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A Singular Nonlinear Second-Order Neumann Boundary Value Problem with Positive Solutions

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Abstract : We study the existence and multiplicity of positive solutions for second-order Neumann boundary value problem $-u'' + a(t)u = h(t)f(t,u),\ t \in (0,1),\ u'(0) = u'(1) = 0$, where coefficient $a(t) : [0,1] \to (-\infty, +\infty)$ is continuous and $\max\limits_{t\in[0,1]} a(t) > 0$, $h(t)$ may be singular at $t = 0$ and 1, moreover $f(t,u)$ may also have singularity at $u = 0$. The first eigenvalue of the relevant linear problem and fixed point index theory are used in this study.

Keywords : Neumann BVP; Singular; Positive solutions; Fixed point index
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1 Introduction

In this paper we are concerned with the existence of positive solutions for the following singular nonlinear second-order ordinary differential equation

$$-u'' + a(t)u = h(t)f(t,u), \quad t \in \ (0,1), \tag{1.1}$$

with Neumann boundary conditions

$$u'(0) = u'(1) = 0 \tag{1.2}$$

under the conditions that coefficient $a(t) : [0,1] \to (-\infty, +\infty)$ is continuous and $\max\limits_{t\in[0,1]} a(t) > 0,\ f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ and $h(t)$ may be singular at $t = 0$ and 1 and f may be singular at $u = 0$.

In the past ten years or so, various boundary value problems for ordinary differential equations have been studied extensively; see, for example, [1–13] and the references therein. Many authors are interested in the existence of positive solutions for second-order Neumann boundary value problem with $a(t) \equiv M > 0$

$$-u'' + Mu = f(t,u), \quad t \in (0,1), \tag{1.3}$$

under boundary conditions (1.2). Using the Guo-Krasnosel'skii fixed point theorem of cone compression-expansion type and the Leggett-Williams fixed point theorem, Jiang and Liu [3] and Sun and Li [4–5] studied the existence of multiple positive solutions to Eq. (1.3) under Neumann boundary conditions (1.2), where the function f has no singularity. In the case where $f(t,u)$ may be singular at $t = 0,1$, but f has no singularity at $u = 0$, Zhang, Sun and Zhong[6] and Yao [7] gave several sufficient conditions for the existence of solutions for the nonlinear second-order equation (1.3)–(1.2).

In this paper, we establish the existence of positive solutions to problem (1.1)–(1.2), by using the first eigenvalue of the relevant linear problem and fixed point index theory which come from Zhang−Sun [8−11] and Cui−Zou [12−13]. Here we emphasize that the Eq. (1.1) is the more general case and we not only allow $h(t)$ to have singularity at $t = 0,1$, but also allow $f(t,u)$ to have singularity at $u = 0$. As far as we are aware, there have been fewer works done for when f has singularity at $u = 0$, so our results are differential in essence from those of [3–7].

The rest of the paper is organized as follows. Some preliminaries and various lemmas are given in Section 2. In Section 3, we give the existence theorems of the *sublinear* singular Neumann boundary value problem. In Section 4, we give the existence theorems of the *superlinear* singular Neumann boundary value problem. In Section 5, we give the existence of multiple positive solutions.

2 Preliminaries and lemmas

In Banach space $C[0,1]$ in which the norm is defined by $\|u\| = \max\limits_{0\le t\le 1} |u(t)|$ for any $u \in C[0,1]$. We set $P = \{u \in C[0,1] | u(t) \ge 0,\ t \in [0,1]\}$ be a cone in $C[0,1]$. We denote by $B_r = \{u \in C[0,1] | \|u\| < r\} (r > 0)$ the open ball of radius r.

The function u is said to be a positive solution of BVP(1.1),(1.2) if $u \in C[0,1] \cap C^2(0,1)$ satisfies (1.1), (1.2) and $u(t) > 0$ for $t \in (0,1)$.

Firstly, we consider the following BVP

$$\begin{cases} -u'' + \mathcal{M}u = h(t)f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases} \tag{2.1}$$

Let $G(t,s)$ be the Green function of the problem (2.1) with $h(t)f(t,u) \equiv 0$ (see [4], [5]), that is,

$$G(t,s) = \begin{cases} \dfrac{\mathrm{ch}(m(1-t))\mathrm{ch}(ms)}{m\mathrm{sh}m}, & 0 \le s \le t \le 1, \\ \dfrac{\mathrm{ch}(m(1-s))\mathrm{ch}(mt)}{m\mathrm{sh}m}, & 0 \le t \le s \le 1, \end{cases}$$

where $m = \sqrt{\mathcal{M}}$, $\mathcal{M} > 0$, $\mathrm{ch}x = \dfrac{e^x + e^{-x}}{2}$, $\mathrm{sh}x = \dfrac{e^x - e^{-x}}{2}$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and $G(t,s) \geq 0$ for $0 \leq t,s \leq 1$. After direct computations we get

$$\frac{1}{m\mathrm{sh}m} = B \leq G(t,s) \leq \widetilde{B} = \frac{\mathrm{ch}^2 m}{m\mathrm{sh}m},\ \forall\ 0 \leq t,s \leq 1. \tag{2.2}$$

Next consider the following BVP, which is equivalent to (1.1),(1.2):

$$\begin{cases} -u'' + \mathcal{M}u = h(t)f(t,u) + \mathcal{M}u - a(t)u, & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases}$$

Obviously, problem (1.1)-(1.2) is equivalently reformulated as the integral equation

$$u(t) = \int_0^1 G(t,s)[h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds, \quad t \in [0,1].$$

We therefore define

$$(Au)(t) = \int_0^1 G(t,s)[h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds, \quad t \in [0,1], \tag{2.3}$$

$$(Tu)(t) = \int_0^1 G(t,s)h(s)u(s)ds, \quad t \in [0,1]. \tag{2.4}$$

We can verify that the nonzero fixed points of the operator A are positive solutions of the problem (1.1)−(1.2).

Define

$$K = \{u \in P | u(t) \geq \gamma\|u\|,\ t \in [0,1]\},$$

where $0 < \gamma = \frac{B}{\widetilde{B}} < 1$. Then K is subcone of P.

We make the following assumptions:

(H_1) $h : (0,1) \to (0,+\infty)$ is continuous, and

$$0 < \int_0^1 h(t)dt < +\infty.$$

(H_2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $a : [0,1] \to (-\infty,+\infty)$ is continuous and $\max\limits_{t\in[0,1]} a(t) = \mathcal{M} > 0$, and for any $0 < c < d < +\infty$,

$$\lim_{n\to\infty} \sup_{u\in K[c,d]} \int_{D(n)} [h(s)f(s,u) + \mathcal{M}u - a(s)u]ds = 0,$$

where $K[c,d] = \{u \in K | c \leq \|u\| \leq d\}$, $D(n) = [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1]$.

Lemma 2.1. *Assume that (H_1), (H_2) hold. Then $A : K[c,d] \to K$ is a completely continuous operator.*

Proof. Let $u \in K$. Since $G(t,s) \geq 0,\ (t,s) \in\ [0,1] \times [0,1]$, by the definition, we have $(Au)(t) \geq 0,\ t \in [0,1]$. On the other hand, by (2.2) we have

$$\begin{aligned} (Au)(t) = &\int_0^1 G(t,s)[h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\ \geq\ & B\int_0^1 [h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|Au\| = &\int_0^1 G(t,s)[h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\ \leq\ & \widetilde{B}\int_0^1 [h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds, \end{aligned} \tag{2.6}$$

for every $t \in [0,1]$, by (2.5) and (2.6) we have

$$(Au)(t) \geq \gamma\|Au\|.$$

Thus, we assert that $A : K[c,d] \to K$.

Next, we prove the continuity of A. Suppose $u_n,\ u \in K[c,d]$ and $u_n \to u(n \to +\infty)$. Then $c \leq \|u_n\| \leq d$ and $c \leq \|u\| \leq d$. For any $\varepsilon > 0$, by (H_2), there exists a natural number $n > 0$ such that

$$\sup_{u\in K[c,d]} \int_{D(n)} [h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds < \frac{\varepsilon}{4\widetilde{B}}. \tag{2.7}$$

On the other hand, for any $t \in [\frac{1}{n}, \frac{n-1}{n}]$, we have

$$\gamma c \leq u_n(t),\ u(t) \leq d.$$

By (H_2), we know that $h(t)f(t,u) + \mathcal{M}u - a(t)u$ is uniformly continuous on $[\frac{1}{n}, \frac{n-1}{n}] \times [\gamma c, d]$. Hence,

$$\lim_{n\to+\infty}[h(t)f(t,u_n)+\mathcal{M}u_n-a(t)u_n] = h(t)f(t,u)+\mathcal{M}u-a(t)u,\ \text{for } t \in [\frac{1}{n}, \frac{n-1}{n}].$$

The Lebesgue dominated convergence theorem yields that

$$\int_{\frac{1}{n}}^{\frac{n-1}{n}} \Big|[h(s)f(s,u_n) + \mathcal{M}u_n - a(s)u_n] - [h(s)f(s,u) + \mathcal{M}u - a(s)u]\Big|ds \to 0.$$

Thus, for the above $\varepsilon > 0$, there exists a natural number N such that for $n > N$, we have

$$\int_{\frac{1}{n}}^{\frac{n-1}{n}} \Big|[h(s)f(s,u_n)+\mathcal{M}u_n-a(s)u_n]-[h(s)f(s,u)+\mathcal{M}u-a(s)u]\Big|ds < \frac{\varepsilon}{2\widetilde{B}}. \tag{2.8}$$

It follows from (2.7) and (2.8) that when $n > N$,

$$\begin{aligned} \|Au_n - Au\| \leq & \int_{\frac{1}{n}}^{\frac{n-1}{n}} \widetilde{B}\Big|[h(s)f(s,u_n) + \mathcal{M}u_n - a(s)u_n] - [h(s)f(s,u) + \mathcal{M}u - a(s)u]\Big|ds \\ & + 2 \sup_{u \in K[c,d]} \int_{\frac{1}{n}}^{\frac{n-1}{n}} \widetilde{B}[h(s)f(s,u) + \mathcal{M}u - a(s)u]ds \\ & < \frac{\varepsilon}{2} + 2 \times \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This implies that $A : K[c,d] \to K$ is continuous.

The compactness of A on $K[c,d]$ can be followed from similar discussion above and Arzela-Ascoli theorem. Thus $A : K[c,d] \to K$ is completely continuous.

In addition, by the same method as in Lemma 2.1 we have that $T : K[c,d] \to K$ is a completely continuous linear operator.

By virtue of Krein−Rutmann theorems, we have(see [8−13]) the following lemma.

Lemma 2.2. *Suppose that $T : C[0,1] \to C[0,1]$ is a completely continuous linear operator and $T(P) \subset P$. If there exists $\psi \in C[0,1]\backslash\{-P\}$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, that is, $\varphi_1 = \lambda_1 T\varphi_1$.*

Lemma 2.3. *Suppose that the condition (H_1) is satisfied, then for the operator T defined by (2.4), the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof. It is obvious that there is $t_1 \in (0,1)$ such that $G(t_1,t_1)h(t_1) > 0$. Thus there exists $[a_1,b_1] \subset (0,1)$ such that $t_1 \in (a_1,b_1)$ and $G(t,s)h(s) > 0, \forall\ t,s \in [a_1,b_1]$. Take $\psi \in C[0,1]$ such that $\psi(t) \geq 0,\ \forall\ t \in [0,1],\ \psi(t_1) > 0$ and $\psi(t) = 0,\ \forall\ t \notin [a_1,b_1]$. Then for $t \in [a_1,b_1]$

$$(T\psi)(t) = \int_0^1 G(t,s)h(s)\psi(s)ds \geq \int_{a_1}^{b_1} G(t,s)h(s)\psi(s)ds > 0.$$

So there exists a constant $c > 0$ such that $c(T\psi)(t) \geq \psi(t), \forall\ t \in [0,1]$. From Lemma 2.2, we know that the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

We also need the following lemmas (see [14]).

Lemma 2.4. *Let E be Banach space, P be a cone in E, and $\Omega(P)$ be a bounded open set in P. Suppose that $A : \overline{\Omega(P)} \to P$ is a completely continuous operator. If there exists $u_0 \in P\backslash\{\theta\}$ such that*

$$u - Au \neq \mu u_0, \quad \forall\ u \in \partial\Omega(P),\ \mu \geq 0,$$

then the fixed point index $i(A, \Omega(P), P) = 0$.

Lemma 2.5. *Let E be Banach space, P be a cone in E, and $\Omega(P)$ be a bounded open set in P with $\theta \in \Omega(P)$. Suppose that $A : \overline{\Omega(P)} \to P$ is a completely* *continuous operator. If*

$$Au \neq \mu u, \quad \forall\ u \in \partial\Omega(P),\ \mu \geq 1,$$

then the fixed point index $i(A, \Omega(P), P) = 1$.

3 Existence results in sublinear case

Theorem 3.1. *Suppose that the conditions* $(H_1) - (H_2)$ *are satisfied, and*

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1, \tag{3.1}$$

$$\limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1, \tag{3.2}$$

where λ_1 *is the first eigenvalue of* T *defined by* (2.4)*. Then the singular Neumann boundary value problem* (1.1) $-$ (1.2) *has at least one positive solution.*

Proof. It follows from (3.1) that there exists $r_1 > 0$ such that

$$f(t,u) \geq \lambda_1 u, \quad \forall\ t \in [0,1],\ 0 \leq u \leq r_1. \tag{3.3}$$

Let u^* be the positive eigenfunction of T corresponding to λ_1, thus $u^* = \lambda_1 T u^*$.

For every $u \in \partial B_{r_1} \cap P$, it follows from (3.3) and $\mathcal{M} = \max\limits_{t\in[0,1]} a(t)$ that

$$\begin{aligned}(Au)(t) = & \int_0^1 G(t,s)[h(s)f(t,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\ \geq & \int_0^1 G(t,s)h(s)f(t,u(s))ds \\ \geq & \ \lambda_1 \int_0^1 G(t,s)h(s)u(s)ds \\ = & \ \lambda_1 (Tu)(t), \quad t \in [0,1].\end{aligned} \tag{3.4}$$

We may suppose that A has no fixed point on $\partial B_{r_1} \cap P$(otherwise, the proof is finished). Now we show that

$$u - Au \neq \tau u^*, \quad \forall\ u \in \partial B_{r_1} \cap P, \tau \geq 0. \tag{3.5}$$

Suppose the contrary, that exist $u_1 \in \partial B_{r_1} \cap P$ and $\tau_1 \geq 0$ such that $u_1 - Au_1 = \tau_1 u^*$. Hence $\tau_1 > 0$ and

$$u_1 = Au_1 + \tau_1 u^* \geq \tau_1 u^*.$$

Put

$$\tau^* = \sup\{\tau | u_1 \geq \tau u^*\}. \tag{3.6}$$

It is easy to see that $\tau^* \geq \tau_1 > 0$ and $u_1 \geq \tau^* u^*$. We find from $T(P) \subset P$ that

$$\lambda_1 T u_1 \geq \tau^* \lambda_1 T u^* = \tau^* u^*.$$

Therefore by (3.4), we have

$$u_1 = Au_1 + \tau_1 u^* \geq \lambda_1 T u_1 + \tau_1 u^* \geq \tau^* u^* + \tau_1 u^* = (\tau^* + \tau_1) u^*,$$

which contradicts the definition of τ^*. Hence (3.5) is true and we have from Lemma 2.4 that

$$i(A, B_{r_1} \cap P, P) = 0. \tag{3.7}$$

It is easy from $\mathcal{M} = \max_{t\in[0,1]} a(t)$ to see that

$$\limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{h(t)f(t,u) + \mathcal{M}u - a(t)u}{h(t)u} = \limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

so by means of (3.2), we have that there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$h(t)f(t,u) + \mathcal{M}u - a(t)u \leq \sigma\lambda_1 u h(t), \ \forall\ u \geq r_2. \tag{3.8}$$

Let $T_1 u = \sigma\lambda_1 T u,\ u \in C[0,1]$, then $T_1 : C[0,1] \to C[0,1]$ is a bounded linear operator and $T_1(P) \subset P$. Denote

$$M^* = \widetilde{B} \sup_{u\in\overline{B}_{r_2}\cap P} \int_0^1 h(s)f(s,u(s))ds. \tag{3.9}$$

It is clear that $M^* < +\infty$. Let

$$W = \{u \in P \mid u = \mu A u,\ 0 \leq \mu \leq 1\}. \tag{3.10}$$

In the following, we prove that W is bounded.

For any $u \in W$, set $\widetilde{u}(t) = \min\{u(t), r_2\}$ and denote $E(t) = \{t \in [0,1] \mid u(t) > r_2\}$, then

$$\begin{aligned}
u(t) = {} & \mu(Au)(t) \leq \int_0^1 G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\
= {} & \int_{E(t)} G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\
& + \int_{[0,1]\setminus E(t)} G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\
\leq {} & \sigma\lambda_1 \int_0^1 G(t,s)h(s)u(s)ds + \widetilde{B}\int_0^1 h(s)f(s,\widetilde{u}(s))ds \\
\leq {} & (T_1 u)(t) + M^*, \quad t \in [0,1].
\end{aligned}$$

Thus $((I-T_1)u)(t) \leq M^*,\ t \in [0,1]$. Since λ_1 is the first eigenvalue of T and $0 < \sigma < 1$, the first eigenvalue of T_1, $(r(T_1))^{-1} > 1$. Therefore, the inverse operator $(I-T_1)^{-1}$ exists and

$$(I-T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots .$$

It follows from $T_1(P) \subset P$ that $(I-T_1)^{-1}(P) \subset P$. So we know that $u(t) \leq (I-T_1)^{-1}M^*,\ t \in [0,1]$ and W is bounded.

Select $r_3 > \max\{r_2,\ \sup W\}$. Then from the homotopy invariance property of fixed point index we have

$$i(A,\ B_{r_3} \cap P,\ P) = i(\theta,\ B_{r_3} \cap P,\ P) = 1. \tag{3.11}$$

By (3.7) and (3.11), we have that

$$i(A,\ (B_{r_3} \cap P)\backslash(\overline{B}_{r_1} \cap P),\ P) = i(A,\ B_{r_3} \cap P,\ P) - i(A,\ B_{r_1} \cap P,\ P) = 1.$$

Then A has at least one fixed point on $(B_{r_3} \cap P)\backslash(\overline{B}_{r_1} \cap P)$. This means that singular Neumann boundary value problem (1.1)$-$(1.2) has at least one positive solution.

Corollary 3.1 *Suppose conditions $(H_1)-(H_2)$ are satisfied, denote*

$$f_0 = \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f^\infty = \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}$$

In addition, assume that $0 \leq f^\infty < f_0 \leq +\infty$,

$$\lambda \in \Big(\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f^\infty}\Big), \tag{3.12}$$

where λ_1 is the first eigenvalue of linear operator T. Then the singular eigenvalue problem

$$\begin{cases} -u'' + a(t)u = \lambda h(t) f(t,u), & 0 < t < 1, \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least one positive solution.

Proof. By (3.12), we know that

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{\lambda f(t,u)}{u} > \lambda_1, \quad \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{\lambda f(t,u)}{u} < \lambda_1.$$

So Corollary 3.1 holds from Theorem 3.1.

4 Existence results in superlinear case

In this section, we give the existence theorem of positive solutions for the superlinear singular Neumann boundary value problem.

Theorem 4.1. *Suppose that the conditions $(H_1)-(H_2)$ are satisfied, and*

$$\liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1, \tag{4.1}$$

$$\limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1, \tag{4.2}$$

where λ_1 is the first eigenvalue of T defined by (2.4)*. Then the singular Neumann boundary value problem* (1.1) − (1.2) *has at least one positive solution.*

Proof. It follows from (4.1) that there exists $\varepsilon > 0$ such that $f(t,u) \geq (\lambda_1 + \varepsilon)u$ when u is sufficiently large. We know from (H_2) that there exists $b_1 \geq 0$ such that

$$f(t,u) \geq (\lambda_1+\varepsilon)u - b_1, \quad \forall\ t \in [0,1],\ 0 \leq u < +\infty. \tag{4.3}$$

Take

$$R > \max\Big\{1, \frac{b_1}{\gamma^2\varepsilon}\Big\}.$$

Then for any $u \in K,\ \|u\| \geq R$, it follows from (4.3) and $\mathcal{M} = \max\limits_{t\in[0,1]} a(t)$ that

$$\begin{aligned}
(Au)(t) = {} & \int_0^1 G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\
\geq {} & \int_0^1 G(t,s)h(s)f(s,u(s))ds \\
\geq {} & (\lambda_1+\varepsilon)\int_0^1 G(t,s)h(s)u(s)ds - b_1\int_0^1 G(t,s)h(s)ds \\
\geq {} & \lambda_1(Tu)(t) + B\varepsilon\int_0^1 h(s)u(s)ds - b_1\widetilde{B}\int_0^1 h(s)ds \\
\geq {} & \lambda_1(Tu)(t) + B\varepsilon\gamma\int_0^1 h(s)ds\|u\| - b_1\widetilde{B}\int_0^1 h(s)ds \\
\geq {} & \lambda_1(Tu)(t), \quad t\in[0,1].
\end{aligned} \tag{4.4}$$

We may suppose that A has no fixed points on $\partial B_R \cap K$(otherwise, the proof is finished). In the following we prove

$$u - Au \neq \tau u^*,\ \forall\ u \in \partial B_R \cap K,\ \tau \geq 0, \tag{4.5}$$

where $u^* \in P$ is the positive eigenfunction of T corresponding to its first eigenvalue λ_1. If otherwise, then there exist $u_2 \in \partial B_R \cap K$ and $\tau_2 \geq 0$ such that

$$u_2 - Au_2 = \tau_2 u^*.$$

Hence $\tau_2 > 0$ and

$$u_2 - Au_2 = \tau_2 u^*.$$

Let

$$\tau^* = \sup\{\tau | u_2 \geq \tau u^*\}.$$

It is easy to see that $\tau^* \geq \tau_2 > 0$ and $u_2 \geq \tau^* u^*$. We find from $T(K) \subset K$ that

$$\lambda_1 T u_2 \geq \tau^*\lambda_1 T u^* = \tau^* u^*,$$

Therefore by (4.3)

$$u_2 = Au_2 + \tau_2 u^* \geq \lambda_1 T u_1 + \tau_2 u^* \geq \tau^* u^* + \tau_2 u^* = (\tau^* + \tau_2)u^*,$$

which contradicts the definition of τ^*. Hence (4.4) is true and we have from Lemma 2.4 that

$$i(A, B_R \cap K, K) = 0. \tag{4.6}$$

It is easy from $\mathcal{M} = \max\limits_{t\in[0,1]} a(t)$ to see that

$$\limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{h(t)f(t,u) + \mathcal{M}u - a(t)u}{h(t)u} = \limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

so by means of (4.2), we have that there exists $0 < r < 1$ such that

$$h(t)f(t,u) + \mathcal{M}u - a(t)u \leq \lambda_1 u h(t), \quad \forall\ t \in [0,1],\ 0 \leq u \leq r. \tag{4.7}$$

Define $T_2 u = \lambda_1 T u, u \in C[0,1]$. Hence $T_2 : C[0,1] \to C[0,1]$ is a bounded linear completely continuous operator and

$$T_2(K) \subset K, \quad r(T_2) = 1.$$

For every $u \in \partial B_r \cap K$, it follows from (4.7) that

$$\begin{aligned} (Au)(t) = &\int_0^1 G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds \\ \leq\ & \lambda_1 \int_0^1 G(t,s)h(s)u(s)ds \\ =\ & (T_2u)(t),\ t \in [0,1], \end{aligned}$$

hence $Au \leq T_2 u$, $\forall\ u \in\ \partial B_r \cap K$. We may also that A has no fixed point on $\partial B_r \cap K$ (otherwise, the proof is finished).

Now we show that

$$Au \neq \mu u, \quad \forall\ u \in \partial B_r \cap K,\ \mu \geq 1. \tag{4.8}$$

If otherwise, there exist $u_3 \in \partial B_r \cap K$ and $\mu_3 \geq 1$ such that $Au_3 = \mu_3 u_3$. Thus $\mu_3 > 1$ and $\mu_3 u_3 = Au_3 \leq T_2 u_3$. By induction, we have $\mu_3^n u_3 \leq T_2^n u_3 (n = 1, 2, \ldots)$. Then

$$\mu_3^n u_3 \leq T_2^n u_3 \leq \|T_2^n\| \|u_3\|,$$

and taking the maximum over [0,1] gives $\mu_3^n \leq \|T_2^n\|$. In view of Gelfand's formula, we have

$$r(T_2) = \lim_{n\to\infty} \sqrt[n]{\|T_2^n\|} \geq \lim_{n\to\infty} \sqrt[n]{\mu_3^n} = \mu_3 > 1,$$

which is a contradiction. Hence (4.8) is true and by Lemma 2.5, we have

$$i(A, B_r \cap K, K) = 1. \tag{4.9}$$

By (4.6) and (4.9) we have

$$i(A,(B_R\cap K)\backslash(\overline{B_r}\cap K),K)=i(A,(B_R\cap K,K)-i(A,B_r\cap K,K)=-1.$$

Then A has at least one fixed point on $(B_R\cap K)\backslash(\overline{B}_r\cap K)$. This means that the singular superlinear Neumann boundary value problem (1.1) − (1.2) has at least one positive solution.

Corollary 4.1 *Suppose conditions* $(H_1)-(H_2)$ *are satisfied, denote*

$$f^0=\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{f(t,u)}{u},\quad f_\infty=\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f(t,u)}{u}$$

In addition, assume that $0\le f^0<f_\infty\le+\infty$,

$$\lambda\in\Big(\frac{\lambda_1}{f_\infty},\frac{\lambda_1}{f^0}\Big), \tag{4.10}$$

where λ_1 *is the first eigenvalue of linear operator* T. *Then the singular eigenvalue problem*

$$\begin{cases} -u''+a(t)u=\lambda h(t)f(t,u), & 0<t<1,\\ u'(0)=u'(1)=0 \end{cases}$$

has at least one positive solution.

Proof. By (4.10), we know that

$$\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{\lambda f(t,u)}{u}>\lambda_1,\quad \limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{\lambda f(t,u)}{u}<\lambda_1.$$

So Corollary 4.1 holds from Theorem 4.1.

5 Existence results of twin positive solutions

In this section we need the following well-know lemma (see [14]).

Lemma 5.1. *Let* E *be a Banach space, and* P *be a cone in* E, *and* $\Omega(P)$ *be a bounded open set in* P. *Suppose that* $A:\overline{\Omega(P)}\to P$ *is a completely continuous operator.*

(i) If $\|Au\|>\|u\|,\ u\in\partial\Omega(P)$, *then the fixed point index* $i(A,\Omega(P),P)=0$.

(ii) If $\theta\in\Omega(P)$ *and* $\|Au\|<\|u\|,\ u\in\partial\Omega(P)$, *then the fixed point index* $i(A,\Omega(P),P)=1$.

Theorem 5.1. *Suppose that conditions* $(H_1)-(H_2)$ *are satisfied. In addition, assume that*

$$\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{f(t,u)}{u}<\lambda_1, \tag{5.1}$$

$$\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u}<\lambda_1, \tag{5.2}$$

where λ_1 is the first eigenvalue of linear operator T. If there exists $r_0 > 0$ such that

$$f(t,u) > \xi r_0, \quad \forall\ t \in [0,1],\ u \in [\gamma r_0, r_0], \tag{5.3}$$

where $\gamma \in (0,1)$, $\xi = \left(B\int_0^1 h(s)ds\right)^{-1}$, then the singular Neumann boundary value problem (1.1) − (1.2) *has at least two positive solutions.*

Proof. It is easy from $\mathcal{M} = \max_{t\in[0,1]} a(t)$ to see that

$$\max_{t\in[0,1]} \frac{h(t)f(t,u) + \mathcal{M}u - a(t)u}{h(t)u} = \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

so by means of (5.1) and (5.2), we have that there exists $0 < r_4 < r_0$ such that $h(t)f(t,u) + \mathcal{M}u - a(t)u \le \lambda_1 uh(t)$ for $0 \le u \le r_4$ and there exist $0 < \sigma < 1$ and $r_5 > r_0$ such that $h(t)f(t,u) + \mathcal{M}u - a(t)u \le \sigma\lambda_1 uh(t)$ for $u \ge r_5$. We may suppose that A has no fixed point on $\partial B_{r_4} \cap K$ and $\partial B_{r_5} \cap K$. Otherwise, the proof is completed.

We have from the proof in Theorem 4.1 and the permanence property of fixed point index that $i(A, B_{r_4} \cap K, K) = 1$. It follows from the proof in Theorem 3.1 that $i(A, B_{r_5} \cap K, K) = 1$.

For every $u \in B_{r_0} \cap K$, we have $\gamma r_0 = \gamma\|u\| \le u(t) \le r_0,\ 0 \le t \le 1$. It follows from $\mathcal{M} = \max_{t\in[0,1]} a(t)$ and (5.3) that

$$\begin{aligned}(Au)(t) = &\int_0^1 G(t,s)[h(s)f(s,u(s)) + \mathcal{M}u(s) - a(s)u(s)]ds\\ \ge &\int_0^1 G(t,s)h(s)f(s,u(s))ds\\ > &\ B\xi r_0\int_0^1 h(s)ds\\ = &\ r_0, \quad t \in [0,1].\end{aligned}$$

Then $\|Au\| > \|u\|$, for any $u \in \partial B_{r_0} \cap K$. Hence we have from Lemma 5.1 that $i(A, B_{r_0} \cap K, K) = 0$.

Therefore,

$$i(A,(B_{r_0} \cap K)\backslash(B_{r_4} \cap K), K) = i(A, B_{r_0} \cap K, K) - i(A, B_{r_4} \cap K, K) = -1,$$

$$i(A,(B_{r_5} \cap K)\backslash(B_{r_0} \cap K), K) = i(A, B_{r_5} \cap K, K) - i(A, B_{r_0} \cap K, K) = 1.$$

Then A has at least two fixed points on $(B_{r_0} \cap K)\backslash(B_{r_4} \cap K)$ and $(B_{r_5} \cap K)\backslash(B_{r_0} \cap K)$. This means that the singular Neumann boundary value problem (1.1)−(1.2) has at least two positive solutions.

Theorem 5.2. *Suppose that conditions $(H_1) - (H_2)$ are satisfied. In addition, assume that*

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1, \tag{5.4}$$

$$\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f(t,u)}{u}>\lambda_1, \tag{5.5}$$

where λ_1 is the first eigenvalue of linear operator T. If there exists $r_0'>0$ such that

$$h(t)f(t,u)+\mathcal{M}u-a(t)u<\xi' r_0' h(t), \quad \forall\ t\in[0,1],\ u\in[\gamma r_0', r_0'], \tag{5.6}$$

where $\gamma\in(0,1)$, $\xi'=\left(\widetilde{B}\int_0^1 h(s)ds\right)^{-1}$, then the singular Neumann boundary value problem $(1.1)-(1.2)$ *has at least two positive solutions.*

Proof. It follows from (5.4) and (5.5) that there exists $0<r_4'<r_0'$ such that $f(t,u)\ge\lambda_1 u$ for $0\le u\le r_4'$ and there exist $r_5'>r_0'$ and $\varepsilon>0$ such that $f(t,u)\ge(\lambda_1+\varepsilon)u$ for $u\ge r_5'$. We may suppose that A has no fixed point on $\partial B_{r_4'}\cap K$ and $\partial B_{r_5'}\cap K$. Otherwise, the proof is completed.

We have from the proof in Theorem 3.1 and the permanence property of fixed point index that $i(A,B_{r_4'}\cap K,K)=0$. It follows from the proof in Theorem 4.1 that $i(A,B_{r_5'}\cap K,K)=0$.

For every $u\in B_{r_0'}\cap K$, we have $\gamma r_0'=\gamma\|u\|\le u(t)\le r_0'$, $0\le t\le 1$. It follows from $\mathcal{M}=\max\limits_{t\in[0,1]}a(t)$ and (5.6) that

$$\begin{aligned}\|Au\| = {} & \max_{t\in[0,1]}(Au)(t)\\ = {} & \max_{t\in[0,1]}\int_0^1 G(t,s)[h(s)f(s,u(s))+\mathcal{M}u(s)-a(s)u(s)]ds\\ < {} & \widetilde{B}\xi' r_0'\int_0^1 h(s)ds\\ = {} & r_0'.\end{aligned}$$

Then $\|Au\|<\|u\|$, for any $u\in\partial B_{r_0'}\cap K$. Hence we have from Lemma 5.1 that $i(A,B_{r_0'}\cap K,K)=1$.

Therefore,

$$i(A,(B_{r_0'}\cap K)\backslash(B_{r_4'}\cap K),K)=i(A,B_{r_0'}\cap K,K)-i(A,B_{r_4'}\cap K,K)=1,$$

$$i(A,(B_{r_5'}\cap K)\backslash(B_{r_0'}\cap K),K)=i(A,B_{r_5'}\cap K,K)-i(A,B_{r_0'}\cap K,K)=-1.$$

Then A has at least two fixed points on $(B_{r_0'}\cap K)\backslash(B_{r_4'}\cap K)$ and $(B_{r_5'}\cap K)\backslash(B_{r_0'}\cap K)$. This means that the singular Neumann boundary value problem (1.1)−(1.2) has at least two positive solutions.

Remark 5.1. Using similar arguments and techniques, the results presented in this paper could be obtained for the following second-order Neumann boundary value problem:

$$\begin{cases} u''+a(t)u=h(t)f(t,u), & 0<t<1,\\ u'(0)=u'(1)=0,\end{cases}$$

where $0 < \max_{t\in[0,1]} a(t) < \frac{\pi^2}{4}$.

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