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# Isomorphism Theorems for Γ-Semigroups and Ordered Γ-Semigroups

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**Abstract :** The notion of Γ-semigroups has been introduced by M. K. Sen and N. K. Saha. Γ-semigroups generalize semigroups. Many classical notions of semigroups have been extended to Γ-semigroups. Ordered Γ-semigroups have been studied by some authors. In this paper, we investigate first and third isomorphism theorems for Γ-semigroups and ordered Γ-semigroups.

**Keywords :** Isomorphism theorems; Γ-semigroups; ordered Γ-semigroups; congruences; pseudo-orders
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## 1 Introduction and Preliminaries

The isomorphism theorems are three theorems that describe the relationship between quotients, homomorphisms, and subobjects. Versions of the theorems exist for groups, rings and various other algebraic structures. The first isomorphism theorem and third isomorphism theorem based on congruences of semigroups [4, page 22-24] and ordered semigroups [7] have been given. In case of ordered semigroups, pseudo-orders play the role of congruences [7].

The notion of Γ-semigroups has been introduced by M. K. Sen and N. K. Saha in [10] and [11]. Many classical notions of semigroups have been extended to Γ-semigroups (see [1], [2], [3], [9], [10] and [11]).

Let $S$ and $\Gamma$ be nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \gamma, b)$ by $a\gamma b$, $S$ is called a Γ-*semigroup* [9] if $S$ satisfies the identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

Let $S$ be an arbitrary semigroup and $\Gamma$ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that $S$ is a

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$\Gamma$-semigroup. Thus a semigroup can be considered to be a $\Gamma$-semigroup.

Let $S$ be a $\Gamma$-semigroup and $\alpha$ be a fixed element in $\Gamma$. We define $a \cdot b = a\alpha b$ for all $a, b \in S$. We can easy to check that $(S, \cdot)$ is a semigroup.

$(S, \Gamma, \leq)$ is called an *ordered* $\Gamma$*-semigroup* [12] if $(S, \Gamma)$ is a $\Gamma$-semigroup and $(S, \leq)$ is a partially ordered set such that

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b \text{ for all } a, b, c \in S \text{ and } \gamma \in \Gamma.$$

We can see some properties of ordered $\Gamma$-semigroups in [5], [8] and [12].

In this paper, we investigate first and third isomorphism theorems for $\Gamma$-semigroups and ordered $\Gamma$-semigroups.

## 2 Isomorphism Theorems for $\Gamma$-semigroups

Let $S$ be a $\Gamma$-semigroup. An equivalence relation $\rho$ on $S$ is called a *right [resp. left] congruence* on $S$ if for each $a, b \in S$, $(a, b) \in \rho$ implies $(a\gamma t, b\gamma t) \in \rho$ [resp. $(t\gamma a, t\gamma b) \in \rho$] for all $t \in S$ and $\gamma \in \Gamma$. An equivalence relation $\rho$ on $S$ is called a *congruence* if $\rho$ is both a right and left congruence on $S$.

Let $S$ be a $\Gamma$-semigroup and $\rho$ be a congruence on $S$. For $a\rho, b\rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a\rho)\gamma(b\rho) = (a\gamma b)\rho$. This is well-defined, since for all $a, a', b, b' \in S$ and $\gamma \in \Gamma$,

$$\begin{aligned} a\rho = a'\rho \text{ and } b\rho = b'\rho &\Rightarrow (a, a'), (b, b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b), (a'\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b)\rho = (a'\gamma b')\rho. \end{aligned}$$

Let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. We have

$$(a\rho\gamma b\rho)\mu c\rho = ((a\gamma b)\rho)\mu c\rho = ((a\gamma b)\mu c)\rho = (a\gamma(b\mu c))\rho = a\rho\gamma(b\mu c)\rho = a\rho\gamma(b\rho\mu c\rho).$$

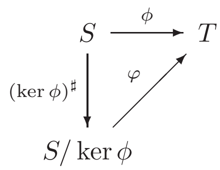
Then the quotient set $S/\rho$ is a $\Gamma$-semigroup.

Let $S$ and $T$ be $\Gamma$-semigroups under same $\Gamma$. The mapping $\phi : S \to T$ is called a $\Gamma$*-homomorphism* if $\phi(x\gamma y) = \phi(x)\gamma\phi(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. A $\Gamma$-homomorphism $\phi$ is called a $\Gamma$*-isomorphism* if $\phi$ is 1-1 and onto. Two $\Gamma$-semigroups $S$ and $T$ are $\Gamma$*-isomorphic* if there exists a $\Gamma$-isomorphism from $S$ onto $T$; it is denoted by $S \cong_\Gamma T$. Let $\phi$ be a $\Gamma$-homomorphism from $S$ into $T$. Let $\ker\phi$ be a relation on $S$ defined by

$$\ker\phi = \phi^{-1} \circ \phi = \{(x, y) \in S \times S \mid \phi(x) = \phi(y)\}.$$

It is easy to see that $\ker\phi$ is a congruence on $S$. The following theorem holds.

**Theorem 2.1.** *Let $S$ and $T$ be $\Gamma$-semigroups under same $\Gamma$ and $\phi : S \to T$ be a $\Gamma$-homomorphism. Then there is a $\Gamma$-monomorphism $\varphi : S/\ker\phi \to T$ such that* *ran* $\varphi =$ *ran* $\phi$ *and the diagram*




*commutes (i.e.* $\varphi \circ (\ker \phi)^\sharp = \phi$*) where the mapping* $(\ker \phi)^\sharp : S \to S/\ker \phi$ *defined by* $(\ker \phi)^\sharp(a) = a \ker \phi$ *for all* $a \in S$*.*

*Proof.* Define $\varphi : S/\ker \phi \to T$ by

$$\varphi(a \ker \phi) = \phi(a) \text{ for all } a \in S.$$

We have

$$a \ker \phi = b \ker \phi \Leftrightarrow (a, b) \in \ker \phi \Leftrightarrow \phi(a) = \phi(b).$$

Then $\varphi$ is well-defined and 1-1. $\varphi$ is a Γ-homomorphism since for all $a, b \in S$ and $\gamma \in \Gamma$,

$$\varphi((a \ker \phi)\gamma(b \ker \phi)) = \varphi((a\gamma b) \ker \phi) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(a \ker \phi)\gamma\varphi(b \ker \phi).$$

It is easy to see that $\operatorname{ran} \phi = \operatorname{ran} \varphi$. We have $\varphi \circ (\ker \phi)^\sharp = \phi$ since

$$(\varphi \circ (\ker \phi)^\sharp)(a) = \varphi((\ker \phi)^\sharp(a)) = \varphi(a \ker \phi) = \phi(a) \text{ for all } a \in S.$$

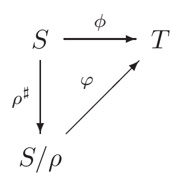
Hence, the theorem is proved. □

The following corollary follows from Theorem 2.1.

**Corollary 2.2.** *(First Isomorphism Theorem for Γ-semigroups)*
*Let* $S$ *and* $T$ *be Γ-semigroups under same* $\Gamma$ *and* $\phi : S \to T$ *be a Γ-homomorphism. Then* $S/\ker \phi \cong_\Gamma$ *ran* $\phi$*.*

The next theorem is concerned with a more general situation.

**Theorem 2.3.** *Let* $S$ *and* $T$ *be Γ-semigroups under same* $\Gamma$ *and* $\phi : S \to T$ *be a Γ-homomorphism. If* $\rho$ *is a congruence on* $S$ *such that* $\rho \subseteq \ker \phi$*, then there is a unique Γ-homomorphism* $\varphi : S/\rho \to T$ *such that ran* $\varphi =$ *ran* $\phi$ *and the diagram*




*commutes (i.e.* $\varphi \circ \rho^\sharp = \phi$*) where the mapping* $\rho^\sharp : S \to S/\rho$ *defined by* $\rho^\sharp(a) = a\rho$ *for all* $a \in S$*.*

*Proof.* Define $\varphi : S/\rho \to T$ by

$$\varphi(a\rho) = \phi(a) \text{ for all } a \in S.$$

We have for all $a, b \in S$,

$$a\rho = b\rho \Rightarrow (a, b) \in \rho \Rightarrow (a, b) \in \ker \phi \Rightarrow \phi(a) = \phi(b).$$

Then $\varphi$ is well-defined. Since for all $a, b \in S$ and $\gamma \in \Gamma$,

$$\varphi((a\rho)\gamma(b\rho)) = \varphi((a\gamma b)\rho) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(a\rho)\gamma\varphi(b\rho),$$

$\varphi$ is a $\Gamma$-homomorphism. It is easy to see that $\operatorname{ran} \phi = \operatorname{ran} \varphi$. For each $a \in S$, we have

$$(\varphi \circ \rho^\sharp)(a) = \varphi(\rho^\sharp(a)) = \varphi(a\rho) = \phi(a).$$

Then $\varphi \circ \rho^\sharp = \phi$. Next, let $\psi : S/\rho \to T$ be any $\Gamma$-homomorphism satisfying $\psi \circ \rho^\sharp = \phi$. Then for all $a \in S$,

$$\psi(a\rho) = \psi(\rho^\sharp(a)) = \psi \circ \rho^\sharp(a) = \phi(a) = \varphi(a\rho).$$

Therefore $\psi = \varphi$.

Hence, the theorem is proved. □

Let $\rho$ and $\sigma$ be congruences on a $\Gamma$-semigroup $S$ with $\rho \subseteq \sigma$. Define the relation $\sigma/\rho$ on $S/\rho$ by

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

To show $\sigma/\rho$ is well-defined, let $x\rho, a\rho, y\rho, b\rho \in S/\rho$ such that $x\rho = a\rho$ and $y\rho = b\rho$. So $(x, a), (y, b) \in \rho$. Since $\rho \subseteq \sigma$, $(x, a), (y, b) \in \sigma$. This implies $(x, y) \in \sigma \Leftrightarrow (a, b) \in \sigma$. The following theorem holds.

**Theorem 2.4.** *(Third Isomorphism Theorem for $\Gamma$-semigroups)* *Let $\rho$ and $\sigma$ be congruences on a $\Gamma$-semigroup $S$ with $\rho \subseteq \sigma$ and*

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

*Then $(i)$ $\sigma/\rho$ is a congruence on $S/\rho$ and $(ii)$ $(S/\rho)/(\sigma/\rho) \cong_\Gamma S/\sigma$.*

*Proof.* (i) Let $a \in S$. Then $(a, a) \in \sigma$, so $(a\rho, a\rho) \in \sigma/\rho$. Next, let $a, b \in S$ such that $(a\rho, b\rho) \in \sigma/\rho$. Then $(a, b) \in \sigma$. Since $\sigma$ is symmetric, $(b, a) \in \sigma$. Therefore $(b\rho, a\rho) \in \sigma/\rho$. Next, let $a, b, c \in S$ such that $(a\rho, b\rho), (b\rho, c\rho) \in \sigma/\rho$. So $(a, b), (b, c) \in \sigma$. Since $\sigma$ is transitive, $(a, c) \in \sigma$. Therefore $(a\rho, c\rho) \in \sigma/\rho$. Finally, let $a, b, c \in S$ and $\gamma \in \Gamma$. Assume $(a\rho, b\rho) \in \sigma/\rho$. Then $(a, b) \in \sigma$. Since $\sigma$ is a congruence on $S$, $(a\gamma c, b\gamma c) \in \sigma$. So $((a\gamma c)\rho, (b\gamma c)\rho) \in \sigma/\rho$. Then $((a\rho)\gamma(c\rho), (b\rho)\gamma(c\rho)) \in \sigma/\rho$. Similarly, $((c\rho)\gamma(a\rho), (c\rho)\gamma(b\rho)) \in \sigma/\rho$. Hence $\sigma/\rho$ is a congruence on $S/\rho$.

(ii) Define $\varphi : (S/\rho)/(\sigma/\rho) \to S/\sigma$ by

$$\varphi((a\rho)(\sigma/\rho)) = a\sigma \text{ for all } a \in S.$$

Clearly, $\varphi$ is onto. We have for all $a, b \in S$,

$$(a\rho)(\sigma/\rho) = (b\rho)(\sigma/\rho) \Leftrightarrow (a\rho, b\rho) \in \sigma/\rho \Leftrightarrow (a, b) \in \sigma \Leftrightarrow a\sigma = b\sigma.$$

Therefore $\varphi$ is well-defined and 1-1. To show $\varphi$ is a Γ-homomorphism, let $a, b \in S$ and $\gamma \in \Gamma$. We have

$$\begin{aligned} \varphi((a\rho)(\sigma/\rho)\gamma(a\rho)(\sigma/\rho)) &= \varphi((a\rho\gamma b\rho)(\sigma/\rho)) \\ &= \varphi((a\gamma b)\rho)(\sigma/\rho)) \\ &= (a\gamma b)\sigma \\ &= (a\sigma)\gamma(b\sigma) \\ &= \varphi((a\rho)(\sigma/\rho))\gamma\varphi((b\rho)(\sigma/\rho)). \end{aligned}$$

Hence $\varphi$ is a Γ-isomorphism. By Corollary 2.2, we have $(S/\rho)/(\sigma/\rho) \cong S/\sigma$. □

# 3 Isomorphism Theorems for Ordered Γ-semigroups

Let $S$ be a Γ-semigroup and $\rho$ be a congruence on $S$, in Section 2, we have that $S/\rho$ is a Γ-semigroup. The following question in natural : If $(S, \Gamma, \leq)$ is an ordered Γ-semigroup and $\rho$ is a congruence on $S$, then is the set $S/\rho$ an ordered Γ-semigroup? A probable order on $S/\rho$ could be the relation $\preceq_\rho$ on $S/\rho$ defined by means of the order $\leq$ on $S$, that is,

$$a\rho \preceq_\rho b\rho \Leftrightarrow \text{ there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } x \leq y.$$

But this relation is not an order, in general. We show it in the following example.

**Example 3.1.** We consider the ordered Γ-semigroup $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$ defined by the multiplication and the order $\leq$ below:

| $\alpha$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $b$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $c$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $e$ | $a$ | $e$ | $c$ | $d$ | $e$ |

| $\beta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $b$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $c$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $e$ | $c$ | $d$ | $e$ |
| $e$ | $a$ | $e$ | $c$ | $d$ | $e$ |

and $\leq = \{(a, a), (a, d), (b, b), (c, c), (c, e), (d, d), (e, e)\}$.

For $x, y, z \in S$ and $\gamma, \mu \in \Gamma$, we have

$$\begin{aligned} &(x\gamma y)\mu a = a = x\gamma(y\mu a), (x\gamma y)\mu c = c = x\gamma(y\mu c) \\ &(x\gamma y)\mu d = d = x\gamma(y\mu d), (x\gamma y)\mu e = e = x\gamma(y\mu e) \\ &(x\gamma y)\alpha b = e = x\gamma(y\alpha b) \\ &(x\gamma y)\beta b = e = x\gamma(y\beta b) \text{ if } y \neq b \\ &(x\gamma b)\beta b = e = x\gamma(b\beta b) \text{ if } x \neq b \\ &(b\alpha b)\beta b = e = b\alpha(b\beta b), (b\beta b)\beta b = b = b\beta(b\beta b). \end{aligned}$$

Then $S$ is a $\Gamma$-semigroup. Since

$$x\gamma a \leq x\gamma d, a\gamma x = d\gamma x, x\gamma c \leq x\gamma e, c\gamma x = e\gamma x \text{ for all } x \in S \text{ and } \gamma \in \Gamma,$$

$S$ is an ordered $\Gamma$-semigroup.

Let $\rho$ be the congruence on $S$ defined as follows:

$$\rho = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,e),(e,a),(c,d),(d,c)\}.$$

Let $\preceq_\rho$ be an order on $S/\rho$ defined by means of the order $\leq$ on $S$, that is,

$$a\rho \preceq_\rho b\rho \Leftrightarrow \text{ there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } x \leq y.$$

We have $a\rho = \{a,e\}, b\rho = \{b\}$ and $c\rho = \{c,d\}$. Also we have $a\rho \preceq_\rho c\rho$ and $c\rho \preceq_\rho a\rho$ but $a\rho \neq c\rho$. Thus $\preceq_\rho$ is not an order relation on $S/\rho$. □

The following question arises : Is there a congruence $\rho$ on an ordered $\Gamma$-semigroup $S$ for which $S/\rho$ is an ordered $\Gamma$-semigroup ? This leads us to the concept of pseudo-orders of ordered $\Gamma$-semigroups.

Now we study pseudo-orders and isomorphism theorems in ordered $\Gamma$-semigroups analogous to pseudo-orders and isomorphism theorems in ordered semigroups considered by Kehayopulu and Tsingelis [6, 7].

Let $(S,\Gamma,\leq)$ be an ordered $\Gamma$-semigroup. A relation $\rho$ on $S$ is called a *pseudo-order* on $S$ if

(i) $\leq \subseteq \rho$,

(ii) for all $a,b,c \in S, (a,b) \in \rho$ and $(b,c) \in \rho$ imply $(a,c) \in \rho$ and

(iii) for all $a,b \in S, (a,b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$ for all $c \in S$ and $\gamma \in \Gamma$.

If $\rho$ is a pseudo-order on $S$, let $\overline{\rho}$ be a relation on $S$ defined by

$$\overline{\rho} = \rho \cap \rho^{-1}.$$

We have that $(a,b) \in \overline{\rho} \Leftrightarrow (a,b) \in \rho$ and $(b,a) \in \rho$.

**Proposition 3.1.** *Let $(S,\Gamma,\leq)$ be an ordered $\Gamma$-semigroup and $\rho$ be a pseudo-order on $S$. Then $\overline{\rho}$ is a congruence on $S$.*

*Proof.* Let $a \in S$. Since $(a,a) \in \leq$ and $\leq \subseteq \rho$, $(a,a) \in \rho$. Then $(a,a) \in \overline{\rho}$. Next, let $a,b \in S$ such that $(a,b) \in \overline{\rho}$. Then $(a,b) \in \rho$ and $(b,a) \in \rho$. This implies that $(b,a) \in \overline{\rho}$. To show that $\rho$ is transitive, let $a,b,c \in S$ such that $(a,b),(b,c) \in \overline{\rho}$. Then $(a,b),(b,a),(b,c),(c,b) \in \rho$. Thus $(a,c),(c,a) \in \rho$. Hence $(a,c) \in \overline{\rho}$. Finally, let $a,b \in S$ such that $(a,b) \in \overline{\rho}$. Then $(a,b),(b,a) \in \rho$. Then $(c\gamma a, c\gamma b),(a\gamma c, b\gamma c),(c\gamma b, c\gamma a),(b\gamma c, a\gamma c) \in \rho$ for all $c \in S$ and $\gamma \in \Gamma$. Therefore $(a\gamma c, b\gamma c),(c\gamma a, c\gamma b) \in \overline{\rho}$ for all $c \in S$ and $\gamma \in \Gamma$. □

Let $S$ be an ordered $\Gamma$-semigroup and $\rho$ be a pseudo-order on $S$. By proposition 3.1, we have that $\overline{\rho}$ is a congruence on $S$. Then $S/\overline{\rho}$ is a $\Gamma$-semigroup. Next, for each $a\overline{\rho}, b\overline{\rho} \in S/\overline{\rho}$, define the order $\preceq_{\overline{\rho}}$ on $S/\overline{\rho}$ by

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Leftrightarrow \text{ there exist } x \in a\overline{\rho} \text{ and } y \in b\overline{\rho} \text{ such that } (x,y) \in \rho.$$

**Proposition 3.2.** *Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup and $\rho$ be a pseudo-order on $S$. The following statements are true.*

(*i*) *For $a, b \in S, a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ if and only if $(a, b) \in \rho$.*

(*ii*) *$\preceq_{\overline{\rho}}$ is an order on $S/\overline{\rho}$.*

*Proof.* (i) If $(a, b) \in \rho$, then clearly, $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$. Conversely, assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$. Then there exist $x \in a\overline{\rho}$ and $y \in b\overline{\rho}$ such that $(x, y) \in \rho$. Since $(x, a) \in \overline{\rho}$ and $(y, b) \in \overline{\rho}$, $(x, a), (a, x), (b, y), (y, b) \in \rho$. Since $(a, x), (x, y), (y, b) \in \rho, (a, b) \in \rho$.

(ii) Let $a, b, c \in S$. Since $(a, a) \in \leq \subseteq \rho$, $a\overline{\rho} \preceq_{\overline{\rho}} a\overline{\rho}$. Assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ and $b\overline{\rho} \preceq_{\overline{\rho}} a\overline{\rho}$. By (i), $(a, b) \in \rho$ and $(b, a) \in \rho$. Then $(a, b) \in \overline{\rho}$. So $a\overline{\rho} = b\overline{\rho}$. Finally, assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ and $b\overline{\rho} \preceq_{\overline{\rho}} c\overline{\rho}$. By (i), $(a, b) \in \rho$ and $(b, c) \in \rho$. Therefore $(a, c) \in \rho$. By (i), $a\overline{\rho} \preceq_{\overline{\rho}} c\overline{\rho}$. Hence $\preceq_{\overline{\rho}}$ is an order on $S/\overline{\rho}$. □

Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup, $\rho$ be a pseudo-order on $S$ and $x, y \in S$ such that $x\overline{\rho} \preceq_{\overline{\rho}} y\overline{\rho}$. Then there exist $a \in x\overline{\rho}$ and $b \in y\overline{\rho}$ such that $(a, b) \in \rho$. Thus $(x, a) \in \overline{\rho}$ and $(y, b) \in \overline{\rho}$. Then $(x, a), (a, x), (y, b), (b, y) \in \rho$. Let $c \in S$ and $\gamma \in \Gamma$. Therefore $(x\gamma c, a\gamma c), (a\gamma c, x\gamma c), (y\gamma c, b\gamma c), (b\gamma c, y\gamma c) \in \rho$. Thus $(x\gamma c, a\gamma c), (y\gamma c, b\gamma c) \in \overline{\rho}$. So $(x\gamma c)\overline{\rho} = (a\gamma c)\overline{\rho}$ and $(y\gamma c)\overline{\rho} = (b\gamma c)\overline{\rho}$. Since $(a, b) \in \rho, (a\gamma c, b\gamma c) \in \rho$. Hence $(x\gamma c)\overline{\rho} \preceq_{\overline{\rho}} (y\gamma c)\overline{\rho}$. Similarly, $(c\gamma x)\overline{\rho} \preceq_{\overline{\rho}} (c\gamma y)\overline{\rho}$. Therefore $(x\overline{\rho})\gamma(c\overline{\rho}) \preceq_{\overline{\rho}} (y\overline{\rho})\gamma(c\overline{\rho})$ and $(c\overline{\rho})\gamma(x\overline{\rho}) \preceq_{\overline{\rho}} (c\overline{\rho})\gamma(y\overline{\rho})$. Thus $S/\overline{\rho}$ is an ordered $\Gamma$-semigroup. Then the following proposition holds.

**Proposition 3.3.** *Let $(S, \Gamma, \leq)$ be a $\Gamma$-semigroup and $\rho$ be a pseudo-order on $S$. Then $S/\overline{\rho}$ is an ordered $\Gamma$-semigroup.*

Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered $\Gamma$-semigroups under same $\Gamma$ and $\phi : S \to T$ be a mapping from $S$ into $T$. $\phi$ is called *isotone* if for $x, y \in S, x \leq_S y$ implies $\phi(x) \leq_T \phi(y)$. $\phi$ is called *reverse isotone* if $x, y \in S, \phi(x) \leq_T \phi(y)$ implies $x \leq_S y$. $\phi$ is called an *ordered $\Gamma$-homomorphism* if $\phi$ is isotone and satisfies $\phi(x\gamma y) = \phi(x)\gamma\phi(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. Each reverse isotone mapping $\phi : S \to T$ is 1-1. Indeed: Let $x, y \in S$ such that $\phi(x) = \phi(y)$. Since $\phi(x) \leq_T \phi(y)$, $x \leq_S y$. Similarly, since $\phi(y) \leq_T \phi(x)$, $y \leq_S x$. Then $x = y$. $\phi$ is called an *ordered $\Gamma$-isomorphism* if it is a $\Gamma$-homomorphism, onto and reverse isotone. Two ordered $\Gamma$-semigroups $S$ and $T$ are *$\Gamma$-isomorphic* if there exists an ordered $\Gamma$-isomorphism from $S$ onto $T$; it is denoted by $S \cong_\Gamma T$.

**Proposition 3.4.** *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered $\Gamma$-semigroups under same $\Gamma$ and $\phi : S \to T$ be an ordered $\Gamma$-homomorphism. Define the relation $\widetilde{\phi}$ on $S$ by*

$$\widetilde{\phi} = \{(a, b) \in S \times S \mid \phi(a) \leq_T \phi(b)\}.$$

*Then $\widetilde{\phi}$ is a pseudo-order on $S$.*

*Proof.* Let $(a, b) \in \leq_S$. Since $a \leq_S b$ and $\phi$ is isotone, $\phi(a) \leq_T \phi(b)$. Then $(a, b) \in \widetilde{\phi}$. Next, let $a, b, c \in S$ such that $(a, b), (b, c) \in \widetilde{\phi}$. So $\phi(a) \leq_T \phi(b), \phi(b) \leq_T \phi(c)$. Then $\phi(a) \leq_T \phi(c)$. This implies $(a, c) \in \widetilde{\phi}$. Finally, let $a, b, c \in S$ and $\gamma \in \Gamma$.

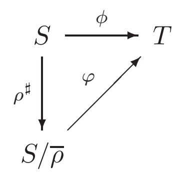
Assume $(a, b) \in \widetilde{\phi}$. Since $\phi(a) \leq_T \phi(b), \phi$ is an ordered $\Gamma$-homomorphism and $T$ is an ordered $\Gamma$-semigroup,

$$\phi(a\gamma c) = \phi(a)\gamma\phi(c) \leq_T \phi(b)\gamma\phi(c) = \phi(b\gamma c).$$

Then $(a\gamma c, b\gamma c) \in \widetilde{\phi}$. Similarly, $(c\gamma a, c\gamma b) \in \widetilde{\phi}$.

Hence $\widetilde{\phi}$ is a pseudo-order on $S$. □

**Theorem 3.5.** *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered $\Gamma$-semigroups under same $\Gamma$, $\phi : S \to T$ be an ordered $\Gamma$-homomorphism. If $\rho$ is a pseudo-order on $S$ such that $\rho \subseteq \widetilde{\phi}$, then the mapping $\varphi : S/\overline{\rho} \to T$ defined by $\varphi(a\overline{\rho}) = \phi(a)$ is a unique ordered $\Gamma$-homomorphism of $S/\overline{\rho}$ into $T$ such that ran $\varphi$ = ran $\phi$ and the diagram*




*commutes (i.e, $\varphi \circ \rho^\sharp = \phi$) where the mapping $\rho^\sharp : S \to S/\overline{\rho}$ defined by $\rho^\sharp(a) = a\overline{\rho}$ for all $a \in S$.*

*Proof.* Define $\varphi : S/\overline{\rho} \to T$ by

$$\varphi(a\overline{\rho}) = \phi(a) \text{ for all } a \in S.$$

We have $\varphi$ is well-defined since for all $a, b \in S$,

$$\begin{aligned} a\overline{\rho} = b\overline{\rho} &\Rightarrow (a, b) \in \overline{\rho} \\ &\Rightarrow (a, b), (b, a) \in \rho \\ &\Rightarrow (a, b), (b, a) \in \widetilde{\phi} \\ &\Rightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a) \\ &\Rightarrow \phi(a) = \phi(b). \end{aligned}$$

Let $a, b \in S$ and $\gamma \in \Gamma$. We have

$$\varphi(a\overline{\rho}\gamma b\overline{\rho}) = \varphi((a\gamma b)\overline{\rho}) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(a\overline{\rho})\gamma\varphi(b\overline{\rho})$$

and

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Rightarrow (a, b) \in \rho \subseteq \widetilde{\phi} \Rightarrow \phi(a) \leq_T \phi(b).$$

Therefore $\varphi$ is an ordered $\Gamma$-homomorphism. For each $a \in S$, we have

$$(\varphi \circ \rho^\sharp)(a) = \varphi(\rho^\sharp(a)) = \varphi(a\overline{\rho}) = \phi(a).$$

Then $\varphi \circ \rho^\sharp = \phi$. Next, let $\psi : S/\overline{\rho} \to T$ be any ordered $\Gamma$-homomorphism such that $\psi \circ \rho^\sharp = \phi$. For all $a \in S$, we have

$$\psi(a\overline{\rho}) = \psi(\rho^\sharp(a)) = (\psi \circ \rho^\sharp)(a) = \phi(a) = \varphi(a\overline{\rho}),$$

so $\psi = \varphi$. Finally, we have $\operatorname{ran}\varphi = \{\varphi(a\overline{\rho}) \mid a \in S\} = \{\phi(a) \mid a \in S\} = \operatorname{ran}\phi$.

Hence the theorem is proved. □

Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ be an ordered Γ-homomorphism. In section 2, we have that $\ker\phi = \{(a,b) \in S \times S \mid \phi(a) = \phi(b)\}$ is a congruence on $S$. Moreover, we have

$$\begin{aligned}(a,b) \in \ker\phi &\Leftrightarrow \phi(a) = \phi(b)\\ &\Leftrightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a)\\ &\Leftrightarrow (a,b) \in \widetilde{\phi} \text{ and } (b,a) \in \widetilde{\phi}\\ &\Leftrightarrow (a,b) \in \overline{\widetilde{\phi}}.\end{aligned}$$

So $\ker\phi = \overline{\widetilde{\phi}}$. Then the following corollary holds.

**Corollary 3.6.** *(First Isomorphism Theorem for ordered Γ-semigroups)* *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ be an ordered Γ-homomorphism. Then $S/\ker\phi \cong_\Gamma ran\,\phi$.*

*Proof.* We apply the first part of Theorem 3.5 for $\rho = \widetilde{\phi}$ and $\ker\phi = \overline{\widetilde{\phi}}$. Then the mapping $\varphi : S/\ker\phi \to T$ defined by $\varphi(a\ker\phi) = \phi(a)$ is an ordered Γ-homomorphism. To show $\varphi$ is reverse isotone, let $a, b \in S$ such that $\phi(a) \leq_T \phi(b)$. Then $(a,b) \in \widetilde{\phi}$. Since $\widetilde{\phi}$ is a pseudo-order on $S$, by Proposition 3.2(i), $a\ker\phi \preceq_{\overline{\widetilde{\phi}}} b\ker\phi$. Then $\varphi$ is reverse isotone. Therefore $\varphi$ is an ordered Γ-isomorphism. □

**Theorem 3.7.** *(Third Isomorphism Theorem for ordered Γ-semigroups)* *Let $\rho$ and $\sigma$ be pseudo-orders on an ordered Γ-semigroup $S$ such that $\rho \subseteq \sigma$. We define a relation $\sigma/\rho$ on $S/\overline{\rho}$ as follows:*

$$\sigma/\rho = \{(a\overline{\rho}, b\overline{\rho}) \in S/\overline{\rho} \times S/\overline{\rho} \mid (a,b) \in \sigma\}.$$

*Then (i) $\sigma/\rho$ is a pseudo-order on $S/\overline{\rho}$ and (ii) $(S/\overline{\rho})/(\overline{\sigma/\rho}) \cong_\Gamma S/\overline{\sigma}$.*

*Proof.* (i) Let $(a\overline{\rho}, b\overline{\rho}) \in \preceq_{\overline{\rho}}$. Then $(a,b) \in \rho$, it implies $(a,b) \in \sigma$. So $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$. Therefore $\preceq_{\overline{\rho}} \subseteq \sigma/\rho$. Next, let $a, b, c \in S$ such that $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$ and $(b\overline{\rho}, c\overline{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$ and $(b,c) \in \sigma$, so $(a,c) \in \sigma$. Therefore $(a\overline{\rho}, c\overline{\rho}) \in \sigma/\rho$. Finally, let $a, b, c \in S$ and $\gamma \in \Gamma$. Assume $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$, thus $(a\gamma c, b\gamma c) \in \sigma$. So $((a\gamma c)\overline{\rho}, (b\gamma c)\overline{\rho}) \in \sigma/\rho$. Therefore $(a\overline{\rho}\gamma c\overline{\rho}, b\overline{\rho}\gamma c\overline{\rho}) \in \sigma/\rho$. Similarly, $(c\overline{\rho}\gamma a\overline{\rho}, c\overline{\rho}\gamma b\overline{\rho}) \in \sigma/\rho$.

(ii) Define $\phi : S/\overline{\rho} \to S/\overline{\sigma}$ by

$$\phi(a\overline{\rho}) = a\overline{\sigma} \text{ for all } a \in S.$$

We have $\phi$ is well-defined since for all $a, b \in S$,

$$a\overline{\rho} = b\overline{\rho} \Rightarrow (a,b) \in \overline{\rho} \Rightarrow (a,b), (b,a) \in \rho \subseteq \sigma \Rightarrow (a,b) \in \overline{\sigma} \Rightarrow a\overline{\sigma} = b\overline{\sigma}.$$

Next, let $a, b \in S$ and $\gamma \in \Gamma$. We have

$$\phi(a\overline{\rho}\gamma b\overline{\rho}) = \phi((a\gamma b)\overline{\rho}) = (a\gamma b)\overline{\sigma} = a\overline{\sigma}\gamma b\overline{\sigma} = \phi(a\overline{\rho})\gamma\phi(b\overline{\rho})$$

and

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Rightarrow (a,b) \in \rho \Rightarrow (a,b) \in \sigma \Rightarrow a\overline{\sigma} \preceq_{\overline{\sigma}} b\overline{\sigma}.$$

Hence $\phi$ is an ordered Γ-homomorphism.

By the definition of $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \{(a\overline{\rho}, b\overline{\rho}) \in S/\overline{\rho} \times S/\overline{\rho} \mid \phi(a\overline{\rho}) \preceq_{\overline{\sigma}} \phi(b\overline{\rho})\}.$$

Thus

$$(a\overline{\rho}, b\overline{\rho}) \in \widetilde{\phi} \Leftrightarrow \phi(a\overline{\rho}) \preceq_{\overline{\sigma}} \phi(b\overline{\rho}) \Leftrightarrow a\overline{\sigma} \preceq_{\overline{\sigma}} b\overline{\sigma} \Leftrightarrow (a,b) \in \sigma \Leftrightarrow (a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho.$$

Then $\widetilde{\phi} = \sigma/\rho$, so $\ker \phi = \overline{\widetilde{\phi}} = \overline{\sigma/\rho}$. It is easy to show that $\operatorname{ran} \phi = S/\overline{\sigma}$. By Corollary 3.6, $(S/\overline{\rho})/(\overline{\sigma/\rho}) \cong_\Gamma S/\overline{\sigma}$. □

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