Some Remarks on Certain Types of Continuity

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Abstract: Infinite matrix transformation theory plays an important role in analysis, specially on the theory of summability. In this paper we introduce the concept of $A-$ continuity of a real valued function associated with a regular infinite matrix A and some result related to $A-$continuous function has been established. Also a characterization of $A-$continuous function is presented after introducing the notion of $A-$ oscillation of a function.

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1 Introduction

Let $A = (a_{mn})$ be an infinite matrix with elements in the real line $\mathbb{R}$ and $\{s_n\}$ be a sequence of real numbers. Let us consider $t_m = \sum_{n=1}^{\infty} a_{mn} s_n$ where it is assumed that the right hand series is convergent for all $m = 0, 1, 2, ...$. Then $\{t_m\}$ represents $A-$ transform of $\{s_n\}$ generated by the matrix $A = (a_{mn})$. If $t_m \to s$ as $m \to \infty$ then the sequence is said to be summable by the limitation method generated by the matrix (a_{mn}) and we shall write $s_n \xrightarrow{A} s$. If $t_m \to s$ as $m \to \infty$ whenever $s_n \to s$ then the matrix $A = (a_{mn})$ is said to be regular. A necessary and sufficient condition for a matrix $A = (a_{mn})$ to be regular is

(i) $\sup \sum_{n=1}^{\infty} |a_{mn}| < \infty$,

(ii) $\lim_{m \to \infty} a_{mn} = 0$ for all n,

(iii) $\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} = 1$.

The matrix $A = (a_{mn})$ satisfying these above conditions (i)-(iii) is called Toeplitz matrix or simply $T-$ matrix [3]. In particular, if $a_{mn} = \frac{1}{m}$, $n \le m$; and $= 0, n > m$ then it is called the first order Cesaro matrix which is designated by $C = (C, 1)$.

Let A be a regular matrix and $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then f is $A-$ continuous at the point $x_o \in \mathbb{R}$ if $f(x_n) \xrightarrow{A} f(x_o)$ whenever $x_n \xrightarrow{A} x_o$.

Otherwise f is said to be $A-$ discontinuous at the point x_o.

R.C.Buck [2] showed that the function $f : \mathbb{R} \to \mathbb{R}$ is linear if f is $C-$ continuous at at least one point of $\mathbb{R}$.

Posner [5] proved the following result:

Let A be a regular matrix and $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that $f(x_n)_n$ is $A-$summable whenever $\{x_n\}$ converges. Then f is continuous on $\mathbb{R}$ A regular matrix $A = (a_{mn})$ is said to satisfy the property (α) if there exists a sequence $\{s_n\}$ of $0's$ and $1's$ such that $A - \lim s_n$ exists and is equal to $\frac{1}{2}$.

2   Main Results

Theorem 2.1 *Let $A = (a_{mn})$ be a regular matrix with property (α). If f is $A-$ continuous at atleast one point x_o then f is continuous.*

Proof. Let f be $A-$ continuous at a point x_o. We first show that f is continuous at any $x \in \mathbb{R}$. Since A is the regular matrix having the property (α) so there exists a sequence α_n of $0's$ and $1's$ which is $A-$ summable to $\frac{1}{2}$. We consider the sequence x_n in $\mathbb{R}$ such that $x_n = \alpha_n x + (1 - \alpha_n)[\frac{x_0 - \frac{x}{2}}{\frac{1}{2}}]$; where $\alpha_n = 0$ or 1 for all n. Therefore $A - \lim x_n = x_o$. If possible, let f be not continuous at x. Then there exists a sequence h_n, $h_n \to 0$ such that $f(x + h_n) \to y \neq f(x)$ as $h_n \to 0$. Now we take another sequence y_n given by $y_n = \alpha_n(x + s_n) + (1 - \alpha_n)[\frac{x_0 - \frac{x}{2}}{\frac{1}{2}}]$. Then for every sequence $\{s_n\}$, $s_n \to 0$, $A - \lim y_n = x_o$. In particular, for $z_n = \alpha_n(x + h_n) + (1 - \alpha_n)[\frac{x_0 - \frac{x}{2}}{\frac{1}{2}}]$; $A - \lim f(z_n) = \frac{1}{2}y + \frac{1}{2}f(2x_0 - x)$. Since f is $A-$ continuous at x_0, $A - \lim f(x_n) = f(x_0) = A - \lim f(z_n)$. Hence $y = f(x)$ which is a contradiction. Hence f is continuous at the point x. $\square$

Note 2.1 The property (α) of the regular matrix A is essential for holding Theorem 2.1. We give the following example in support of this.

Example 2.1 Let A be an infinite identity matrix. Then clearly A is regular. But A does not have the property (α).

Now we consider the function f defined by
$$f(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2}, \\ 1 & \text{if } x \geq \frac{1}{2}. \end{cases}$$
Then f is discontinuous.

We shall show that f is $A-$continuous at the point 0,i.e. $A - \lim f(x_n) = f(0) = 0$ whenever $A - \lim x_n = 0$.

Now $A - \lim x_n = 0$ implies $x_n \to 0$. To show that $f(x_n) \to f(0) = 0$. There are two possibilities:

i)$| x_n |< \frac{1}{2}$ and ii)$| x_n |\geq \frac{1}{2}$.

i) $| x_n |< \frac{1}{2} \Rightarrow -\frac{1}{2} < x_n < \frac{1}{2}$. In this case $f(x_n) = f(0) = 0$. Hence f is $A-$ continuous at 0.

ii)$| x_n | \geq \frac{1}{2}$. As $x_n \to 0$, this case is impossible.
Hence the result.

Theorem 2.2 *Let $A = (a_{mn})$ be a regular matrix with the property (α). If f is $A-$continuous at every point of $\mathbb{R}$, then f is linear.*

Proof. By property (α) of A, there exists a sequence v_n of $0's$ and $1's$ such that $A - \lim t_n = \frac{(x+y)}{2}$. It can be shown that $A - \lim f(t_n) = \frac{[f(x)+f(y)]}{2}$.

Since f is $A-$ continuous at $\frac{(x+y)}{2}$, so $f(\frac{x+y}{2}) = \frac{[f(x)+f(y)]}{2}$ i.e. f is half-point linear. By theorem 2.1, f is also continuous. Hence by a well-known result of functional equations, we conclude that f is linear. $\qquad\square$

Note 2.2 In Theorem 2.2, $A-$ continuity of f at every point is essential for the linearity of f. In this regard we mention the example given by Anatoni and Salat [1], of the function which is everywhere continuous but not linear and which fails to be $A-$ continuous at more than one point.

Example 2.2 Let us consider the regular matrix,

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots\cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots\cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots \end{pmatrix}$$

Let the function be defined by

$$f(x) = \begin{cases} -1 & \text{if } x \leq -1, \\ x & \text{if } -1 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

The matrix A is regular with property (α). The function is continuous but nonlinear. Also note that the function f is $A-$continuous only at one point $x = 0$.

Theorem 2.3 *Let $A = (a_{mn})$ be a regular matrix. If $f : \mathbb{R} \to \mathbb{R}$ and the graph of f is a G_δ set, then the set of points of $A-$discontinuity of f is a closed set.*

Proof. Let G denote the graph of f. Since G is G_δ therefore $G = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open. Now $\widetilde{G} \setminus G = \bigcup_{n=1}^{\infty} (\widetilde{G} \setminus G_n)$. Each $\overline{G} - G_n$ is nowhere dense in $\widetilde{G}$. Therefore $\widetilde{G} \setminus G$ is a set of first category in$\widetilde{G}$. For any $x \in \widetilde{G} \setminus G$, we have $x + G \subset \widetilde{G} \setminus G$. Hence $x + G$ is of the first category and so also is G. Thus $\widetilde{G} = (\widetilde{G} \setminus G) \bigcup G$ is a set of first category in $\overline{G}$. Since $\widetilde{G} \subset \mathbb{R}$ and $\mathbb{R}^2$ is complete,

$\widetilde{G}$ is a set of second category in $\widetilde{G}$. Thus we arrive at a contradiction. Hence $\widetilde{G} = G$ i.e. G is closed. Therefore D_f, the set of points of discontinuity of f is a closed set.

Let D_{fA} denote the set of points of $A-$discontinuity of f. We will show that D_{fA} is precisely the set D_f. Let $x_o \in D_f$, then there exists a sequence $\{x_n\}$ of real numbers such that $x \to x_o$ but $f(x_n) \not\to f(x_o)$. If possible, let $x_o \notin D_{fA}$. If $t_n \to x_o$ is an arbitrary sequence then $t_n \to x_o$ and since $x_o \notin D_{fA}$ we get $f(t_n) \xrightarrow{A} f(x_0)$. Hence by Posner's result [5], f is continuous. Thus x_o is a point of continuity, which contradicts the choice of x_o. Hence D_f is a subset of D_{fA}. Again suppose that $x \in D_{fA}$. Let $x_n \to x$. Since A is regular we have $x_n \to x$. As $x \in D_{fA}$, $f(x_n)$ is not $A-$ summable to $f(x)$. If $x \notin D_f$ then x is a point of continuity of f. Therefore $x_n \to x$ implies $f(x_n) \to f(x)$ and this in fact means that $x_n \xrightarrow{A} x$ implies $f(x_n) \xrightarrow{A} f(x)$. This shows that $x \notin D_{fA}$; a contradiction. Hence D_{fA} is also a subset of D_f. Thus $D_{fA} = D_f$. $\quad\square$

Theorem 2.4 *If $S \subseteq \mathbb{R}$ be an F_σ set then there exists a regular infinite matrix A and a real function f such that $S = \{x \in \mathbb{R} : f$ is A- discontinuous at $x\}$.*

Proof. Since S is an F_σ set it is possible to find a function f such that $D_f = S$. Let us consider a subset B of $\mathbb{N}$, the set of natural number such that $\mathbb{N} \setminus B$ is an infinite set. We now order the set B as $n_1 < n_2 < n_3 ...$.

Now we define a regular matrix $T = (a_{mn})$ as follows

$$a_{mn} = \begin{cases} 0 & \text{if } n \neq n_m (m = 1, 2, ...), \\ 1 & \text{if } n = n_m. \end{cases}$$

Let $x \notin S$. Then x is a point of continuity of f. Let $x_n \xrightarrow{A} x$. Then $t_m = \sum_{n=1}^{\infty} a_{mn} x_n = x_{mn} \to x$. As f is continuous at x, hence $f(x_{mn}) \to f(x)$. This shows that x is a point of $A-$ continuity of f. Now $x \in S$. Then x is a point of discontinuity of f. Then there exists a sequence $\{x_n\}$ such that $x_n \xrightarrow{A} x$ but $f(x_n) \xrightarrow{A} f(x)$. However this means that there exists a sequence x_n such that $x_n \xrightarrow{A} x$ but $f(x_n) \xrightarrow{A} f(x)$. Hence f is $A-$ discontinuous at x and the theorem follows. $\quad\square$

Note 2.3 We have seen that if f is $A-$ continuous at at least one point then f is continuous. Now a question arises: Does any $A-$ continuous function posses derivative. We give an affirmative answer in the following theorem.

Theorem 2.5 *If f is $A-$ continuous at a point $x_o \in \mathbb{R}$ with A having property (α) then f is differentiable almost everywhere on a dense open subset of an interval.*

Proof. We have shown on the proof of theorem 1 that f is continuous on $\mathbb{R}$. Therefore f is continuous on an open interval $I \subset \mathbb{R}$. We know that every horizontal line meets the graph of a continuous function on some interval J of I in finite number of points. $\quad\square$

Cech [3] proved that such function must be monotone on some interval. Therefore we can decompose the interval J in countably many closed intervals in each of which f is monotone. Thus by Baire category theorem, there exists a sequence of intervals I_n whose union is dense in J, so that f is monotone in each I_n. Hence f is differentiable almost everywhere on dense open subset of I.

We now study the properties of some local types of uniform convergence which are sufficient for proving the $A-$ continuity at one point of the limit of a sequence of $A-$ continuous functions and also introduced the concept of the $A-$ oscillation of the function to give a characterization of the $A-$continuity of a function.

Definition 2.1 Let A be a regular matrix and $f : \mathbb{R} \to \mathbb{R}$ be a real valued function. Let $x_o \in \mathbb{R}$. Let $K = \{\{x_n\} : x_n \xrightarrow{A} x_o\}$. Then f is said to be locally $A-$ continuous at the point $x_o \in \mathbb{R}$ if there exists a neighborhood $V(x_o)$ of x_o such that for any sequence $\{x_n\}$ in $V(x_o) \bigcap K, f(x_n) \to f(x_0)$, i.e. there exists $V(x_o)$ and for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for all $n > m$ and all $\{x_n\} \in V(x_o) \bigcap K, | \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(x_o) | < \epsilon.$

The following example shows that a function which is locally $A-$ continuous at a point may not be $A-$ continuous at the point .

Example 2.3 Let $A = (a_{nk})$, where $a_{nk} = \frac{1}{n}$, if $k \geq n$ and $a_{nk} = 0$ if $k > n$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \geq 0 \\ -\frac{1}{2} & \text{if } x < 0 \end{cases}$$

Let us consider the point 0, then $f(0) = \frac{1}{2}$. Now consider the sequence $\{x_n\}$ where $x_n = (-1)^n$, with $x_n \xrightarrow{A} 0$. But $f(x_n) \xrightarrow{A} 0 \neq f(0) = \frac{1}{2}$. Thus f is not $A-$ continuous at the point 0.

Now let us take the neighborhood $[0, \infty)$ of 0,and $\{x_n\}$ any sequence in $[0, \infty) \bigcap K$, where $K = \{\{x_n\} : x_n \xrightarrow{A} 0\}$. Then $f(x_n) \to f(0) = \frac{1}{2}$. Thus f is locally $A-$ continuous at 0.

Definition 2.2 A sequence of functions $\{f_n\}$ is said to be uniformly convergent to a function f at a point if for a given $\epsilon > 0$, $\exists m \in \mathbb{N}$ such that for all $x \in N(a)$ (some neighborhood of a), $n \geq m$, $| f_n(x) - f(x) |< \epsilon$.

Theorem 2.6 *Let A be a regular infinite matrix and $\{f_n\}$ is uniformly convergent to f at a point a.If infinitely many f_n are locally $A - $ continuous at a, then f is also locally $A-$ continuous at a.*

Proof. $\{f_n\}$ being uniformly convergent to f at a, given $\epsilon > 0$, $\exists m \in \mathbb{N}$ and some neighborhood $N(a)$ of a, such that for all $x \in N(a)$, $n \geq m$,

$$| f_n(x) - f(x) |< \frac{\epsilon}{3} \tag{1}$$

As infinitely many f_n are locally $A-$ continuous at a, we can choose $s > m$ such that f_s is locally $A-$ continuous at a . Therefore there exists a neighborhood $V(a)$ of a such that for any sequence $\{x_n\}$ in $V(a) \bigcap K$ we have , $f_s(x_n) \xrightarrow{A} f_s(a_)$, i.e. , $\exists m_o \in \mathbb{N}$ such that $\forall n > m_0$,

$$| \sum_{n=1}^{\infty} a_{nk} f_s(x_k) - f_s(a) | < \frac{\epsilon}{3} \qquad (2)$$

Again A being regular, $\exists m_1 \in \mathbb{N}$ and a constant G such that $\forall n > m_1$,

$$\sum_{n=1}^{\infty} | a_{nk} | < G \qquad (3)$$

Let $M = max\{m_o, m_1\}$.

Now, for each $\{x_k\} \subset V(a) \bigcap N(a) \bigcap K$ and for $n > M$ we have

$| \sum_{k=1}^{\infty} a_{nk} f(x_k) - f(a) |$

$\leq | \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) | + | \sum_{k=1}^{\infty} a_{nk} f(x_k) - f_s(a) | + | f_s(a) - f(a) |$

$< | \sum_{k=1}^{\infty} a_{nk} f_s(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k) | + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$= | \sum_{k=1}^{\infty} a_{nk} [f_s(x_k) - f(x_k)] | + 2\frac{\epsilon}{3}$

$\leq \sum_{k=1}^{\infty} | a_{nk} | | [f_s(x_k) - f(x_k)] | + 2\frac{\epsilon}{3}$

$\leq G\frac{\epsilon}{3} + 2\frac{\epsilon}{3} = \epsilon\frac{(G+2)}{3}.$

Thus, f is locally $A-$continuous at a. $\qquad\square$

The concept of quasi-uniform convergence of a sequence of functions is well known. This concept plays an important part in the formulation of conditions for continuity of limit functions of sequence of continuous functions. By analogy with definition1, we give the following local version of it.

Definition 2.3 A sequence $\{f_n\}$ is said to be quasi-uniformly convergent to f at a point a, if there exists a neighborhood $N(a)$ of a such that $f_n(a)$ converges pointwise to $f(a)$ in $N(a)$, and for every $\epsilon > 0$, for each $n \in \mathbb{N}$, there exists $r(n) \in \mathbb{N}$ such that $min_{0 \leq i \leq r(n)} | f_{n+i}(x) - f(x) | < \epsilon$ for all $x \in N(a)$.

Theorem 2.7 *Let A be a regular infinite matrix. If $\{f_n\}$ is a sequence of functions locally $A-$continuous at the point a and converges quasi-uniformly to f at the point a, then f is also locally $A - $continuous at a.*

Proof. A being regular, $\exists m_0 \in \mathbb{N}$ and a constant G such that

$$\forall n > m, \sum_{k=1}^{\infty} |a_{nk}| < G \tag{4}$$

Let $\epsilon > 0$. Since, $f_n(a) \to f(a)$ as $n \to \infty$, $\exists m_1 \in \mathbb{N}$ such that

$$|f_n(a) - f(a)| < \frac{\epsilon}{3} \forall n \geq m_1 \cdots \tag{5}$$

Let $m = max(m_o, m_1)$.

Then, by quasi-uniform convergence of $\{f_n\}$ at the point a, $\exists$ a neighborhood $N(a)$ of $p \geq 1$ such that $\forall x \in N(a), \exists t \leq p$ for which

$$|f_{m=t}(x) - f(x)| < \frac{\epsilon}{3} \tag{6}$$

Again, by locally $A-$ continuity of functions f_{m+t} at the point a, $\exists$ a neighborhood $V(a)$ of a such that for all $\{x_n\}$ in $V(a) \bigcap K$,

$$|\sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - f_{m+t}(a)| < \frac{\epsilon}{3} \tag{7}$$

Let $\{x_k\}$ be an arbitrary sequence taken from $N(a) \bigcap V(a) \bigcap K$. Then we have,

$$|\sum_{k=1}^{\infty} a_{nk} f(x_k) - f(a)|$$

$$\leq |\sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k)|$$

$$+ |\sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - f_{m+t}(a)| + |f_{m+t}(a) - f(a)|$$

$$< |\sum_{k=1}^{\infty} a_{nk} f_{m+t}(x_k) - \sum_{k=1}^{\infty} a_{nk} f(x_k)| + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$\leq \sum_{k=1}^{\infty} |a_{nk}| |[f_s(x_k) - f(x_k)]| + 2\frac{\epsilon}{3}$$

$$\leq G\frac{\epsilon}{3} + 2\frac{\epsilon}{3} = \epsilon\frac{(G+2)}{3}.$$

The local $A-$continuity of f at the point a follows. $\square$

CHARACTERIZATION OF A-CONTINUOUS FUNCTION BY A-OSCILLATION:

We can give characterization of the points of $A-$continuity of a function .

Let f be a real-valued function and $A = (a_{mn})$ be a regular infinite matrix. For each point $a \in \mathbb{R}$ and an open neighborhood $V(a)$ of a and every natural

number N, we define the number $w_A(a, V(a), N) = \sup | \sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a) |$, where the supremum is taken over all $n \geq N$ and all sequence $\{x_k\}$ in $V(a)$ for which the transformed sequence $\{\sum_{k=1}^{\infty} a_{nk}x_k\}_n$ also belong to $V(a)$.

If $N < N'$ then obviously $w_A(a, V(a), N') \leq w_A(a, v(a)k, N)$. Then,

$$w_A(a, V(a), N) = inf w_A(a, V(a), N).$$

Definition 2.4 Let $U(a)$ be a system of all open neighborhood of a. Then $w_A(a) = inf w_A(a, V(a))$ is called the $A-$oscillation of the function f at the point a.

Theorem 2.8 *Let $r > 0$ and f be a function defined on $\mathbb{R}$. Then, the set $H_r = \{x \in \mathbb{R} : w_A(x) < r\}$ is an open set.*

Proof. Let x_o be any point of H_r. We shall show that x_o is an interior point of H_r. Since, $w_A(x_o) < r$, there is an open neighborhood $N(x_o)$ of x_0 such that $w_A(x_o, N(x_0)) < r$. By definition of $w_A(x_o, N(x_O))$, $\exists$ an integer $m \geq 1$ such that $w_A(x_o, N(x_o), m) = sup | \sum_{k=1}^{\infty} a_{nk}f(x_k) - f(x_o) |< r$. Since for each $x \in N(x_o)$, the open set $N(x_o)$ is a neighborhood of x. So, we have , $w_A(x, N(x_o), m) < r$, and this implies $w_A(x, N(x_o)) < r$ and hence $w_A(x) < r$, for each $x \in N(x_o)$. Hence $N(x_o) \subset H_r$, which shows that x_o is an interior point of H_r and therefore H_r is an open set. $\square$

Theorem 2.9 *A function f is $A-$continuous at the point a iff $w_A(a) = 0$.*

Proof. Let f be $A-$continuous at a. Then $f(x_n) \xrightarrow{A} f(a)$ whenever $x_n \xrightarrow{A} a$. So, given $\epsilon > 0, \exists n_o \in \mathbb{N}$ and $V \in U(A)$ such that for $n \geq n_o$,

$| \sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a) |< \epsilon$ whenever $\{x_n\} \in V(a) \bigcap K$.

From this we get immediately $w_A(a, V, n_o) \leq \epsilon$. Then $w_A(a, V) \leq \epsilon$ and hence $w_A(a) \leq \epsilon$. Now ϵ being arbitrary small $w_A(a) = 0$.

Conversely, let $w_A(a) = 0$. Then for each $\epsilon > 0 \exists N(a) \in U(a)$ such that $w_A(a, n(a)) < \epsilon$. Then by definition of $w_A(a, N(a)) \exists$ a natural number n_o such that $w_A(a, N(A), n_o) < \epsilon$ i.e. $sup | \sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a) |< \epsilon$ where the supremum is taken for all $n \geq n_o$ and all $\{x_k\} \in N(a) \bigcap K$ i.e. $| \sum_{k=1}^{\infty} a_{nk}f(x_k) - f(a) |< \epsilon$ for all $n \geq n_o$ and all $\{x_k\} \in N(a) \bigcap K$. Hence f is $A-$continuous at the point a. $\square$

Theorem 2.10 *The set of points of* $A-$*continuity of a function is of type* G_δ *in* $\mathbb{R}$.

Proof. Let C_f represent the set of points of $A-$continuity of f in $\mathbb{R}$. Then $C_f = \{x \in \mathbb{R} : w_A(x) = 0\}$.

By the above theorem we have $C_f = \bigcap U_k$ where $U_k = \{x \in \mathbb{R} : w_A(x) < \frac{1}{k}\} k \geq 1$.

But each U_k is open set in X. Then C_f is of type G_δ in $\mathbb{R}$. $\qquad\square$

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