phi} = \frac{J}{E}\Gamma_{11}^{2}, & \alpha_{\psi} = \frac{J}{E}\Gamma_{12}^{2} \end{array}\right\} \tag{2.21}$$

where

$$\left.\begin{array}{l} \Gamma_{11}^{2} = \frac{-FE_{\phi} + 2EF_{\phi} - EE\psi}{2W^{2}} \\ \Gamma_{12}^{2} = \frac{EG_{\phi} - FE_{\psi}}{2W^{2}} \\ W = \sqrt{EG - F^{2}} \end{array}\right\} \tag{2.22}$$

The three functions E, F, G of of ϕ, ψ satisfy the Gauss equation:

$$K = \frac{\left(\frac{W\Gamma_{11}^{2}}{E}\right)_{\psi} - \left(\frac{W\Gamma_{12}^{2}}{E}\right)_{\phi}}{W} \tag{2.23}$$

where K is the Gaussian curvature.

3 Transformation of Basic Flow Equations in The Streamlined Coordinate System (ϕ, ψ)

If the arbitrary curve ϕ(x, y) = constant and the streamlines ψ(x, y) = constant, generate a curvilinear net in the physical frame, the system of (2.8-2.13) is transformed to the following system:

$$q = \frac{\sqrt{E}}{W} \tag{3.1}$$

$$\begin{aligned} J\omega = & -JL_{\psi} + \left(\frac{\left(F^{2} - J^{2}\right)\sin 2\alpha}{2E} + \frac{FJ\cos 2\alpha}{E}\right)A_{\phi} - (F\sin\alpha\cos\alpha \\ & + J\cos^{2}\alpha)A_{\psi} + \left(\frac{2FJ\sin 2\alpha - F^{2}\cos 2\alpha + J^{2}\cos 2\alpha}{E}\right)M_{\phi} \\ & + (F\cos 2\alpha - J\sin 2\alpha)M_{\psi} + \frac{J}{\sqrt{E}}\left[F\left(F_{1}\cos\alpha\right.\right. \\ & \left.\left. + F_{2}\sin\alpha\right) + J\left(F_{2}\cos\alpha + F_{1}\sin\alpha\right)\right] \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 = & -JL_{\phi} + \left(F\sin\alpha\cos\alpha - J\sin^{2}\alpha\right)A_{\phi} - E\sin\alpha\cos\alpha A_{\psi} \\ & + (J\sin 2\alpha - F\cos 2\alpha)M_{\phi} + E\cos 2\alpha M_{\psi} \\ & + J\sqrt{E}\left(F_{1}\cos\alpha + F_{2}\sin\alpha\right) \end{aligned} \tag{3.3}$$

$$\left[\frac{\left(\frac{GT_{\phi}}{j} - \frac{FT_{\psi}}{j}\right)_{\phi} - \left(\frac{ET_{\psi}}{j} - \frac{FT_{\phi}}{j}\right)_{\psi}}{JR_{e}P_{r}}\right] = -\frac{E_{c}R_{e}\left(A^{2} + 4M^{2}\right)}{4\mu} + \frac{qT_{\phi}}{\sqrt{E}} \tag{3.4}$$

$$\omega = \left(\frac{\left(\frac{F}{W}\right)_\phi - \left(\frac{E}{W}\right)_\psi}{W} \right) \tag{3.5}$$

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \tag{3.6}$$

$$\mu = \mu(T) \tag{3.7}$$

wherein ϕ and ψ are considered as independent variables. This is a system of seven equations in ten unknown functions E, F, G, W, L, T, q, μ, F_1, F_2. In (3.2-3.3), the functions A and M are given by

$$A = \frac{4\mu}{JR_e} \begin{bmatrix} \left(\frac{F\sin\alpha + J\cos\alpha}{\sqrt{E}}\right)(q_\phi\cos\alpha - q\alpha_\phi\sin\alpha) \\ -\sqrt{E}\sin\alpha\,(q_\psi\cos\alpha - q\alpha_\psi\sin\alpha) \end{bmatrix} \tag{3.8}$$

$$M = \frac{\mu}{JR_e} \begin{bmatrix} q_\phi\left(\frac{-F\cos 2\alpha + J\sin 2\alpha}{\sqrt{E}}\right) + q_\psi\sqrt{E}\cos 2\alpha \\ +q\left(\frac{F\sin 2\alpha + J\cos 2\alpha}{\sqrt{E}}\right)\alpha_\phi \\ -q\alpha_\psi\sqrt{E}\sin 2\alpha \end{bmatrix} \tag{3.9}$$

Recently Labropulu et al. [8] presented a new approach for the determination of exact solutions of steady plane infinitely conducting MHD aligned flows. In their approach (ξ,ψ)-system and (η,ψ)-system are used to obtain exact solution of these flows where coordinates $\psi(x,y)$ is the stream function and $\omega = \xi(x,y) + \iota\eta(x,y)$ is an analytic function of $z = x+iy$. Labropulu and Chandna, following Martin [9] transform, their flow equations in (ϕ,ψ)-system where $\psi =$ constant represents family of streamline and $\phi =$ constant is an arbitrary family of curves. The system of flow equations becomes undetermined due to arbitrariness of the coordinate lines $\phi =$ constant Labropulu and Chandna made the system determinate by taking $\phi = \xi(x,y)$ or $\phi = \eta(x,y)$ where $\xi(x,y)$ and $\eta(x,y)$ are real and imaginary part of the analytical functions ω which is outlined blow:

Let

$$\omega = \xi(x,y) + \iota\eta(x,y) \tag{3.10}$$

be an analytic function of $z = x+iy$ where $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Since ω is analytic function of x and y, then real and imaginary part must satisfy Cauchy-Riemann equations.

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}, \qquad \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x} \tag{3.11}$$

The equations

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{3.12}$$

can be solved to get

$$x = x(\xi, \eta), \qquad y = y(\xi, \eta) \tag{3.13}$$

such that

$$\frac{\partial x}{\partial \xi} = J^* \frac{\partial \eta}{\partial y}, \ \frac{\partial x}{\partial \eta} = -J^* \frac{\partial \xi}{\partial y}, \ \frac{\partial y}{\partial \xi} = -J^* \frac{\partial \eta}{\partial x}, \ \frac{\partial y}{\partial \eta} = J^* \frac{\partial \xi}{\partial x} \tag{3.14}$$

provided that $0 <|J^*| < \infty$ where $J^*$is given by

$$J^* = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{3.15}$$

Equation (3.15), utilizing (3.11) and (3.14), gives

$$J^* = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \tag{3.16}$$

Equation (3.16), employing (3.12) and (3.14), yields

$$ds^2 = J^* \left(d\xi^2 + d\eta^2\right) \tag{3.17}$$

To analyse whether a family of curves $\frac{\eta - f(\xi)}{g(\xi)}$ = constant, can or can't be streamline in (ξ, ψ) coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\frac{\eta - f(\xi)}{g(\xi)} = \gamma(\psi) \qquad \gamma'(\psi) \neq 0 \tag{3.18}$$

$$\begin{aligned} ds^2 = \; & J^* \left\{1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2\right\} d\xi^2 \; + 2J^* \{f'(\xi) \\ & + g'(\xi)\gamma(\psi)\} g(\xi)\gamma'(\psi)\, d\xi d\psi + J^* g^2(\xi)\gamma'^2 d\psi^2 \end{aligned} \tag{3.19}$$

$$E = J^* \left\{1 + \left[f'(\xi) + g'(\xi)\gamma(\psi)\right]^2\right\} \tag{3.20}$$

$$G = J^* g^2(\xi)\gamma'^2(\psi) \tag{3.21}$$

$$F = J^* \left[f'(\xi) + g'(\xi)\gamma(\psi)\right] g(\xi)\gamma'(\psi) \tag{3.22}$$

$$W = J^* g(\xi)\gamma'(\psi) \tag{3.23}$$

$$J = J^* g(\xi)\gamma'(\psi) \tag{3.24}$$

Similarly to analyse whether a family of curves $\frac{\xi - f(\eta)}{m(\eta)}$ = constant, can or can't be streamline in (η, ψ) coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\frac{\xi - f\left(\eta\right)}{m\left(\eta\right)} = \gamma(\psi) \qquad \gamma(\psi) \neq 0 \tag{3.25}$$

$$ds^2 = \begin{array}{c} J^* \left\{1 + \left[k'\left(\eta\right) + m'\left(\eta\right)\gamma\left(\psi\right)\right]^2\right\} d\eta^2 + 2J^* \left\{k'(\eta)\right. \\ \left. + m'\left(\eta\right)\gamma\left(\psi\right)\right\} m\left(\eta\right)\gamma'\left(\psi\right) d\eta d\psi \\ + J^* m^2\left(\eta\right)\gamma'^2 d\psi^2 \end{array} \tag{3.26}$$

$$E = J^* \left\{1 + \left[k'\left(\eta\right) + m'\left(\eta\right)\gamma\left(\psi\right)\right]^2\right\} \tag{3.27}$$

$$G = J^* m^2\left(\eta\right)\gamma'^2\left(\psi\right) \tag{3.28}$$

$$F = J^* \left[k'\left(\eta\right) + m'\left(\eta\right)\gamma\left(\psi\right)\right] m\left(\eta\right)\gamma'\left(\psi\right) \tag{3.29}$$

$$W = J^* m\left(\eta\right)\gamma'\left(\psi\right) \tag{3.30}$$

$$J = -J^* m\left(\eta\right)\gamma'\left(\psi\right) \tag{3.31}$$

4 Solutions

In this section, we determine the solution of the flow equations by assuming analytical function $\omega = \xi + \iota\eta$ as follows:

(1) Let

$$\omega = \xi + \iota\eta = lnz \tag{4.1}$$

$$\left.\begin{array}{c} \xi = \frac{1}{2}\ln\left(x^2 + y^2\right) \\ \eta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}\right\} \tag{4.2}$$

Let us now determine the solution of the flow (3.1-3.7) along the curves defined in (3.18) and (3.25) as follows:

Example 4.1. *(Flows with $\xi =$ constant as streamlines)*

We let [8]

$$\xi = \gamma(\psi) \qquad \gamma'(\psi) \neq 0 \tag{4.3}$$

where $\gamma\left(\psi\right)$ is an unknown function and ξ is given by (4.2). Comparing (4.3) with (3.25), we get

$$k(\eta) = 0 \qquad m(\eta) = 1 \tag{4.4}$$

Utilizing (4.4) in (3.27-3.31),we get

$$\left.\begin{array}{c} E = J^* \\ F = 0 \\ G = J^* \gamma'^2(\psi) \\ J = -J^* \gamma'(\psi) \\ W = J^* \gamma'(\psi) \end{array}\right\} \tag{4.5}$$

where

$$J^* = e^{2\gamma(\psi)} \tag{4.6}$$

Equations (3.1-3.7), utilizing (4.5) and (4.6), become

$$q = \frac{1}{\gamma'(\psi) e^{\gamma(\psi)}} \tag{4.7}$$

$$\begin{array}{c} \omega = -L_\psi - \frac{\gamma' \sin 2\eta}{2} A_\eta - A_\psi \sin^2\eta + \gamma' M_\eta \cos 2\eta + M_\psi \sin 2\eta \\ -\gamma' e^{\gamma(\psi)} \left[F_1(\eta,\psi)\cos\eta - F_2(\eta,\psi)\sin\eta\right] \end{array} \tag{4.8}$$

$$\begin{array}{c} 0 = -L_\eta - A_\eta \cos^2\eta - A_\psi \frac{\sin\eta\cos\eta}{\gamma'} - M_\eta \sin 2\eta + M_\psi \frac{\cos 2\eta}{\gamma'} \\ +e^{\gamma(\psi)} \left[F_2(\eta,\psi)\cos\eta - F_1(\eta,\psi)\sin\eta\right] \end{array} \tag{4.9}$$

$$-\gamma'^3 T_{\eta\eta} - \gamma'^2 R_e P_r T_\eta - \gamma' T_{\psi\psi} + \gamma'' T_\psi = \frac{\gamma'^3 e^{2\gamma} R_e^2 P_r E_c \left(A^2 + 4M^2\right)}{4\mu} \tag{4.10}$$

$$\omega = \frac{\gamma''}{\gamma'^3 e^{2\gamma}} \tag{4.11}$$

wherein (4.8 - 4.10), the functions A and $M$are given by

$$A = \frac{2\mu}{R_e \gamma'(\psi) e^{2\gamma(\psi)}} \left(2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) \sin 2\eta \tag{4.12}$$

$$M = -\frac{\mu}{R_e \gamma'(\psi) e^{2\gamma(\psi)}} \left(2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) \cos 2\eta \tag{4.13}$$

In order to determine the solutions of the partial differential (4.8-4.10), we make use of the compatibility condition $L_{\eta\psi} = L_{\psi\eta}$ and this yield

$$\left.\begin{array}{c} \gamma'^2 Z_{\eta\eta} - \gamma' \left(2 - \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) Z_\psi - Z_{\psi\psi} = e^{\gamma(\psi)}[(-\gamma' F_{1_\eta}(\eta,\psi) \\ -F_{2_\psi}(\eta,\psi)\cos\eta + \left(2\gamma' F_1(\eta,\psi) + \gamma' F_{2_\eta}(\eta,\psi)\right. \\ \left. + F_{1_\psi}(\eta,\psi)\sin\eta\right] \end{array}\right\} \tag{4.14}$$

where

$$Z = \frac{\mu}{R_e e^{2\gamma}\gamma'} \left(2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) \tag{4.15}$$

Equation (4.14) is the equation which the viscosity μ and $\gamma(\psi)$ must satisfy for the flow under consideration.

Equation (4.14) possesses many solutions and we consider only those solutions for which the exact solution of (4.10) can be determined. These solutions are for the following cases:

(i) Case I $\gamma'' = 0$

(ii) Case II $\gamma'' \neq 0$

We study these cases separately

Case I

When $\gamma'' = 0$ we get

$$\gamma = a\psi + b \tag{4.16}$$

Equation (4.14) provides

$$\left.\begin{array}{c} a^2\chi_{\eta\eta} - 2a\chi_\psi - \chi_{\psi\psi} = e^{\gamma(\psi)}[(-aF_{1_\eta}(\eta,\psi) - F_{2_\psi}(\eta,\psi))\cos\eta \\ + \left(2aF_1(\eta,\psi) + aF_{2_\eta}(\eta,\psi) + F_{1_\psi}(\eta,\psi)\right)\sin\eta] \end{array}\right\} \tag{4.17}$$

where

$$\chi(\eta,\psi) = \frac{2e^{-2\gamma}\mu(\eta,\psi)}{aR_e} \tag{4.18}$$

In general, it is not possible to obtain the solution of (4.17). However if we assume

$$\left.\begin{array}{c} F_1(\eta,\psi) = e^{b_1\gamma(\psi)}F_{11}(\eta) \\ F_2(\eta,\psi) = e^{b_1\gamma(\psi)}F_{22}(\eta) \\ \chi(\eta,\psi) = e^{(1+b_1)\gamma(\psi)}H(\eta,\psi) \end{array}\right\} \tag{4.19}$$

then the solution of (4.17) is possible and (4.17), utilizing (4.19), yields

$$\left.\begin{array}{c} a^2H_{\eta\eta}(\eta,\psi) - H_{\psi\psi}(\eta,\psi) - (4a+2ab_1)\,H_\psi(\eta,\psi) - [2a^2(1 \\ +b_1) + a^2(1+b_1)^2]H(\eta,\psi) = e^{(1+b_1)\gamma(\psi)}Q(\eta) \end{array}\right\} \tag{4.20}$$

where $Q(\eta)$ is

$$\left.\begin{array}{c} Q(\eta) = [-aF_{11}'(\eta) - b_1aF_{22}(\eta)]\cos\eta + [2aF_{11}(\eta) \\ +aF_{22}'(\eta) + ab_1F_{11}(\eta)]\sin\eta \end{array}\right\} \tag{4.21}$$

The solution of (4.20) is

$$\left.\begin{array}{c} H(\eta,\psi) = d_2e^{\sqrt{3+4b_1+b_1^2}\eta} + d_3e^{-\sqrt{3+4b_1+b_1^2}\eta} \\ +\frac{1}{2a^2\sqrt{3+4b_1+b_1^2}}[e^{-\sqrt{3+4b_1+b_1^2}\eta}\ \{e^{2\sqrt{3+4b_1+b_1^2}\eta} \\ \int e^{-\sqrt{3+4b_1+b_1^2}\eta}(Q(\eta)+d_1)d\eta - \int e^{\sqrt{3+4b_1+b_1^2}\eta}(Q(\eta) \\ +d_1)d\eta\}] + e^{-a\psi(3+b_1)}(d_4 + d_5e^{2a\psi}) + \frac{d_1}{a^2(3+4b_1+b_1^2)} \end{array}\right\} \tag{4.22}$$

where $b_1, d_1, d_2, d_3, d_4, d_5$ are all non-zero arbitrary constant. We note that in (4.22) the function $Q(\eta)$ is arbitrary as $F_{11}$and F_{22} are both arbitrary, and therefore we can generate a large number of expressions for $H(\eta,\psi)$ and viscosity μ.

The temperature distribution T, for this example, satisfies

$$a^2 T_{\eta\eta}(\eta,\psi) - aR_eP_rT_\eta(\eta,\psi) + T_{\psi\psi}(\eta,\psi) = -2aR_eP_rE_cZ(\eta,\psi) \quad (4.23)$$

In general it is impossible to determine the solution of (4.23), however if we assume

$$T(\eta,\psi) = e^{(1+b_1)\gamma(\psi)}\left[T_1(\eta) + T_2(\psi)\right] \quad (4.24)$$

the (4.23) transforms into a simple equation whose solution can easily be determined. The transformed equation is

$$\left.\begin{array}{c} a^2T_{1\eta\eta}(\eta) - aR_eP_rT_{1\eta}(\eta) + T_{2\psi\psi}(\psi) + 2a(1+b_1)T_2(\psi) + a^2(1 \\ +b_1)^2[T_1(\eta)+T_2(\psi)] = -2aE_cR_eP_r\left[H_1(\eta) + H_2(\psi)\right] \end{array}\right\} \quad (4.25)$$

It is obvious from the equation that it possesses solution of the form

$$T(\eta,\psi) = T_1(\eta) + T_2(\psi) \quad (4.26)$$

and is given by

$$\left.\begin{array}{c} T(\eta,\psi) = e^{-\frac{\eta\left(-P_rR_e+\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}\right)}{2a}}\left(d_7 + d_8e^{\frac{\eta\left(\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}\right)}{a}}\right) \\ +\frac{e^{-\frac{\eta\left(P_rR_e-\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}\right)}{2a}}}{a\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}}\,(-\int e^{\eta(-P_rR_e+\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2})}\,(d_6 \\ -2aE_cP_rR_eH_1(\eta))d\eta + e^{\frac{\eta\left(\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}\right)}{a}}\times \\ \int e^{\frac{-\eta\left(P_rR_e+\sqrt{-4a^2(1+b_1)^2+P_r^2R_e^2}\right)}{2a}}\left(d_6 - 2aE_cP_rR_eH_1(\eta)\right)d\eta) \\ +\left(d_9+\psi d_{10}\right)e^{-a\psi(1+b_1)} + \frac{1}{a^2}(e^{-a\psi(1+b_1)}(-\psi\int e^{a\psi(1+b_1)}\,(d_6 \\ +2aE_cP_rR_eH_2(\psi))d\psi + \int\psi e^{a\psi(1+b_1)}\left(d_6 + 2aE_cP_rR_eH_2(\psi)\right)d\psi)) \end{array}\right\} \quad (4.27)$$

where

$$\left.\begin{array}{c} H_1(\eta) = d_2e^{\eta\sqrt{3+4b_1+b_1^2}} + d_3e^{-\eta\sqrt{3+4b_1+b_1^2}} \\ +\frac{1}{2a^2\sqrt{3+4b_1+b_1^2}}[e^{-\eta\sqrt{3+4b_1+b_1^2}}\{e^{2\eta\sqrt{3+4b_1+b_1^2}}\times \\ \int e^{-\eta\sqrt{3+4b_1+b_1^2}}(Q(\eta)+d_1)d\eta \\ -\int e^{\eta\sqrt{3+4b_1+b_1^2}}(Q(\eta)+d_1)d\eta\}] \end{array}\right\} \quad (4.28)$$

$$H_2(\psi) = e^{-a\psi(3+b_1)}(d_4 + d_5 e^{2a\psi}) + \frac{d_1}{a^2\left(3+4b_1+b_1^2\right)} \tag{4.29}$$

and $d_6, d_7, d_8, d_9, d_{10}$ are all non-zero arbitrary constants. We mention that $T(\eta,\psi)$ in (4.27) involves arbitrary $Q(\eta)$ and hence we can construct infinite number of expression for $T(\eta,\psi)$.The generalised energy function L is determined from (4.8) and (4.9). Equations (4.8) and (4.9), utilizing expression for A and M, become

$$L_\psi = -a\chi_\eta - \chi_\psi \sin 2\eta - ae^{\gamma(1+b_1)}\kappa(\eta) \tag{4.30}$$

$$L_\eta = -\sin 2\eta\ \chi_\eta - 4\cos^2\eta\ \chi - \frac{1}{a}\chi_\psi + e^{\gamma(1+b_1)}\kappa_1(\eta) \tag{4.31}$$

where

$$\kappa(\eta) = F_{11}(\eta)\cos\eta - F_{22}(\eta)\sin\eta \tag{4.32}$$

$$\kappa_1(\eta) = F_{22}(\eta)\cos\eta - F_{11}(\eta)\sin\eta \tag{4.33}$$

and $\chi(\eta,\psi)$ is defined by (4.18). The solution of (4.30) and (4.31) is

$$L = -\sin 2\eta\ \chi - 2\int \chi d\eta - \frac{1}{a}\eta\chi_\psi + \int e^{\gamma(1+b_1)}\kappa_1(\eta)\, d\eta \tag{4.34}$$

The pressure distribution can easily be determined from the definition of generalized energy function L from (2.7).

When $\gamma'' = 0$, we see that the expressions for viscosity μ, temperature T, and generalized energy function involve arbitrary functions and this indicates that the flow equations admit an infinite number of solutions.

Case II

When $\gamma'' \neq 0$, then $2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \neq 0$

Since, $2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0$, leads to the previous case. If we let $\gamma' = g$,then the compatibility Equation (4.14) becomes:

$$\left.\begin{array}{c} g^2 Z_{\eta\eta} - g\left(2 - \frac{g'(\psi)}{g^2(\psi)}\right) Z_\psi - Z_{\psi\psi} = e^{\gamma(\psi)}[(-gF_{1_\eta}(\eta,\psi) \\ -F_{2_\psi}(\eta,\psi))\cos\eta\ + (2gF_1(\eta,\psi) + gF_{2_\eta}(\eta,\psi) \\ +F_{1_\psi}(\eta,\psi))\sin\eta] \end{array}\right\} \tag{4.35}$$

The general solution of (4.35) is not possible. However, we can transform this equation into an equation whose solution is possible by substituting

$$\left.\begin{array}{r} F_1(\eta,\psi) = e^{b_2\gamma(\psi)}F_{11}(\eta) \\ F_2(\eta,\psi) = e^{b_2\gamma(\psi)}F_{22}(\eta) \\ Z(\eta,\psi) = e^{(1+b_2)\gamma(\psi)}\mathcal{H}(\eta,\psi) \end{array}\right\} \tag{4.36}$$

Equation (4.35), utilizing (4.36) transforms to the equation

$$\mathcal{H}_{\eta\eta}-\frac{1}{g^2}\mathcal{H}_{\psi\psi}-\frac{1}{g}\left[2\left(2+b_2\right)-\frac{g'}{g^2}\right]\mathcal{H}_{\psi}-\left(1+b_2\right)\left(3+b_2\right)\mathcal{H}=\frac{1}{g}\mathcal{G}(\eta) \quad (4.37)$$

where

$$\mathcal{G}\left(\eta\right)=\left[-F_{11}'\left(\eta\right)-b_2F_{22}\left(\eta\right)\right]\cos\eta+\left[F_{11}\left(\eta\right)+F_{22}'\left(\eta\right)+b_2F_{11}'\left(\eta\right)\right]\sin\eta \quad (4.38)$$

Equation (4.37) can further be simplified by defining a function

$$\frac{1}{g}=\mathcal{F} \quad (4.39)$$

On introducing the function in (4.37), we get

$$\mathcal{H}_{\eta\eta}-\mathcal{F}^2\mathcal{H}_{\psi\psi}-\mathcal{F}[4+2b_2+\mathcal{F}']\mathcal{H}_{\psi}-\left(1+b_2\right)\left(3+b_2\right)\mathcal{H}=\mathcal{F}\mathcal{G}\left(\eta\right) \quad (4.40)$$

Since our interest is in finding the solutions of the flow equations for which the function $\mathcal{G}\left(\eta\right)$ containing the force term is non-zero, we consider only those cases for which $\mathcal{G}\left(\eta\right)\neq 0$. The (4.37) for $\mathcal{G}\left(\eta\right)\neq 0$, has many solutions and here we consider some of them as follows:

When $b_2=-1,-3$

the (4.37) become

$$\mathcal{H}_{\eta\eta}-\mathcal{F}^2\mathcal{H}_{\psi\psi}-\mathcal{F}[2+\mathcal{F}']\mathcal{H}_{\psi}=\mathcal{F}\mathcal{G}\left(\eta\right) \qquad \text{for } b_2\text{=-1} \quad (4.41)$$

and

$$\mathcal{H}_{\eta\eta}-\mathcal{F}^2\mathcal{H}_{\psi\psi}-\mathcal{F}[-2+\mathcal{F}']\mathcal{H}_{\psi}=\mathcal{F}\mathcal{G}\left(\eta\right) \qquad \text{for } b_2\text{=-3} \quad (4.42)$$

Two solutions of (4.41), are determined and these are

$$\left.\begin{array}{c}\mathcal{H}\left(\eta,\psi\right)=\mathcal{F}\{t_1e^{2\eta\sqrt{2}}+t_2e^{-2\eta\sqrt{2}}+e^{-2\eta\sqrt{2}}[e^{4\eta\sqrt{2}}\int\frac{\mathcal{G}(\eta)e^{-2\eta\sqrt{2}}}{4\sqrt{2}}d\eta \\ -\int\frac{\mathcal{G}(\eta)e^{2\eta\sqrt{2}}}{4\sqrt{2}}d\eta]\}+\int e^{t_3-4\gamma(\psi)}d\psi+t_4\end{array}\right\} \quad (4.43)$$

and

$$\mathcal{H}=e^{\lambda\eta}\left[\frac{t_5(2\psi+t_6)}{8-\lambda^2}+(2\psi+t_6)^{\frac{-1+\sqrt{1+\lambda^2}}{2}}\left(t_7+t_8(2\psi+t_6)^{-\sqrt{1+\lambda^2}}\right)\right] \quad (4.44)$$

provided

$$\mathcal{F}=2\psi+t_6 \text{ and } \mathcal{G}(\eta)=-t_5e^{\lambda\eta} \quad (4.45)$$

The temperature distribution T, for $b_2 = -1$, satisfy the equation

$$T_{\eta\eta} - R_e P_r (2\psi + t_6) T_\eta + (2\psi + t_6)^2 T_{\psi\psi} + 2(2\psi + t_6) T_\psi = 0 \quad (4.46)$$

Two solutions of (4.46) are determined and these solutions are

$$\left.\begin{array}{c} T(\eta,\psi) = \eta + \text{BesselK}\left[0, \sqrt{P_r R_e\,(2\psi + t_6)}\right] t_{10} \\ +t_9 \text{Hypergeometric0F1Regularized}\left[1, \frac{1}{4} P_r R_e\,(2\psi + t_6)\right] \end{array}\right\} \quad (4.47)$$

$$\left.\begin{array}{c} T(\eta,\psi) = e^{-v\eta}[e^{-\psi\sqrt{\lambda P_r R_e}}(t_{11}\text{HypergeometricU}\left[\frac{1}{2}\right. \\ -\frac{\lambda^{\frac{3}{2}}}{4\sqrt{P_r R_e}}, 1, (2\psi + t_6)\sqrt{\lambda P_r R_e}\ ] \\ +t_{12}\text{LaguerreL}\left[\frac{1}{4}\left(-2 + \frac{\lambda^{\frac{3}{2}}}{\sqrt{\lambda P_r R_e}}\right), \sqrt{\lambda P_r R_e}\,(2\psi + t_6)\right])] \end{array}\right\} \quad (4.48)$$

where t_i, $i = 1, 2, ..., 12$, are all non-zero arbitrary constants.

The (4.42) admits solution for $\mathcal{F} = 2\psi + t_{13}$ and is given by

$$\mathcal{H}(\eta,\psi) = (2\psi + t_{13})\left[\mathcal{G}(\eta)\frac{\psi^2}{2} + \psi t_{14}\right] + t_{15} \quad (4.49)$$

The temperature distribution T in this case, satisfies the equation

$$T_{\eta\eta} - R_e P_r (2\psi + t_{13}) T_\eta + (2\psi + t_{13})^2 T_{\psi\psi} - 2(2\psi + t_{13}) T_\psi + 4T = 0 \quad (4.50)$$

The solution of (4.50) is

$$\left.\begin{array}{c} T(\eta,\psi) = [\frac{1}{4}\lambda(\text{BesselJ}[-\iota\lambda, \sqrt{\lambda P_r R_e\,(2\psi + t_{13})}] t_{17}\text{Gamma}[1 \\ -\iota\lambda] + \text{BesselJ}[\iota\lambda, \sqrt{\lambda P_r R_e\,(2\psi + t_{13})}] t_{16}\text{Gamma}[1 \\ +\iota\lambda]) P_r R_e\,(2\psi + t_{13})] e^{\lambda^{*}\eta} \end{array}\right\} \quad (4.51)$$

where $t_{13}, t_{14}, t_{15}, t_{16}, t_{17}$ are all non-zero arbitrary constants.

We note that $\mathcal{G}(\eta)$ involves the force components and they are arbitrary and this indicates that for the flows with $\xi =$ constant as streamlines the flow equations admit an infinite set of solutions.

Example 4.2. *(Flows with $\eta =$ constant as streamlines)*

Assume [8]

$$\eta = \gamma(\psi) \quad (4.52)$$

where $\gamma(\psi)$ is unknown function and η is given by (4.2).
Equation (4.52) and (3.18), give

$$f(\xi) = 0 \qquad g(\xi) = 1 \quad (4.53)$$

Equation (3.20-3.24),employing (4.53) yield

$$\left.\begin{array}{c} E = e^{2\xi} \\ F = 0 \\ G = e^{2\xi}\gamma'^2(\psi) \\ J = e^{2\xi}\gamma'(\psi) \\ W = e^{2\xi}\gamma'(\psi) \end{array}\right\} \tag{4.54}$$

Equation (3.1-3.7), employing (4.54), give

$$q = \frac{1}{e^{2\xi}\gamma'(\psi)} \tag{4.55}$$

$$\begin{array}{c} \omega = -L_\psi - \cos^2\gamma(\psi) A_\psi - M_\psi \sin 2\gamma(\psi) - \frac{\gamma'(\psi)\sin 2\gamma(\psi)}{2} A_\xi \\ +\gamma' M_\xi \cos 2\gamma(\psi) + \gamma' e^{\gamma(\psi)}[F_2(\xi,\psi)\cos\gamma(\psi) \\ +F_1(\xi,\psi)\sin\gamma(\psi)] \end{array} \tag{4.56}$$

$$\begin{array}{c} 0 = -L_\xi - A_\psi \frac{\cos\gamma(\psi)\sin\gamma(\psi)}{\gamma'(\psi)} + M_\psi \frac{\cos 2\gamma(\psi)}{\gamma'(\psi)} - \sin^2\gamma(\psi) A_\xi \\ +M_\xi \sin 2\gamma(\psi) + e^{\gamma(\psi)}[F_1(\xi,\psi)\cos\eta + F_2(\xi,\psi)\sin\eta] \end{array} \tag{4.57}$$

$$\gamma' T_{\xi\xi} + \left(\frac{T_\psi}{\gamma'(\psi)}\right)_\psi = -\frac{e^{2\xi}\gamma'(\psi) R_e^2 P_r E_c \left(A^2 + 4M^2\right)}{4\mu} + R_e P_r T_\xi \tag{4.58}$$

$$\omega = \frac{\gamma''(\psi)}{e^{2\xi}\gamma'^3(\psi)} \tag{4.59}$$

where the functions A and M are given by

$$A = \frac{4\mu}{R_e\gamma'(\psi)e^{2\xi}}\left(-\cos 2\gamma(\psi) + \frac{\gamma''(\psi)}{2\gamma'^2(\psi)}\sin 2\gamma(\psi)\right) \tag{4.60}$$

$$M = -\frac{2\mu}{R_e\gamma'(\psi)e^{2\xi}}\left(\sin 2\gamma(\psi) + \frac{\gamma''(\psi)}{2\gamma'^2(\psi)}\cos 2\gamma(\psi)\right) \tag{4.61}$$

Equations (4.56) and (4.57), employing (4.60) and (4.61), can be rewritten as

$$\left.\begin{array}{c} \frac{\gamma''}{\gamma'^3 e^{2\xi}} = -L_\psi - 4\gamma' X \sin 2\gamma - \gamma' Y_\xi + 4\cos^2\gamma\ X_\psi \\ -4\gamma'\cos^2\gamma\ Y - \sin 2\gamma\ Y_\psi \\ +\gamma' e^\gamma[\cos\gamma(\psi) F_2(\xi,\psi) \\ +F_1(\xi,\psi)\sin\gamma(\psi)] \end{array}\right\} \tag{4.62}$$

$$\left.\begin{array}{c} 0 = -\gamma' L_\xi - 4\gamma'\sin^2\gamma X_\xi - 4\gamma' X - \gamma'\sin 2\gamma\ Y_\xi - Y_\psi \\ +e^\gamma[\cos\eta F_1(\xi,\psi) + F_2(\xi,\psi)\sin\eta] \end{array}\right\} \tag{4.63}$$

where

$$X = \frac{\mu}{R_e e^{2\psi}\gamma'} \tag{4.64}$$

$$Y = \frac{\gamma''}{R_e e^{2\xi}\gamma'^3}\mu \tag{4.65}$$

The (4.62) and (4.63), employing compatibility conditions $L_{\psi\xi} = L_{\xi\psi}$ yield

$$\left.\begin{array}{c} \frac{2e^{-2\xi}\gamma''}{\gamma'} + 4\gamma'^2 X_\psi - \gamma'' Y_\psi + \gamma' Y_{\psi\psi} - \gamma'^3 Y_\xi + 4\gamma'^2 Y_{\xi\psi} \\ -\gamma'^3 Y_{\xi\xi} = \gamma'^2 e^{\xi}[(F_{2_\psi} - \gamma' F_{1_\xi} - 2\gamma' F_1)\sin\gamma \\ +(F_{1_\psi} - \gamma' F_{2_\xi})\cos\gamma] \end{array}\right\} \tag{4.66}$$

In general, in literature, there exists no method of finding the solution of (4.66). However, (4.66), can easily be transformed into an equation through transformations or substitutions whose some solutions are possible to determine using existing methods/ techniques. The substitutions

$$X(\xi,\psi) = e^{-2\xi+\lambda_1\psi} X_1(\xi,\psi) \tag{4.67}$$

$$Y(\xi,\psi) = e^{-2\xi+\lambda_2\psi} Y_1(\xi,\psi) \tag{4.68}$$

$$F_1(\xi,\psi) = e^{-3\xi} F^*_{11}(\psi) \tag{4.69}$$

$$F_2(\xi,\psi) = e^{-3\xi} F^*_{22}(\psi) \tag{4.70}$$

transform (4.66) into equation

$$\left.\begin{array}{c} -4\gamma'^2\left(\lambda_1 X_1 + X_{1_\psi}\right)e^{\lambda_1\psi} + 4\gamma'^2\left(X_{1_{\psi\xi}} + \lambda_1 X_{1_\xi}\right)e^{\lambda_1\psi} \\ +[\left(2\gamma'\lambda_2 - \gamma''\right)Y_{1_\psi} + \left(\gamma'\lambda_2 - \gamma''\right)\lambda_2 Y_1 + \gamma' Y_{1_{\psi\psi}} \\ +2\gamma'^3 Y_{1_\xi} - \gamma'^3 Y_{1_{\xi\xi}}]e^{\lambda_2\psi} = 2\frac{\gamma''}{\gamma'} + \gamma'^2\{[F^{*\prime}_{22}(\psi) \\ +\gamma' F^*_{11}(\psi)]\sin\gamma + [F^{*\prime}_{11}(\psi) + 3\gamma' F^*_{22}(\psi)]\cos\gamma\} \end{array}\right\} \tag{4.71}$$

Equation (4.71) following example 4.1 possesses for $\gamma'' = 0$ and $\gamma'' \neq 0$.

The solutions of (4.71), for these two cases are determined as follows:

Case-I

For $\gamma'' = 0$, the compatibility (4.71) becomes

$$\left.\begin{array}{c} 4a^2\left(-\lambda_1 X_1 - X_{1_\psi} + X_{1_{\psi\xi}} + \lambda_1 X_{1_\xi}\right)e^{\lambda_1\psi} + (2a\lambda_2 Y_{1_\psi} + a\lambda_2^2 Y_1 \\ +aY_{1_{\psi\psi}} + 2a^3 Y_{1_\xi} - a^3 Y_{1_{\xi\xi}})e^{\lambda_2\psi} = a^2\{[F^{*\prime}_{22}(\psi) \\ +aF^*_{11}(\psi)]\sin\gamma + [F^{*\prime}_{11}(\psi) + 3aF^*_{22}(\psi)]\cos\gamma\} \end{array}\right\} \tag{4.72}$$

The (4.72) admits solutions when $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$, we consider these as sub-cases, separately, as follows:

Sub-Case-I

For $\lambda_1 = \lambda_2$ and $Y_1 = Y_1(\psi)$, the (4.72) becomes

$$-\lambda_1 X_1 - X_{1_\psi} + X_{1_{\psi\xi}} + \lambda_1 X_{1_\xi} = \frac{\tau(\psi)\, e^{-\psi\lambda_1}}{4a^2} \tag{4.73}$$

where

$$\left.\begin{array}{c}\tau(\psi) = a^2\{[F_{22}^{*\prime}(\psi) + aF_{11}^{*}(\psi)]\sin\gamma + [F_{11}^{*\prime}(\psi) + 3aF_{22}^{*}(\psi)]\cos\gamma \\ +(2a\lambda_2 Y_{1_\psi} + a\lambda_2^2 Y_1 + aY_{1_{\psi\psi}})e^{\lambda_2\psi}\end{array}\right\} \tag{4.74}$$

The solution of (4.73) is

$$X_1(\xi,\psi) = \xi - \int \frac{\tau(\psi)}{4a^2} d\psi + h \quad \text{for} \quad \lambda_1 = \lambda_2 = 0 \tag{4.75}$$

and

$$X_1(\xi,\psi) = -\left[\int \frac{\tau(\psi)}{4a^2} d\psi + h_1\right] e^{-\lambda_1\psi} \text{ for } \lambda_1 = \lambda_2 \neq 0$$

$$\text{and } X_1(\xi,\psi) = X_1(\psi) \tag{4.76}$$

where h, h_1 are both non-zero arbitrary constants.

Sub-Case-II

To determine solution for $\lambda_1 \neq \lambda_2$, we set

$$\lambda_1 X_1 + X_{1\psi} = \theta(\psi) \tag{4.77}$$

$$Y_1 = d_1\xi + d_2 + Y_2(\psi) \tag{4.78}$$

On utilizing (4.77) and (4.78) in (4.72) we find

$$X_1 = e^{-\lambda_1\psi}\left[\int e^{-\lambda_1\psi}\theta(\psi)\, d\psi + h_2\right] \tag{4.79}$$

and $Y_2(\psi)$ satisfies

$$Y_2''(\psi) + 2\lambda_2 Y_2'(\psi) + \lambda_2^2 Y_2(\psi) = \varphi(\psi) - \lambda_2^2 d_1\xi \tag{4.80}$$

where

$$\left.\begin{array}{c}\varphi(\psi) = \frac{1}{ae^{\lambda_2\psi}}[a^2\{[F_{22}^{*\prime}(\psi) + aF_{11}^{*}(\psi)]\sin\gamma + [F_{11}^{*\prime}(\psi) \\ +3aF_{22}^{*}(\psi)]\cos\gamma\} + 4a^2 e^{\lambda_1\psi}\theta(\psi) - \left(a\lambda_2^2 d_2 \right. \\ \left. +2a^3 d_1\right)e^{\lambda_2\psi}\} - \lambda_2^2\xi d_1\end{array}\right\} \tag{4.81}$$

where $\varphi(\psi)$ is an unknown function . The solution of (4.80) is

$$Y_1 = d_1\xi + d_2 + \int\int \varphi(\psi)\, d\psi d\psi + \psi d_3 + d_4 \quad \text{for } \lambda_2 = 0 \tag{4.82}$$

and

$$\left.\begin{array}{c} Y_1 = e^{-\lambda_2\psi} d_5 + \psi e^{-\lambda_2\psi} d_6 + e^{-\lambda_2\psi}[\psi \int e^{\lambda_2\psi}\theta(\psi)\, d\psi \\ -\int \psi e^{\lambda_2\psi}\theta(\psi)\, d\psi] \qquad \text{for } \lambda_2 \neq 0, d_1 = 0 \end{array}\right\} \tag{4.83}$$

The temperature distribution T, for $\gamma'' = 0$, is

$$aT_{\xi\xi}(\xi,\psi) + \frac{1}{a}T_{\psi\psi}(\xi,\psi) - R_e P_r T_\xi(\xi,\psi) = -4E_c R_e P_r X(\xi,\psi) \tag{4.84}$$

where

$$X = \frac{\mu}{aR_e e^{2\xi}} \tag{4.85}$$

Two solution of (4.84), are obtained, and these are

$$T = \left.\begin{array}{c} \int\left[\int\left(\frac{h3 - 4E_c R_e P_r \xi}{a}\right) e^{-\frac{R_e P_r \psi}{a}} d\psi + h_4\right] e^{\frac{R_e P_r \psi}{a}} d\psi \\ +\int\left[\int (h_6 - 4E_c R_e P_r Z(\psi))\, a d\psi \right. \\ \left. + h_7\right] d\psi + h_9 \quad \text{for} \quad \lambda_1 = 0 \end{array}\right\} \tag{4.86}$$

and

$$T = h_{10}\xi + \int\left[\int aR_e P_r (h_{11} - 4E_c X_1(\psi)) d\psi + h_{12}\right] d\psi$$
$$+ h^* \text{ for } \lambda_1 \neq 0 \tag{4.87}$$

where $h_3, h_4, h_5, h_6, h_7, h_9, h_{10}, h_{11}, h_{12}, h^*$ are all non-zero arbitrary constant and $h_9 = h_8 + h_5$. We mentioned that in obtaining the solution in (4.86), we set $X(\xi,\psi) = \xi + Z(\psi)$ where $Z(\psi)$ is an unknown function. For solution (4.87), $X(\xi,\psi)$ is considered only function of ψ and we set $X(\xi,\psi) = X_1(\psi)$ For sub case-II, the solutions can easily be determined in the same manner as in example 4.1 and sub case-I.

Case-II

When $\gamma'' \neq 0$, compatibility equation is

$$\left.\begin{array}{c} -4\gamma'^2\left(\lambda_1 X_1 + X_{1_\psi}\right) e^{\lambda_1\psi} + 4\gamma'^2\left(X_{1_{\psi\xi}} + \lambda_1 X_{1_\xi}\right) e^{\lambda_1\psi} \\ +[(2\gamma'\lambda_2 - \gamma'')Y_{1_\psi} \; + [(\gamma'\lambda_2 - \gamma'')\lambda_2 Y_1 + \gamma' Y_{1_{\psi\psi}} \\ +2\gamma'^3 Y_{1_\xi} - \gamma'^3 Y_{1_{\xi\xi}}] e^{\lambda_2\psi} = H^*(\psi) \end{array}\right\} \tag{4.88}$$

where

$$H^*(\psi) = 2\frac{\gamma''}{\gamma'} + \gamma'^2\{[F_{22}^{*\prime}(\psi) + \gamma' F_{11}^*(\psi)]\sin\gamma$$

$$+[F_{11}^{*\prime}(\psi) + 3\gamma' F_{22}^*(\psi)]\cos\gamma\} \quad (4.89)$$

The (4.88) admits solutions when X_1, Y_1 both are function of ψ alone and when $X_1 = X_{11}(\psi), Y_1 = h_{13}\xi + Y_{11}(\psi)$.These solutions are

$$X_1 = e^{-\lambda_1\psi}\left[\int -\frac{h_{14} + H^*(\psi)}{4\gamma'^2}d\psi + h_{15}\right] \quad (4.90)$$

$$Y_1 = \int\left[\int \frac{h_{14}e^{\lambda_2\psi}}{\gamma'}d\psi + h_{16}\right]e^{-\lambda_2\psi}d\psi + h_{17} \quad (4.91)$$

$$X_1(\psi) = X_{11}(\psi) = e^{-\lambda_1\psi}\left[\int -\frac{h_{18} + H^*(\psi)}{4\gamma'^2}d\psi + h_{19}\right] \quad (4.92)$$

$$Y_1(\xi,\psi) = h_{13}\xi + \int e^{\lambda_2\psi}\left[\int \frac{h_{18} - 2\gamma'^3 h_{13}}{\gamma'}d\psi + h_{20}\right]d\psi + h_{21} \quad (4.93)$$

where $h_i, i = 13, \ldots, 21$ are all non-zero arbitrary constants. For (4.90) and (4.91), the function $\gamma(\psi)$ is

$$\gamma(\psi) = \frac{e^{\lambda_2\psi + h_{21}}}{\lambda_2} + h_{22} \quad (4.94)$$

where h_{22} is an non-zero arbitrary constant.

Following the procedure as in example 4.1, the temperature distribution T is given by

$$T = h_{23}\xi + \int \gamma'\left[\int G^*(\psi)\left(X_1(\psi) + \frac{\gamma'^2}{\gamma''}Y_1(\psi)\right)d\psi\right.$$

$$\left.+h_{23}R_eP_r\gamma'\psi + h_{24}\right]d\psi + h_{25} \quad (4.95)$$

where h_{23}, h_{24}, h_{25} are all non-zero arbitrary constant. The temperature distribution T for $X_1 = X_{11}(\psi), Y_1 = h_{13}\xi + Y_{11}(\psi)$ is given by

$$T = h_{27}\xi + \int\left[e^{\psi h_{26} + h_{28}}\int G^*(\psi)\,(X_{11} + \frac{\gamma'^2}{\gamma''}Y_1(\xi,\psi)\right.$$

$$\left.+h_{27}R_eP_r)d\psi + h_{29}e^{\psi h_{26}}\right]d\psi + h_{30} \quad (4.96)$$

where h_{26}, h_{27}, h_{28} and h_{29} are all non-zero arbitrary constant and $\gamma(\psi)$ for (4.96) is

$$\gamma(\psi) = \frac{1}{h_{26}} e^{(\psi h_{26} + h_{31})} + h_{32} \tag{4.97}$$

where h_{30}, h_{31}, h_{32} are all non-zero arbitrary constants and the function $G^*(\psi)$ is

$$G^*(\psi) = -\frac{E_c R_e P_r}{2} \left(4 + \frac{\gamma''^2}{\gamma'^4} \right) \tag{4.98}$$

We note here that the function $X_1 = X_{11}(\psi)$,and $Y_{11}(\psi)$ are both arbitrary.

(2) Assume

$$\omega = \xi + i\eta = a^* z + b^* \tag{4.99}$$

where $a^* = a_1 + ia_2$, $b^* = b_1 + ib_2$

Example 4.3. *(Flows with $\eta - \lambda\xi =$ constant as streamlines)*

Proceeding in the same manner as in examples 4.1 and 4.2, we find

$$\left.\begin{aligned} E &= \tfrac{1+\lambda^2}{a_1^2+a_2^2} \\ F &= \tfrac{\lambda\gamma'(\psi)}{a_1^2+a_2^2} \\ G &= \tfrac{\gamma'^2(\psi)}{a_1^2+a_2^2} \\ J^* &= \tfrac{1}{a_1^2+a_2^2} \\ W &= \tfrac{\gamma'(\psi)}{a_1^2+a_2^2} \\ J &= \tfrac{\gamma'(\psi)}{a_1^2+a_2^2} \end{aligned}\right\} \tag{4.100}$$

For this example, the (3.1 - 3.7), employing (4.100), become

$$q = \frac{\sqrt{(1+\lambda^2)\left(a_1^2 + a_2^2\right)}}{\gamma'(\psi)} \tag{4.101}$$

$$\left.\begin{aligned} \tfrac{(1+\lambda^2)\gamma''}{\gamma'^2} &= -\gamma'\beta_5 L_\psi + \gamma'^2\beta_7 A_\xi - \gamma'\beta_6 A_\psi + \gamma'^2\beta_8 M_\xi \\ &\quad + \gamma'\beta_9 M_\psi + \gamma'\beta_{10} F_1(\xi,\psi) + \gamma'\beta_{11} F_2(\xi,\psi) \end{aligned}\right\} \tag{4.102}$$

$$\left.\begin{aligned} 0 &= -\gamma'\beta_5 L_\xi + \gamma'\beta_1 A_\xi + \beta_3 A_\psi + \beta_2\gamma'(\psi) M_\xi + \beta_4 M_\psi \\ &\quad + \beta_{12}\gamma' F_1(\xi,\psi) + \gamma'\beta_{13} F_2(\xi,\psi) \end{aligned}\right\} \tag{4.103}$$

$$\left(\gamma' T_\xi - \lambda T_\psi\right)_\xi + \left(\frac{1+\lambda^2}{\gamma'(\psi)} T_\psi - \lambda T_\xi\right)_\psi - R_e P_r T_\xi = -\frac{\beta_5 \gamma' R_e^2 P_r E_c \left(A^2 + 4M^2\right)}{4\mu} \tag{4.104}$$

where

$$A = -\frac{4a_1 a_2 \gamma''}{R_e \gamma'^3}\mu \tag{4.105}$$

$$M = -\frac{\left(a_1^2 - a_2^2\right)\gamma''}{R_e \gamma'^3 \mu} \tag{4.106}$$

and, β_i, $i = 1, 2, ..., 13$ are all given in appendix-A.

The (4.102-4.104), employing (4.105) and (4.106), become

$$L_\psi = -\beta_{18}\frac{\gamma''}{\gamma'^3} + \beta_{19}\gamma' X_\xi + \beta_{20} X_\psi + \beta_{21} F_1(\xi,\psi) + \beta_{22} F_2(\xi,\psi) \tag{4.107}$$

$$L_\xi = -\beta_{14} X_\xi - \frac{\beta_{15}}{\gamma'} X_\psi + \beta_{16} F_1(\xi,\psi) + \beta_{17} F_2(\xi,\psi) \tag{4.108}$$

$$\gamma' T_{\xi\xi} - 2\lambda T_{\xi\psi} + \frac{1+\lambda^2}{\gamma'} T_{\psi\psi} - \frac{\gamma''\left(1+\lambda^2\right)}{\gamma'^2} T_\psi - R_e P_r T_\xi = -\frac{\gamma' R_e^2 P_r E_c X^2}{\beta_5 \mu} \tag{4.109}$$

where

$$X(\xi,\psi) = \frac{\gamma''(\psi)}{R_e \gamma'^3(\psi)}\mu(\xi,\psi) \tag{4.110}$$

and β_j, $j = 14, ..., 17$ are all given in appendix-A.

The compatibility condition in this case is

$$\left.\begin{array}{c} \frac{\beta_{15}}{\gamma'} X_{\psi\psi} + \beta_{19}\gamma' X_{\xi\xi} + (\beta_{20} + \beta_{14}) X_{\psi\xi} - \frac{\beta_{15}\gamma''}{\gamma'^2} X_\psi = -\beta_{21} F_{1\xi} \\ +\beta_{16} F_{1_\psi} + \beta_{17} F_{2_\psi} - \beta_{22} F_{2_\xi} \end{array}\right\} \tag{4.111}$$

where X is defined in (4.110) and β_j, $j = 18, ..., 22$ are all given in appendix-A.

We note here that in the previous example the flow equation possesses solutions for $\gamma''= 0$, and $\gamma''\neq 0$. If we consider the same cases in this example we see that when $\gamma'' = 0$, the (4.111) becomes

$$0 = -\beta_{21} F_{1\xi} + \beta_{16} F_{1_\psi} + \beta_{17} F_{2_\psi} - \beta_{22} F_{2_\xi} \tag{4.112}$$

We have already mentioned that we are interested in those solutions of the flow equations for which the expression containing the components of force is not equal to zero. The (4.112) contains the force component and therefore, we do consider the case for γ''=0, . For this example we consider only the case $\gamma''\neq 0$. When $\gamma''\neq 0$ the (4.111), using transformations, can be transformed into a differential equation whose solutions are easily determinable. These transformations are

$$\left.\begin{array}{c} X(\xi,\psi)=\theta_1\xi+X_1(\psi) \\ F_1=F_{11}(\psi) \\ F_2=F_{22}(\psi) \end{array}\right\} \tag{4.113}$$

and

$$\left.\begin{array}{c} X(\xi,\psi)=e^{\theta_1\xi}Z_1(\psi) \\ F_1=e^{\theta_2\xi}Q_1(\psi) \\ F_2=e^{\theta_2\xi}Q_2(\psi) \end{array}\right\} \tag{4.114}$$

For transformation (4.113), the (4.111) becomes

$$X_{1_{\psi\psi}}-\frac{\gamma''}{\gamma'}X_{1_\psi}=\frac{\gamma'}{\beta_{15}}\left[\beta_{16}F'_{11}(\psi)+\beta_{17}F'_{22}(\psi)\right] \tag{4.115}$$

whose solution is

$$X=\theta_1\xi+\int[\ \frac{\gamma'}{\beta_{15}}\int\{\beta_{16}F'_{11}(\psi)+\beta_{17}F'_{22}(\psi)\,d\psi\}\ d\psi+k_1\psi+k_2 \tag{4.116}$$

where $\gamma(\psi)$ is arbitrary.

For transformation (4.114), the (4.111), takes the form

$$\beta_{15}Z''_1(\psi)+\left[\gamma'(\beta_{20}+\beta_{14})\theta_2-\beta_{15}\frac{\gamma''}{\gamma'}\right]Z'_1(\psi)+\beta_{19}\gamma'^2\theta_2^2Z_1(\psi)=k(\psi) \tag{4.117}$$

where

$$k(\psi)=\gamma'\left[-\beta_{21}\theta_2Q_1(\psi)+\beta_{16}Q'_1(\psi)+\beta_{17}Q'_2(\psi)-\beta_{22}\theta_2Q_2(\psi)\right] \tag{4.118}$$

and the solution of (4.117) is

$$\left.\begin{array}{l} Z_1(\psi)=(k_3\beta_{15}+\psi(\beta_{14}-\beta_{20})\theta_2)^{\frac{1}{2}-\frac{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}}{2(\beta_{14}-\beta_{20})}}(k_5 \\ \quad +k_6(k_3\beta_{15}+\psi(\beta_{14}+\beta_{20})\theta_2)^{\sqrt{\frac{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}{(\beta_{14}+\beta_{20})}}} \\ \quad +\frac{1}{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}\theta_2}[(k_3\beta_{15}+\psi(\beta_{14} \\ \quad +\beta_{20})\theta_2)^{\frac{1}{2}-\frac{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}}{2(\beta_{14}+\beta_{20})}}(-\int(k_3\beta_{15}+\psi(\beta_{14} \\ \quad +\beta_{20})\theta_2)^{\frac{1}{2}(1+\frac{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}}{\beta_{14}+\beta_{20}})}\mathcal{K}(\psi)d\psi+(\int(k_3\beta_{15} \\ \quad +\psi(\beta_{14}+\beta_{20})\theta_2)^{\frac{1}{2}-\frac{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}}{2(\beta_{14}+\beta_{20})}}\mathcal{K}(\psi)d\psi)(k_3\beta_{15} \\ \quad +\psi(\beta_{14}+\beta_{20})\theta_2)^{\frac{\sqrt{-4\beta_{15}\beta_{19}+(\beta_{14}+\beta_{20})^2}}{\beta_{14}+\beta_{20}}})) \end{array}\right\} \tag{4.119}$$

provided

$$\gamma=-\frac{\beta_{15}}{\theta_2\left(\beta_{20}+\beta_{14}\right)}\ln\left|k_3-\frac{\theta_2\left(\beta_{20}+\beta_{14}\right)}{\beta_{15}}\psi\right|+k_4 \qquad (4.120)$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are all non-zero arbitrary constants.

The temperature distribution T, for $X\left(\xi,\psi\right)=\theta_1\xi+X_1(\psi)$ is

$$\left.\begin{array}{l} T=l_1\left(\frac{(m-R_eP_rl_2)\beta_5\gamma'^2}{2l_1\beta_5R_eP_r\gamma'^2-E_cR_eP_r\gamma''\theta_1}\right)^2+l_2\left(\frac{(m-R_eP_rl_2)\beta_5\gamma'^2}{2l_1\beta_5R_eP_r\gamma'^2-E_cR_eP_r\gamma''\theta_1}\right) \\ +\int\gamma'\left[\int\frac{(m-2\gamma'l_1)\beta_5\gamma'^2-E_cR_eP_r\gamma''}{(1+\lambda^2)\beta_5\gamma'^2}d\psi+l_3\right]d\psi+l_4 \quad \text{for} \quad \theta_1\neq 0 \end{array}\right\} \qquad (4.121)$$

When $\theta_1=0$ the temperature distribution T is

$$T=\int\gamma'\left[\int\frac{-E_cR_eP_r\gamma''}{\left(1+\lambda^2\right)\beta_5\gamma'^2}X_1\left(\psi\right)d\psi+l_5\right]d\psi+l_6 \qquad (4.122)$$

where $l_1, l_2, l_3, l_4, l_5, l_6$ are all non-zero arbitrary constants. In (4.122) the function $\gamma\left(\psi\right)$ is arbitrary.

When $X=e^{\theta_2\xi}Z1(\psi)$, the temperature distribution T, satisfies the differential equation

$$T_{\xi\xi}-\frac{2\lambda}{\gamma'}T_{\xi\psi}+\frac{1+\lambda^2}{\gamma'}T_{\psi\psi}-\frac{\gamma''\left(1+\lambda^2\right)}{\gamma'^2}T_{\psi}-\frac{R_eP_r}{\gamma'}T_{\xi}=-\frac{R_eP_rE_c\gamma''e^{\theta_2\xi}Z_1(\psi)}{\beta_5\gamma'^3} \qquad (4.123)$$

which on using $T=e^{\theta_2\xi}T_1(\psi)$, transforms into differential equation

$$\zeta^2T_{1_{\xi\xi}}+2\left(\frac{\theta_2\lambda}{\left(1+\lambda^2\right)l_7}+\frac{1}{2}\right)\zeta T_{1_{\xi}}+\left(\frac{\theta_2^2-R_eP_r\theta_2\zeta}{\left(1+\lambda^2\right)l_7}\right)T_1=-\frac{R_eP_rE_c\zeta Z_1(\psi)}{\beta_5\left(1+\lambda^2\right)l_7} \qquad (4.124)$$

where $Z_1(\psi)$ is an unknown function and $\zeta=c_1\psi+c_2$.The solution of (4.124) can easily be determined for given $Z_1(\psi)$. However the form of $Z_1(\psi)$, if not given which satisfies (4.124) can easily be determined by arranging the L.H.S of (4.124) in such a way that the combination of some of the terms vanish when T therein is replaced by a special function. For example, the (4.124) can be rewritten as

$$\zeta^2T_{1_{\xi\xi}}+2\left(\frac{\theta_2\lambda}{\left(1+\lambda^2\right)l_7}+\frac{1}{2}\right)\zeta T_{1_{\xi}}+\left(\zeta^2-d^2\right)T_1$$
$$+\left(d^2-\zeta^2+\frac{\theta_2^2-R_eP_r\theta_2\zeta}{\left(1+\lambda^2\right)l_7}\right)T_1=-\frac{R_eP_rE_c\zeta Z_1(\psi)}{\beta_5\left(1+\lambda^2\right)l_7} \qquad (4.125)$$

If we set

$$\zeta^2 T_{1_{\xi\xi}} + 2\zeta T_{1_\xi} + \left(\zeta^2 - d^2\right) T_1 = 0 \tag{4.126}$$

which is a Bessel equation in T_1 of order d whose solution is

$$T_1 = \frac{\text{BesselJ}\left[\frac{1}{2}\sqrt{1+4d^2}, \zeta\right] l_8}{\sqrt{\zeta}} + \frac{\text{BesselY}\left[\frac{1}{2}\sqrt{1+4d^2}, \zeta\right] l_9}{\sqrt{\zeta}} \tag{4.127}$$

On using (4.127) in (4.125), we get the expression for $Z_1(\psi)$ and is

$$Z_1(\psi) = \frac{1}{\zeta^{\frac{3}{2}} E_c l_7 P_r R_e}\left(\left(BesselJ\left[\frac{1}{2}\sqrt{1+4d^2}, \zeta\right] l_8\right.\right.$$

$$\left. +BesselY\left[\frac{1}{2}\sqrt{1+4d^2}, \zeta\right] l_9\right)\beta_5 \left(\begin{array}{c} \left(\zeta^2 - d^2\right)\left(1+\lambda^2\right) l_7^2 \\ +\left(\zeta P_r R_e - \theta_2\right)\theta_2 \end{array}\right) \tag{4.128}$$

provided $\frac{\theta_2 \lambda}{(1+\lambda^2) l_7} = \frac{1}{2}$, and l_7, l_8, l_9 are all non-zero arbitrary constant.

Following the above procedure we can find $Z_1(\psi)$in terms of other special functions. We indicate that as in previous solutions, the solutions in example 4.3 involve arbitrary functions and therefore the flow equations for flows with η-$\lambda\xi$= constant as streamlines admits a large number of exact solutions.

5 Conclusion

In this paper some exact solutions governing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an external force for an arbitrary state equation are presented. It is indicated that all solutions involve arbitrary function(s) for the streamline pattern of the form $\frac{\eta - f(\xi)}{g(\xi)}$= constant or $\frac{\xi - f(\eta)}{g(\eta)}$= constant and this arbitrariness indicates that the flow equations admits infinite set of solutions for these streamline patterns.

A Appendix

$$\beta_1 = -\frac{a_2(\lambda a_1 + a_2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \tag{A.1}$$

$$\beta_2 = -\frac{(\lambda a_1^2 + 2a_1a_2 - \lambda a_2^2)}{(1+\lambda^2)(a_1^2+a_2^2)^2} \tag{A.2}$$

$$\beta_3 = \frac{a_1a_2}{(a_1^2+a_2^2)^2} \tag{A.3}$$

$$\beta_4 = \frac{(a_1^2-a_2^2)}{(a_1^2+a_2^2)^2} \tag{A.4}$$

$$\beta_5 = \frac{1}{a_1^2+a_2^2} \tag{A.5}$$

$$\beta_6 = \frac{a_1(a_1-\lambda a_2)}{(1+\lambda^2)(a_1^2+a_2^2)^2} \tag{A.6}$$

$$\beta_7 = \frac{(\lambda a_1 + a_2)(a_1-\lambda a_2)}{(1+\lambda^2)^2(a_1^2+a_2^2)^2} \tag{A.7}$$

$$\beta_8 = -\frac{((-1+\lambda^2)a_1^2 + 4\lambda a_1a_2 - (-1+\lambda^2)a_2^2)}{(1+\lambda^2)^2(a_1^2+a_2^2)^2} \tag{A.8}$$

$$\beta_9 = \frac{(\lambda a_1^2 + 2a_1a_2 - \lambda a_2^2)}{(1+\lambda^2)(a_1^2+a_2^2)^2} \tag{A.9}$$

$$\beta_{10} = \frac{\lambda a_1 - a_2}{1+\lambda^2} \tag{A.10}$$

$$\beta_{11} = \frac{a_1 - \lambda a_2}{1+\lambda^2} \tag{A.11}$$

$$\beta_{12} = \frac{a_1}{(a_1^2+a_2^2)^2} \tag{A.12}$$

$$\beta_{13} = -\frac{a_2}{(a_1^2+a_2^2)^2} \tag{A.13}$$

$$\beta_{14} = \frac{4a_1a_2\beta_1 + \beta_2(a_1^2-a_2^2)}{\beta_5} \tag{A.14}$$

$$\beta_{15} = \frac{4a_1a_2\beta_3 + \beta_4(a_1^2-a_2^2)}{\beta_5} \tag{A.15}$$

$$\beta_{16} = \frac{\beta_{12}}{\beta_5} \tag{A.16}$$

$$\beta_{17} = \frac{\beta_{13}}{\beta_5} \tag{A.17}$$

$$\beta_{18} = \frac{1+\lambda^2}{\beta_5} \tag{A.18}$$

$$\beta_{19} = \frac{4a_1a_2\beta_7 + \beta_8(a_1^2 - a_2^2)}{\beta_5} \tag{A.19}$$

$$\beta_{20} = \frac{4a_1a_2\beta_6 - \beta_9(a_1^2 - a_2^2)}{\beta_5} \tag{A.20}$$

$$\beta_{21} = \frac{\beta_{10}}{\beta_5} \tag{A.21}$$

$$\beta_{22} = \frac{\beta_{11}}{\beta_5} \tag{A.22}$$

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