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Regularity in Ternary Semihypergroups Induced by Subsets of Ternary Semigroups

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Abstract Ternary semihypergroups, which can be considered as a generalization of arbitrary ternary semigroups, are algebraic hyperstructures together with one ternary associative hyperoperation. In this article, we construct a new algebraic hyperstructure which can be formed as a ternary semihypergroup. In particularly, such a ternary semihypergroup is induced by a nonempty subset of the base set of the ternary semigroup. We investigate the regularity of the ternary semihypergroups. Furthermore, we present some of its characterizations via significant conditions.

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1. Introduction

The definition and the theory of ternary algebraic systems (including ternary semigroups), which is called triplexes, were introduced by D. H. Lehmer [1] in 1932. Previously, in 1904 these algebraic structures were studied by E. Kanser [2] who presented the idea of n-ary algebras, where n is a natural number. According to the study of Lehmer, the algebraic structures turn out to be commutative ternary groups. Moreover, the notion of ternary semigroups was first known to S. Banach who gave an example of a ternary semigroup which does not necessarily reduced to a (binary) semigroup (cf. [3]).

Nowadays, the algebraic structure of ternary semigroups and its interesting properties have been studied by many authors in both primary and advance ways. In 1955, J. Los [4] showed that each ternary semigroup can be embedded into a semigroup and also studied some of its algebraic properties. In 1936, the notion of regularity was introduced and studied by J. Von Neumann see, [5]. Based on the concept of regularity, F. Sioson [6] presented the notion of regular ternary semigroups in 1963. Moreover, he introduced the concept of ideals and redicals of ternary semigroups, see [7]. In 1990, M. L. Santiago [8]

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studied the notion of ideals of ternary semigroups and also investigated some properties of regular ternary semigroups. Up to 2008, T. K. Dutta et. al. [9] characterized regular ternary semigroups, completely regular ternary semigroups and intra-regular ternary semigroups via various ideals in ternary semigroups. Subsequently, a characterization of regular ternary semigroups, which was studied by using a concept of idempotent pairs, was studied by M. L. Santiago and S. Sri Bala [10] in 2010. Based on the well-known results the so-called Cayley's theorem, the authors studied the Cayley's theorem for ternary semigroups induced by semigroups, see [11]. Furthermore, the notion of ternary Menger algebras of rank n, which can be regarded as a suitable generalization of arbitrary ternary semigroups, was first introduced by the authors in 2022, see [12]. That is, the structures will be reduced to ternary semigroups by setting a special condition that a natural number n is equal to 1. For more information related to ternary semigroups, see [13–16].

In 1934, F. Marty [17] who have been credited as an initiator in the study of hyperstructure theory, when he defined hypergroups by using the notion of hyperoperations. The algebraic hyperstructures can be considered as a generalization of classical algebraic structures. On classical algebraic structures, the composition of two elements (or n elements) will be mapped to an element. While on algebraic hyperstructures, the composition of two elements (or n elements) will be mapped to a nonempty subset of the elements set. As a result, a number of various aspects of pure and applied mathematics related to hyperstructures are widely studied from the theoretical point of view and for its applications by many mathematicians in the following decades and nowadays. Furthermore, there are several books which contain aspects related to the algebraic hyperstructures such as P. Corsini and V. Leoreanu wrote a book on hyperstructure, which have been pointed out on some applications of hyperstructures in graph theory, hypergraph theory, fuzzy set theory, cryptography, codes, automata [18]. For another related book, see [19].

Analogous to semigroup and ternary semigroup theory, some algebraic properties and interesting results on semihypergroups and ternary semihypergroups have been investigated. Regularity of semihypergroups derived from subsets of semigroups are investigated by A. Asokkumar and M. Velrajan, see [20]. In 2010, B. Davvazz and V. Leoreanu-Fotea [21], studied binary relations on ternary semihypergroups and investigated some primary properties of compatible relations on ternary semihypergroups. In 2011, K. Naka and K. Hila [22] studied algebraic properties of regular, completely regular, and intra-regular ternary semihypergroups. In 2013, they presented and studied some classes of hyperideals in ternary semihypergroups, see [23]. Up to 2014, K. Naka et. al. [24] introduced and studied prime left, semiprime left and irreducible left hyperideals in ternary semihypergroups. In 2020, A. Talee et. al. [25] introduced a new approach towards int-soft hyperideals in ordered ternary semihypergroups. In addition, the intra-regular and regular po-ternary semihpergroups by means of int-soft hyperideal were characterized. Recently, the authors studied representations, translations and reductions on ternary semihypergroups, see [26]. For more results on ternary semihypergroups, see [27–29].

The main purposes of this article are to introduce ternary semihypergroups which are induced by some nonempty subsets of the base set of ternary semigroups and study regularity of such ternary semihypergroups. Furthermore, by using some significant conditions on the base set of ternary semigroups, some characterizations of regularity ternary semihypergroups are investigated in Section 3. Finally, we complete the article by giving conclusions and future works in Section 4.

2. Preliminaries

In this section, we recall the definitions and some algebraic properties of ternary semigroups and ternary semihyperoups. Furthermore, in order to revive our main results, some of their special elements are presented.

A ternary semigroup $(T, \bullet)$ is a pair of a nonempty set T together with a ternary operation $\bullet : T \times T \times T \longrightarrow T$ such that it satisfies the so-called a *ternary associative law*, i.e.,

$$\bullet(\bullet(a, b, c), d, e) = \bullet(a, \bullet(b, c, d), e) = \bullet(a, b, \bullet(c, d, e)),$$

for all $a, b, c, d, e \in T$. Analogous to the algebraic structure of semigroups, some special elements of ternary semigroups are introduced as follows:

Definition 2.1. ([11]). Let $(T, \bullet)$ be a ternary semigroup. An element $e \in T$ is said to be

(i) a left identity element if $\bullet(e, e, x) = x$ for all $x \in T$;
(ii) a right identity element if $\bullet(x, e, e) = x$ for all $x \in T$;
(iii) a lateral identity element if $\bullet(e, x, e) = x$ for all $x \in T$;
(iv) an identity element if $\bullet(e, e, x) = \bullet(x, e, e) = \bullet(e, x, e) = x$ for all $x \in T$.

Let $(T, \bullet)$ be a ternary semigroup with an identity element e. For $t \in T$, if there exist $u, v \in T$ (resp. $x, y \in T$) such that $\bullet(u, v, t) = e$ (resp. $\bullet(t, x, y) = e$), then the element t is called a *left (right) invertible element* of $(T, \bullet)$. In addition, an element $a \in T$ is called a *regular element* if there exists $x \in T$ such that $a = \bullet(a, x, a)$.

Definition 2.2. ([9]). Let I be a nonempty subset of a ternary semigroup $(T, \bullet)$. A subset I is said to be:

(i) a left ideal if $\bullet(T, T, I) \subseteq I$;
(ii) a right ideal if $\bullet(I, T, T) \subseteq I$;
(iii) a lateral ideal if $\bullet(T, I, T) \subseteq I$;
(iv) an ideal if it is a left, a right, and a lateral ideal of T.

An ideal I is called a proper ideal of T if it a proper subset of T, i.e., $I \neq T$.

Remark 2.3. According to the algebraic structure of semigroups and ternary semigroups, every semigroup can be induced to a ternary semigroup, but some ternary semigroups do not necessarily reduce to semigroups, which can be shown as follows:

Example 2.4. Some examples of ternary semigroups were already provided in [11, 14].

(i) Let $(T, \cdot)$ be a semigroup. Define a ternary operation $\bullet$ on the base set T by
$$\bullet(x, y, z) = \cdot(\cdot(x, y), z) \quad \text{for all } x, y, z \in T.$$
Then, an algebraic structure $(T, \bullet)$ forms a ternary semigroup.

(ii) Consider on the unit interval $I = [0, 1]$. Define a ternary operation $\bullet$ on I as follows:
$$\bullet(x, y, z) = \min\{x, y, z\} \quad \text{for all } x, y, z \in I.$$
Hence, $(I, \bullet)$ forms a ternary semigroup with an identity element 0.

(iii) Consider on a set $T = \{-i, i\} \subseteq \mathbb{C}$. Then, T together with a usual multiplication of complex numbers is not a semigroup. However, we can define a ternary operation $\bullet$ on T by $\bullet(x, y, z) = \cdot(\cdot(x, y), z)$ for all $x, y, z \in T$. Thus, $(T, \bullet)$ is a ternary semigroup.

(iv) On the set of all real numbers $\mathbb{R}$, we can define a ternary operation $\bullet$ as follows:

$$\bullet(x, y, z) = x - y + z \quad \text{for all } x, y, z \in \mathbb{R}.$$

So, $(\mathbb{R}, \bullet)$ is a ternary semigroup, while it can not be reduced to a semigroup.

Now, let T be a nonempty set. The set T with one mapping $\circ : T \times T \times T \longrightarrow P^*(T)$, where $P^*(T)$ is referred to the set of all nonempty subsets of T, is called a *ternary hypergroupoid* and the mapping $\circ$ is called a *ternary hyperoperation* on T. For nonempty subsets $X_i, i = 1, 2, 3$ of T,

$$\circ(X_1, X_2, X_3) = \bigcup_{x_i \in X_i} \circ(x_1, x_2, x_3).$$

A ternary hypergroupoid $(T, \circ)$ is called a *ternary semihypergroup* if the ternary hyperoperation $\circ$ satisfies the ternary associative law, i.e., $\circ(\circ(a, b, c), d, e) = \circ(a, \circ(b, c, d), e) = \circ(a, b, \circ(c, d, e))$ for all $a, b, c, d, e \in T$. It means that

$$\bigcup_{x \in \circ(a,b,c)} \circ(x, d, e) = \bigcup_{y \in \circ(b,c,d)} \circ(a, y, e) = \bigcup_{z \in \circ(c,d,e)} \circ(a, b, z).$$

According to algebraic structures of ternary semigroups and ternary semihypergroups, [19] each ternary semigroup can be constructed to be a ternary semihypergroup which follows from the following remark.

Remark 2.5. Let $(T, \bullet)$ be a ternary semigroup. Define a ternary hyperoperation $\circ$ on T by

$$\circ(x, y, z) = \{\bullet(x, y, z)\} \quad \text{for all } x, y, z \in T.$$

Thus, $(T, \circ)$ forms a ternary semihypergroup.

Similar to the relationship between semigroups and ternary semigroups, every semihypergroup can be induced to a ternary semihypergroup, while there are some ternary semihypergroups which do not necessarily reduced to semihypergroups. This statement follows from Remark 2.3 and Remark 2.5. Moreover, Example 2.4 (iii) – (iv) satisfies the statement, since we can identify the set $\{x\}$ with the element x.

In order to get our main results, some special elements on ternary semihypergroups are introduced. On a ternary semihypergroup $(T, \circ)$, an element $a \in T$ is called a *regular element* if there exists $x \in T$ such that $a \in \circ(a, x, a)$. A ternary semihypergroup with each element is regular, is called a *regular ternary semihypergroup*.

Here are some examples of ternary semihypergroups.

Example 2.6. (i) Let I be a unit interval $[0, 1]$. Define a ternary hyperoperation $\circ$ on I by

$$\circ(x, y, z) = \left[0, \tfrac{x \times y \times z}{3}\right] \quad \text{for all } x, y, z \in I,$$

where $\times$ is a usual multiplication over real numbers. So, $(I, \circ)$ is a ternary semihypergroup.

(ii) Consider on the set of all natural numbers $\mathbb{N}$ under the following ternary hyperoperation:

$$\circ(x, y, z) = \{m \in \mathbb{N} \mid m \geq \max\{x, y, z\}\} \quad \text{for all } x, y, z \in \mathbb{N}.$$

Then, $(\mathbb{N}, \circ)$ forms a ternary semihypergroup. Furthermore, all elements of $\mathbb{N}$ are regular, and hence $(\mathbb{N}, \circ)$ is regular.

Example 2.7. ([23]). Let $T = \{a, b, c, d, e, g\}$. Define a ternary hyperoperation $\circ$ on T by $\circ(x, y, z) = *(*(x, y), z)$ for all $x, y, z \in T$, where the binary hyperoperation $*$ is given as follows:

$*$	a	b	c	d	e	f
a	$\{a\}$	$\{a,b\}$	$\{c\}$	$\{c,d\}$	$\{e\}$	$\{e,f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c,d\}$	$\{c\}$	$\{c,d\}$	$\{c\}$	$\{c,d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e,f\}$	$\{c\}$	$\{c,d\}$	$\{e\}$	$\{e,f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

Then, $(T,\circ)$ forms a ternary semihypergroup.

3. Regularity of Ternary Semihypergroups Induced by Subsets of Ternary Semigroups

In this section, we construct a new algebraic hyperstructure which can be formed as a ternary semihypegroup. Then, we investigate regularity of the ternary semihypergroups. Moreover, some characterizations via some specific conditions of regularity of the ternary semihypergroups are presented.

Proposition 3.1. *Let $(T,\bullet)$ be a ternary semigroup and P be a nonempty subset of T. Define a ternary hyperoperation $\bullet_P$ on T by*

$$\bullet_P(x,y,z) = \bullet(\bullet(x,P,y),P,z) \quad \text{for all } x,y,z \in T, \tag{3.1}$$

i.e., $\bullet_P(x,y,z) = \{\bullet(\bullet(x,p,y),q,z) \mid p,q \in P,\ x,y,z \in T\}$. Then $(T,\bullet_P)$ forms a ternary semihypergroup.

Proof. Obviously, a mapping $\bullet_P$, which is defined as in (3.1), is a ternary hyperoperation from T into $P^*(T)$. Now, let $a,b,c,d,e \in T$ and $\emptyset \neq P \subseteq T$. Then

$$\begin{aligned}
\bullet_P(\bullet_P(a,b,c),d,e) &= \bullet(\bullet(\bullet(\bullet(a,P,b),P,c),P,d),P,e) \\
&= \{\bullet(\bullet(\bullet(\bullet(a,p,b),q,c),r,d),s,e) \mid p,q,r,s \in P,\ a,b,c,d,e \in T\} \\
&= \{\bullet(\bullet(\bullet(a,p,\bullet(b,q,c)),r,d),s,e) \mid p,q,r,s \in P,\ a,b,c,d,e \in T\} \\
&= \{\bullet(\bullet(a,p,\bullet(\bullet(b,q,c),r,d)),s,e) \mid p,q,r,s \in P,\ a,b,c,d,e \in T\} \\
&= \bullet(\bullet(a,P,\bullet(\bullet(b,P,c),P,d)),P,e) \\
&= \bullet_P(a,\bullet_P(b,c,d),e).
\end{aligned}$$

Similary to the above argument, we can show that

$$\bullet_P(\bullet_P(a,b,c),d,e) = \bullet_P(a,b,\bullet_P(c,d,e)).$$

Consequently, the structure $(T,\bullet_P)$ forms a ternary semihypergroup. ∎

According to Proposition 3.1, for every ternary semigroup $(T,\bullet)$, we can construct a new algebraic hyperstructure $(T,\bullet_P)$, where a ternary hyperoperation $\bullet_P$ is given as in (3.1) via a nonempty subset of the base set of a ternary semigroup $(T,\bullet)$, and then it forms a ternary semihypergroup. We call such a ternary semihypergroup $(T,\bullet_P)$, a *ternary semihypergroup induced by a nonempty subset of a ternary semigroup.*

Now, let $(T,\bullet)$ be a ternary semigroup. If $a \in T$ is regular, then we denote the set $\{x \in T \mid a = \bullet(a,x,a)\}$ by V_a. By using the set V_a, we have some characterizations of regularity of ternary semihypergroups induced by subsets of ternary semigroups as follows:

Proposition 3.2. *Let $(T, \bullet)$ be a ternary semigroup and $\emptyset \neq P \subseteq T$. An element $a \in T$ is regular in a ternary semihypergroup $(T, \bullet_P)$ if and only if the element a is regular in $(T, \bullet)$ and $V_a \cap \bullet(P, T, P) \neq \emptyset$.*

Proof. ($\Longrightarrow$) Assume that an element $a \in T$ is a regular element in $(T, \bullet_P)$. Then, there exists $x \in T$ such that $a \in \bullet_P(a, x, a)$, which means that $a \in \bullet(\bullet(a, P, x), P, a)$. So, there are $p, q \in P$ such that

$$a = \bullet(\bullet(a, p, x), q, a) = \bullet(a, \bullet(p, x, q), a) = \bullet(a, y, a),$$

where $y = \bullet(p, x, q) \in \bullet(P, T, P)$. It implies that a is a regular element in $(T, \bullet)$. Moreover, $y \in \bullet(P, T, P)$ and $V_a \cap \bullet(P, T, P) \neq \emptyset$.

($\Longleftarrow$) Assume that an element $a \in T$ is regular in $(T, \bullet)$ and $V_a \cap \bullet(P, T, P) \neq \emptyset$. By the assumption, there exists $x \in V_a \cap \bullet(P, T, P)$. So, we immediately obtain that $a = \bullet(a, x, a)$ and $x = \bullet(p, y, q)$ for some $p, q \in P$, $y, \in T$ and hence

$$a = \bullet(a, x, a) = \bullet(a, \bullet(p, y, q), a) = \bullet(\bullet(a, p, y), q, a) = \bullet_P(a, y, a).$$

Therefore, the element $a \in T$ is regular in $(T, \bullet_P)$. ∎

Corollary 3.3. *Let $(T, \bullet)$ be a ternary semigroup and $\emptyset \neq P \subseteq T$. If $a \in T$ is regular in a ternary semihypergroup $(T, \bullet_P)$, then the element a is regular in $(T, \bullet)$.*

Proof. The proof follows from Proposition 3.2. ∎

Corollary 3.4. *Let $(T, \bullet)$ be a ternary semigroup and $\emptyset \neq P \subseteq T$. If $(T, \bullet_P)$ is a regular ternary semihypergroup, then $(T, \bullet)$ is regular.*

Proof. The proof follows from Corollary 3.3. ∎

The following example is given to show that the converse of Corollary 3.3 need not be true.

Example 3.5. Let $\mathbb{N}$ be the set of all natural numbers. Define a ternary operation $\bullet$ on $\mathbb{N}$ as follows:

$$\bullet(a, b, c) = \max\{a, b, c\} \quad \text{for all } a, b, c \in \mathbb{N}.$$

It is easy to see that $(\mathbb{N}, \bullet)$ forms a ternary semigroup. Indeed, for each $a \in \mathbb{N}$, we have

$$\bullet(a, a, a) = \max\{a, a, a\} = a,$$

which means that each element of $\mathbb{N}$ is regular, i.e., $a = \bullet(a, a, a)$ for all $a \in \mathbb{N}$. Thus, $(\mathbb{N}, \bullet)$ is a regular ternary semigroup. Now, let us consider on the set $P = \{m + 1, m + 2, \ldots, m + m\}$, where m is a fixed element in $\mathbb{N}$. We see that the element $m \in \mathbb{N}$ is not regular in a ternary semihypergroup $(\mathbb{N}, \bullet_P)$. Moreover, $V_m = \{1, 2, \ldots, m\}$ and $V_m \cap \bullet(P, \mathbb{N}, P) = \emptyset$. It means that there are subset P of the base set of the ternary semigroup $(\mathbb{N}, \bullet)$ such that the converse of Corollary 3.3 need not be true.

Theorem 3.6. *Let $(T, \bullet)$ be a ternary semigroup and $\emptyset \neq P \subseteq T$. A ternary semihypergroup $(T, \bullet_P)$ is regular if and only if $(T, \bullet)$ is regular and $V_a \cap \bullet(P, T, P) \neq \emptyset$ for all $a \in T$.*

Proof. The proof follows from Proposition 3.2. ∎

Next, by using some sufficient conditions on a ternary semigroup $(T, \bullet)$ and a nonempty subset P of T, the converse of Corollary 3.3 holds.

Proposition 3.7. *If an element $a \in T$ is regular in a ternary semigroup $(T, \bullet)$, then the element a is regular in a ternary semihypergroup $(T, \bullet_T)$.*

Proof. Suppose that $a \in T$ is regular in a ternary semigroup $(T, \bullet)$. There is $x \in T$ such that $a = \bullet(a, x, a)$. So, $a = \bullet(a, a, a) = \bullet(\bullet(a, x, a), x, a) = \bullet(a, \bullet(x, a, x), a)$, which yields that $\bullet(x, a, x) \in V_a \cap \bullet(T, T, T)$. Hence, $V_a \cap \bullet(T, T, T) \neq \emptyset$. By Proposition 3.2, a ternary semihypergroup $(T, \bullet_T)$ is regular. ∎

Corollary 3.8. *If a ternary semigroup $(T, \bullet)$ is regular, then a ternary semihypergroup $(T, \bullet_T)$ is regular.*

Proof. The proof follows from Proposition 3.7. ∎

Proposition 3.9. *If an element $a \in T$ is regular in a ternary semigroup $(T, \bullet)$, then the element a is regular in a ternary semihypergroup $(T, \bullet_{V_a})$.*

Proof. Assume that an element $a \in T$ is regular in a ternary semigroup $(T, \bullet)$. It immediately implies that there exists $x \in T$ such that $a = \bullet(a, x, a)$. So, $x \in V_a$ and hence $\emptyset \neq V_a \subseteq T$. Thus, a ternary semihypergroup $(T, \bullet_{V_a})$ exists. Since $a = \bullet(a, x, a) = \bullet(\bullet(a, x, a), x, a) = \bullet(a, \bullet(x, a, x), a)$, we get $\bullet(x, a, x) \in V_a \cap \bullet(V_a, T, V_a)$. Thus, $V_a \cap \bullet(V_a, T, V_a) \neq \emptyset$. By Proposition 3.2, $(T, \bullet_{V_a})$ forms a regular ternary semihypergroup. ∎

Proposition 3.10. *Let $(T, \bullet)$ be a ternary semigroup and P, Q be nonempty subsets of T such that $P \subseteq Q$. If an element $a \in T$ is regular in a ternary semihypergroup $(T, \bullet_P)$, then the element a is regular in a ternary semihypergroup $(T, \bullet_Q)$.*

Proof. Assume that $a \in T$ is regular in $(T, \bullet_P)$. By Proposition 3.2, $V_a \cap \bullet(P, T, P) \neq \emptyset$. Since $P \subseteq Q$, we have $V_a \cap \bullet(Q, T, Q) \neq \emptyset$. Again, by Proposition 3.2, the element a is regular in $(T, \bullet_Q)$. ∎

Corollary 3.11. *Let $(T, \bullet)$ be a ternary semigroup and P, Q be nonempty subsets of T such that $P \subseteq Q$. If a ternary semihypergroup $(T, \bullet_P)$ is regular, then a ternary semihypergroup $(T, \bullet_Q)$ is regular.*

Proof. The proof follows from Proposition 3.10. ∎

Proposition 3.12. *If a ternary semigroup $(T, \bullet)$ is regular, then $(T, \bullet_{\cup_{a \in T} V_a})$ forms a regular ternary semihypergroup.*

Proof. Assume that the hypothesis holds. Let $a \in T$. By our assumption, a is a regular element in $(T, \bullet)$. It is obvious that $\emptyset \neq V_a \subseteq \cup_{a \in T} V_a$. Thus, ternary semihypergroups $(T, \bullet_{V_a})$ and $(T, \bullet_{\cup_{a \in T} V_a})$ exist. By Proposition 3.9, a ternary semihypergroup $(T, \bullet_{V_a})$ is regular. Finally, by Proposition 3.10, the element a is regular in $(T, \bullet_{\cup_{a \in T} V_a})$ and hence $(T, \bullet_{\cup_{a \in T} V_a})$ forms a regular ternary semihypergroup. ∎

Theorem 3.13. *Let P be a right ideal and Q be a left ideal of a ternary semigroup $(T, \bullet)$ such that $P \cap Q \neq \emptyset$. If an element $a \in T$ is regular in ternary semihypergroups $(T, \bullet_P)$ and $(T, \bullet_Q)$, then the element a is regular in a ternary semihypergroup $(T, \bullet_{P \cap Q})$.*

Proof. Let $a \in T$. Assume that a is regular in $(T, \bullet_P)$ and $(T, \bullet_Q)$. Then $a \in \bullet_P(a, x, a)$ and $a \in \bullet_Q$ for some $x \in T$. Hence, $a \in \bullet(\bullet(a, P, x), P, a)$ and $a \in \bullet(\bullet(a, Q, x), Q, a)$. It implies that $a = \bullet(\bullet(a, p, x), p', a)$ and $a = \bullet(\bullet(a, q, y), q', a)$ for some $p, p' \in P$, $q, q' \in Q$. Now, by using ternary associativity of the ternary operation $\bullet$ on T, we have

$$\begin{aligned}
a &= \bullet(\bullet(a, p, x), p', a) \\
&= \bullet(\bullet(a, p, x), p', \bullet(\bullet(a, q, y), q', a)) \\
&= \bullet(a, \bullet(\bullet(\bullet(p, x, p'), a, q), y, q'), a) \\
&= \bullet(a, \bullet(\bullet(\bullet(p, x, p'), a, q), y, q'), \bullet(\bullet(a, p, x), p', a)) \\
&= \bullet(a, \bullet(\bullet(p, x, p'), a, q), \bullet(y, q', \bullet(\bullet(a, p, x), p', a))) \\
&= \bullet(a, \bullet(\bullet(p, x, p'), a, q), \bullet(\bullet(y, q', \bullet(a, p, x)), p', a)) \\
&= \bullet(a, \bullet(\bullet(p, x, p'), a, q), \bullet(\bullet(y, q', \bullet(a, p, x)), p', \bullet(\bullet(a, q, y), q', a))) \\
&= \bullet(a, \bullet(\bullet(p, x, p'), a, q), \bullet(\bullet(y, q', \bullet(a, p, x)), \bullet(p', \bullet(a, q, y), q'), a)) \\
&= \bullet(a, m, \bullet(n, m', a)),
\end{aligned}$$

where $m = \bullet(\bullet(p, x, p'), a, q), n = \bullet(y, q', \bullet(a, p, x))$ and $\bullet(p', \bullet(a, q, y), q')$. Since P and Q are a right ideal and a left ideal of $(T, \bullet)$, respectively, we have $m, m' \in P \cap Q$. Moreover, we also have $n \in T$. Thus,

$$\begin{aligned}
a &= \bullet(a, m, \bullet(n, m', a)) \\
&= \bullet(\bullet(a, m, n), m', a) \\
&\in \bullet(\bullet(a, P \cap Q, n), P \cap Q, a) \\
&= \bullet_{P \cap Q}(a, n, a),
\end{aligned}$$

which follows that a is a regular element in a ternary semihypergroup $(T, \bullet_{P\cap Q})$. ■

Corollary 3.14. *Let P be a right ideal and Q be a left ideal of a ternary semigroup $(T, \bullet)$ such that $P \cap Q \neq \emptyset$. If $(T, \bullet_P)$ and $(T, \bullet_Q)$ are regular ternary semihypergroups, then a ternary semihypergroup $(T, \bullet_{P\cap Q})$ is regular.*

Proof. The proof follows from Theorem 3.13. ■

Theorem 3.15. *Let $(T, \bullet)$ be a regular ternary semigroup with an identity element e and $\emptyset \neq P \subseteq T$. Then a ternary semihypergroup $(T, \bullet_P)$ is regular if and only if the nonempty subset P of T has a right invertible element and a left invertible element of the ternary semigroup $(T, \bullet)$.*

Proof. ($\Longrightarrow$) Assume that $(T, \bullet_P)$ is a regular ternary semihypergroup. By the assumption, the identity element e is regular in $(T, \bullet_P)$. Thus, there exists $x \in T$ such that $e \in \bullet_P(e, x, e)$. So,

$$e = \bullet(\bullet(e, p, x), q, e) = \bullet(e, \bullet(p, x, q), e) = \bullet(p, x, q)$$

for some $p, q \in P$. It means that the element p and q in P are a right invertible element and a left invertible element, respectively.

($\Longleftarrow$) Suppose that P contains a right invertible element p and a left invertible element q. Then, there are $u, v, x, y \in T$ such that $\bullet(p, u, v) = e$ and $\bullet(x, y, q) = e$. Let $a \in T$.

Since $(T, \bullet)$ is regular, we have $a = \bullet(a, b, a)$ for some $b \in T$. Then,

$$\begin{aligned} a &= \bullet(a, b, a) \\ &= \bullet(a, \bullet(e, b, e), a) \\ &= \bullet(a, \bullet(\bullet(p, u, v), b, \bullet(x, y, q)), a) \\ &= \bullet(a, \bullet(p, \bullet(\bullet(u, v, b), x, y), q), a) \\ &\in \bullet(\bullet(a, P, m), P, a) \\ &= \bullet_P(a, m, a), \end{aligned}$$

where $m = \bullet(\bullet(u, v, b), x, y) \in T$. It implies that a is regular in the ternary semihypergroup $(T, \bullet_P)$. ∎

Corollary 3.16. *Let $(T, \bullet)$ be a regular ternary semigroup with an identity element e amd every one-sided invertible element of $(T, \bullet)$ is invertible, $\emptyset \neq P \subseteq T$. Then, $(T, \bullet_P)$ is a regular ternary semihypergroup if and only if the nonempty subset P of T contains a invertible element of $(T, \bullet)$*

Proof. The proof follows from Theorem 3.15. ∎

Finally, we will complete the article by showing that the regularity of a ternary semihypergroup induced by a nonempty subset of a ternary semigroup can be preserved under the isomorphic structures of the ternary semigroup.

Theorem 3.17. *Let $(S, \circ)$ and $(T, \bullet)$ be ternary semigroups. Let π be an isomorphism from $(S, \circ)$ onto $(T, \bullet)$. Then, the following statements hold:*

(i) a ternary semihypergroup $(S, \circ_P)$ is isomorphic to a ternary semihypergroup $(T, \bullet_{\pi(P)})$ for every nonempty subset P of S;

(ii) if an element $a \in S$ in a ternary semihypergroup $(S, \circ_P)$ is regular for every nonempty subset P of S, then an element $\pi(a) \in T$ is regular in a ternary semihypergroup $(T, \bullet_{\pi(P)})$,

where $\pi(P) = \{y \in T \mid \pi(x) = y$ for some $x \in S\}$.

Proof. (i) Since π is an isomorphism from $(S, \circ)$ onto $(T, \bullet)$, we obtain that π is bijective from S onto T, which are the base sets of ternary semihypergroups $(S, \circ_P)$ and $(T, \bullet_{\pi(P)})$. For each $x, y, z \in S$, we have

$$\begin{aligned} \pi(\circ_P(x, y, z)) &= \pi(\circ(\circ(x, P, y), P, z)) \\ &= \{\pi(\circ(\circ(x, p, y), q, z)) \in T \mid p, q \in P\} \\ &= \{\bullet(\bullet(\pi(x), \pi(p), \pi(y)), \pi(q), \pi(z)) \in T \mid \pi(p), \pi(q) \in \pi(P)\} \\ &= \bullet(\bullet(\pi(x), \pi(P), \pi(y)), \pi(P), \pi(z)) \\ &= \bullet_{\pi(P)}(\pi(x), \pi(y), \pi(z)). \end{aligned}$$

So, π is an isomorphism from $(S, \circ_P)$ and $(T, \bullet_{\pi(P)})$, which means that $(S, \circ_P) \cong (T, \bullet_{\pi(P)})$.

(ii) Assume that an element $a \in S$ is regular in a ternary semihypergroup $(S, \circ_P)$. By the assumption, there exists $x \in S$ such that $x \in S$ such that $a \in \bullet_P(a, x, a) = \circ(\circ(a, P, x), P, a)$. Then, there are $p, q \in P$ such that $a = \circ(\circ(a, p, x), q, a)$. So, we observe that

$$\begin{aligned}\pi(a) &= \pi(\circ(\circ(a,p,x),q,a))\\ &= \bullet(\bullet(\pi(a),\pi(x),\pi(x)),\pi(q),\pi(a))\\ &\in \bullet(\bullet(\pi(a),\pi(P),\pi(x)),\pi(P),\pi(a))\\ &= \bullet_{\pi(P)}(\pi(a),\pi(x),\pi(a)).\end{aligned}$$

It implies that $\pi(a)$ is a regular element in the ternary semihypergroup $(T,\bullet_{\pi(P)})$. ∎

Corollary 3.18. *Let $(S,\circ)$ and $(T,\bullet)$ be ternary semigroups. Let π be an isomorphism from $(S,\circ)$ onto $(T,\bullet)$. If a ternary semihypergroup $(S,\circ_P)$ is regular, then a ternary semihypergroup $(T,\bullet_{\pi(P)})$ is regular for every nonempty subset P of S.*

Proof. The proof follows from Theorem 3.17 (ii). ∎

4. Conclusions and Future Works

In this article, we firstly started with recalling some important notions related to ternary semigroups and ternary semihypergroups. In order to get our main results, special elements of ternary semigroup and ternary semihypergroups were also provided. Secondly, we constructed the so-called a *ternary semihypergroup induced by a nonempty subset of a ternary semigroup*. Then, we investigated the regularity of the ternary semihypergroups. Moreover, we given some characterizations of the regularity of the ternary semihypergroups by using some significant conditions on subsets of the base set of ternary semigroups. Finally, we completed the article by showing that the regularity of the ternary semihypergroups can be preserved under the isomorphic structures of the ternary semigroups.

Based on the main results of this article and the algebraic structure of n-ary semihypergroups, which can be regarded as a natural generalization of ternary semihypergroups, there are several interesting research questions to study in the future works.

(i) Can we extend the results of the regularity of ternary semihypergroups induced by subsets of ternary semigroups to study in n-ary semihypergroups?

(ii) Can we study other regularity such as intra-regular, left regular and right regular etc. on ternary semihypergroups induced by subsets of ternary semigroups which were defined as in this article?

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