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Möbius Flowers

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Abstract If you bisect conjoined two Möbius bands along each centerline, some of them end in interlocking hearts, and the others end in two separate hearts. What makes the difference between the happy outcome and the unhappy one? In this paper, we unravel this "Möbius Love-Fate problem" by using the concept of knot theory, as well as generalizing this theorem in various ways.

MSC: 97A40
Keywords: Möbius bands; linking number; knot theory

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1. Introduction

Let us prepare an elongated strip of paper. By gluing together its two ends, we obtain a band called an annulus as shown in FIGURE 1(a). If we give one end a half-twist (180 degree rotation) before gluing together its two ends, we obtain a twisted band called a Möbius band as shown in FIGURE 1(b).

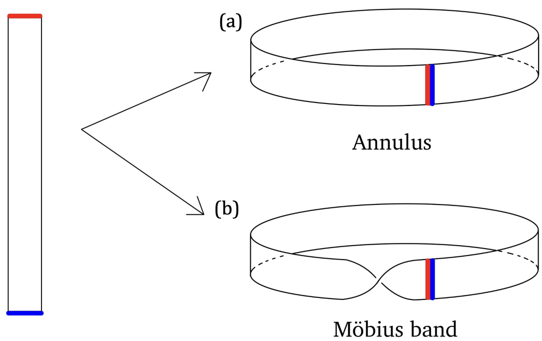

Figure 1. Annulus and Möbius band

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A Möbius band is known for a non-orientable surface with only one face and only one boundary. Cutting a Möbius band along the center line yields one band twice as long as the original length, and with two full-twists (2 360 twist). Here is a paper cross (see FIGURE 2(a)). We make a half-twist on one strip, then glue the two ends together, i.e., we create a Möbius band as shown in FIGURE 2(b). Next, do the same with another strip (see FIGURE 2(c)). The resulting FIGURE is called a conjoined Möbius.

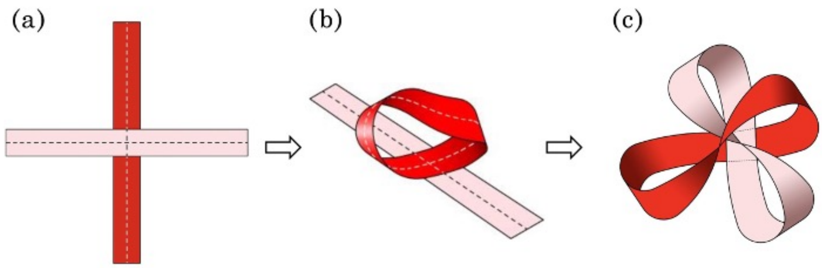


FIGURE 2. Creating a conjoined Möbius

If we bisect conjoined Möbius bands along each centerline, some of them end up as interlocking hearts, while the rest end up as two separated hearts as shown in FIGURE 3(a) and 3(b).

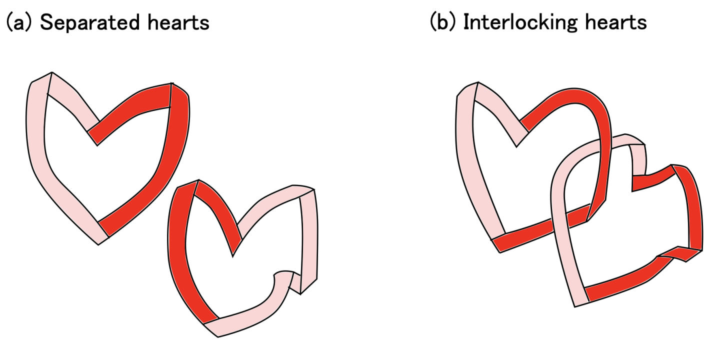


FIGURE 3. Separated hearts and interlocking hearts

The question now arises: What indicators allow us to differentiate between the two cases? This problem is called Möbius Love-Fate problem. We will reveal the mystery behind this problem and generalize the number of conjoined Möbius bands to an arbitrary natural number greater than 2.

Remark 1.1. In this paper, we treat the problem basically with non-topological words or ideas. We often use the rigid transformations so that the readers could easily understand what is going on here, by trying to tinker with them by themselves.

2. Preliminaries

2.1. Δ-Check and Möbius Flowers

There exist two types of Möbius bands depending on the direction of the twist. Here, we provide a method called a **Δ-check** with which we can distinguish between the two types of bands.

Definition 2.1. (Δ-check and two types of Möbius bands). First, we collapse a Möbius band into a triangle shape called a Möbius delta (Δ) as shown in FIGURE 4. The Möbius delta has three layers: the topmost layer (both ends are visible), the middle layer (only one end is visible), and the bottom layer (both ends are invisible). Label the topmost layer as I, the middle layer as II, and the bottom layer as III (See FIGURE 5). A Möbius band is called **Type** α, denoted by $\Delta\alpha$, if the direction of I, II, and III is clockwise (FIGURE 5(a)). On the other hand, a Möbius band is called **Type** β, denoted by $\Delta\beta$, if the direction of I, II and III is counter-clockwise (FIGURE 5(b)). Note that the direction of I, II, and III in both types remains unchanged even if we flip or rotate a Möbius delta.

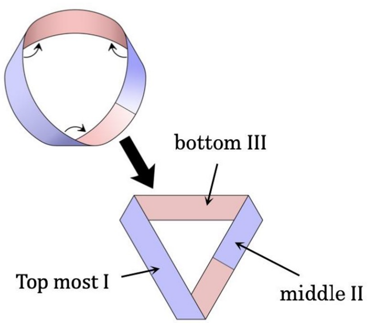


Figure 4. Δ-check

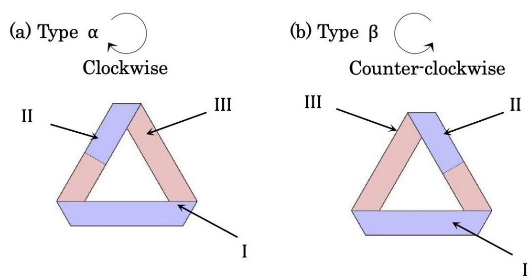


Figure 5. Two types of Möbius bands

Remark 2.2. Topologically, $\Delta\alpha$ is called Möbius band with -1 half twist, $\Delta\beta$ is called Möbius band with +1 half twist as in FIGURE 6 and 7.

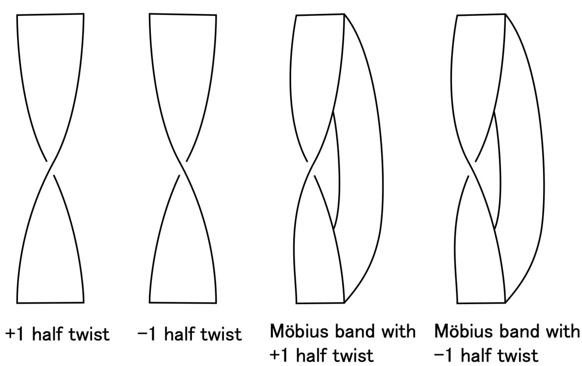


Figure 6. +1 / -1 half twist and Möbius band with +1 / -1 half twist

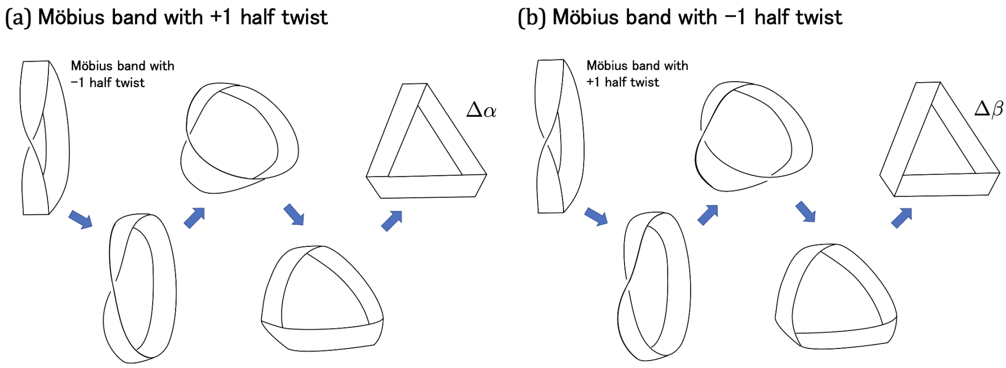


FIGURE 7. Transformation from Möbius band with +1 / -1 half twist to $\Delta\alpha$ / $\Delta\beta$

Definition 2.3. (An N-star and an N-flower). Let four vertices and the centerline of a rectangle strip S_i be $A_i, B_i, A_{i+N}, B_{i+N}$ (labeled clockwisely), and l_i, respectively for $i = 0, \cdots, N-1$ as shown in FIGURE 8(a). Arrange S_x $(x = 0, \cdots, N-1)$ such that l_x coincides with the clockwisely-rotated l_0 around its center by $\frac{x\pi}{N}$, and joint overlapping parts of S_i as shown in FIGURE 8(b). Then, glue the overlapped center parts all together as shown in FIGURE 8(c). The object generated by this procedure is called an N**-star**. Make N Möbius bands M_xs from an N strips S_xs of an N-star by making a half-twist and gluing together the ends of each strip (Note that A_x is attached to A_{x+N} and B_x is attached to B_{x+N}). We call the obtained object an N**-flower** (For example, 5-flower is illustrated in FIGURE 9).

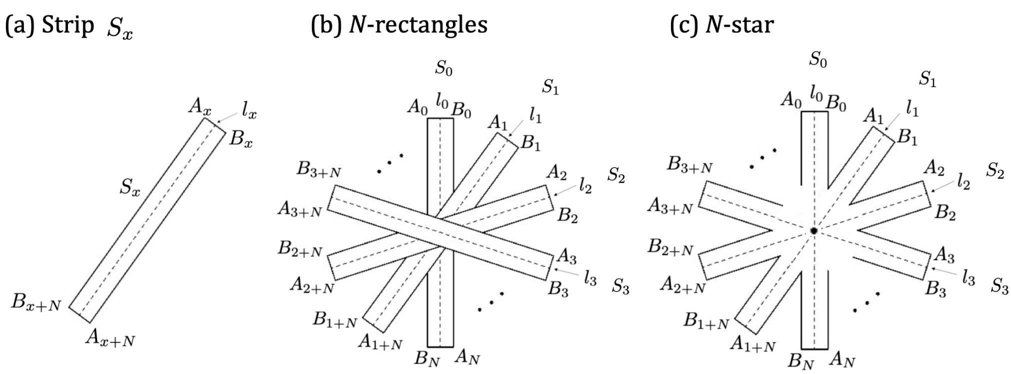


FIGURE 8. Strip S_x, N-rectangles and an N-star

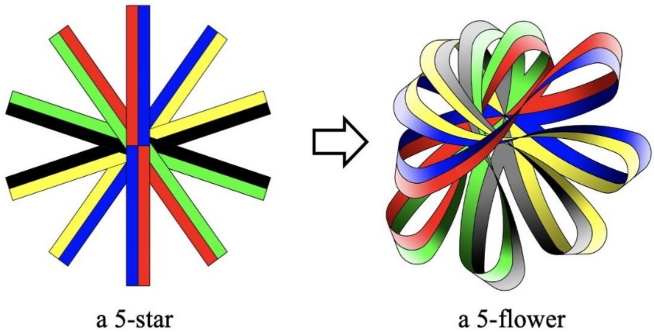


FIGURE 9. A 5-flower created from a 5-star

Note that these N Möbius bands of N-flowers are layered (from bottom to top) according to the order of gluing the rectangular strips into Möbius bands, or what we will refer to as gluing order. Let $(p_0, p_1, p_2, ..., p_{N-1})$ be a permutation of $(0, 1, 2, ..., N-1)$. The gluing order $S_{p_0} \to S_{p_1} \to S_{p_2} \to \cdots \to S_{p_{N-1}}$ can be denoted simply by $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$. The N-flower created according to the gluing order $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$ is referred to as an N-flower with $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$. FIGURE 10 shows two distinct 3-flowers created from a 3-star (FIGURE 10(a)); one is a 3-flower with $0 \to 1 \to 2$ (FIGURE 10(b)) and the other is a 3-flower with $1 \to 0 \to 2$ (FIGURE 10(c)). Note that from bottom to top, 1st layer $\to$ 2nd layer $\to$ 3rd layer of each of these 3-flowers is $M_0 \to M_1 \to M_2$, $M_1 \to M_0 \to M_2$ which coincides with its gluing order $0 \to 1 \to 2$, $1 \to 0 \to 2$, respectively.

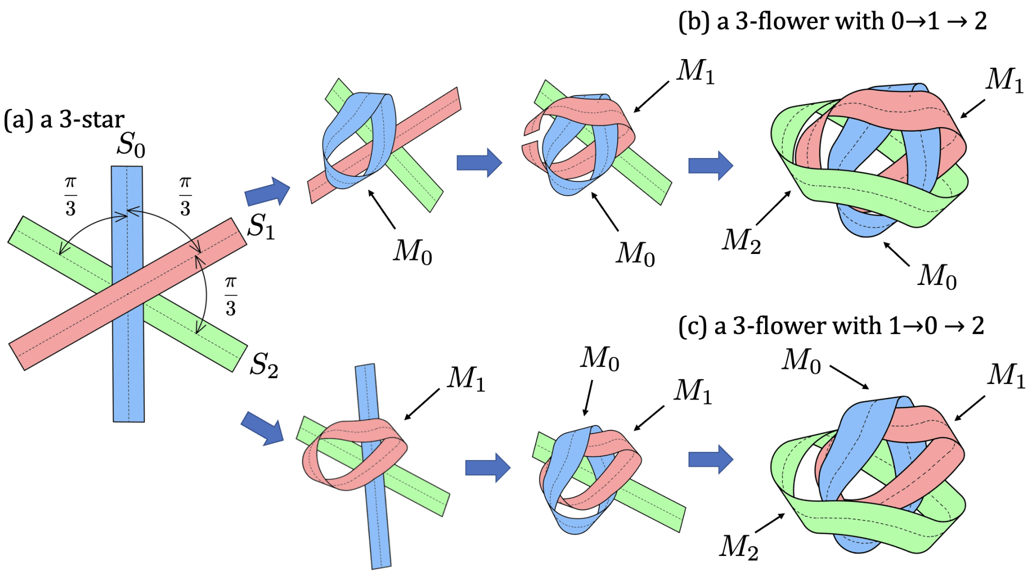


FIGURE 10. Two distinct 3-flowers created from a 3-star

Let the map $f : \{M_i\} \to \{\alpha, \beta\}$, which indicates the type of Möbius band M_i. The flower type of an N-flower is defined by its gluing order and $S = (f(M_0)f(M_1)\cdots f(M_{N-1}))$. That is, N-flowers can be classified depending on the gluing order and S. Then, the N-flower created according to the gluing order $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$ and $S = (f(M_0)f(M_1)\cdots f(M_{N-1}))$ is referred to as an N-flower with $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$ for $(f(M_0)f(M_1)\cdots f(M_{N-1}))$.

2.2. Linking Number

Definition 2.4. A **knot** is an embedding of the circle S_1 into three-dimensional Euclidean space, $\mathbb{R}_3$. Generally two knots are considered equivalent if there exists a continuous deformation of $\mathbb{R}_3$ which takes one knot to the other. A **link** is a collection of knots which do not intersect, but which may be linked (or knotted) together. A knot can be described as a link with one component.

A **link diagram** is basically a picture of a projection of a link onto a plane, in which only a finite number of overlapped points appear, and at each overlapped point (which we call a crossing point), just two curves cross each other transversally. Here below is the way to draw the diagram around the crossing point. At the crossing point on the projection of a link onto a plane, the branch lying above it is called an overpass and the branch lying below it an underpass. In the link diagram, you should create a break in the underpass. The resulting diagram is an immersed plane curve with the additional data of which curve is over and which is under at each crossing point.

Definition 2.5. At a crossing point, c, of an oriented link diagram, as shown in FIGURE 11, we have two possible configurations. In the left case, we assign sign$(c) = +1$ to the crossing point c, while in the right case, we assign sign$(c) = -1$. The left crossing point is said to be positive, while the right one is said to be negative.

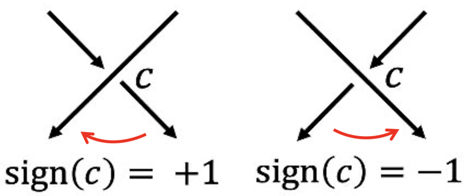


Figure 11. Signs at crossings

Let L be the link with 2 components K_1, K_2, say, $L = (K_1, K_2)$. Suppose that the crossing points of D at which the projection of K_1 and K_2 intersect are: $D = \{c_1, c_2, \ldots, c_m\}$.

Remark 2.6. We ignore the crossing points of the projections of K_1, and K_2, which are self intersections of the knot component.

Then, $\frac{1}{2}(\text{sign}(c_1) + \text{sign}(c_2) + \ldots + \text{sign}(c_m))$ is called the **linking number** of K_1 and K_2, which we shall denote by $lk(L)$ or $lk(K_1, K_2)$. Indeed, it is an invariant of the link L, for a proof, see [4].

Theorem 2.7. (Linking Number Theorem). *Let $L = (K_1, K_2)$ be the link with two knot components K_1 and K_2. If $lk(L) = lk(K_1, K_2) \neq 0$, K_1 and K_2 are interlocked. If the knot components K_1 and K_2 are separated, $lk(L) = lk(K_1, K_2) = 0$. However, the converse is not always true (ref. Whitehead Link).*

3. Möbius Love-Fate Problem

The mixed-conjoined Möbius is a conjoined Möbius composed of two different types of bands, i.e., one is $\Delta\alpha$ (with -1 half twist) and the other is $\Delta\beta$ (with +1 half twist) after the Δ-check. On the other hand, single-conjoined Möbius is a conjoined Möbius composed of two bands that are of the same type, i.e., both $\Delta\alpha$ or both $\Delta\beta$ after the Δ-check.

Theorem 3.1. (Möbius Love-Fate Theorem). *If a mixed-conjoined Möbius is bisected, two bands (which we call petals) produced when bisecting 2-flower are interlocked (FIGURE 12(a)). If a single-conjoined Möbius is bisected, two bands(petals) produced when bisecting 2-flower are separated (FIGURE 12(b)).*

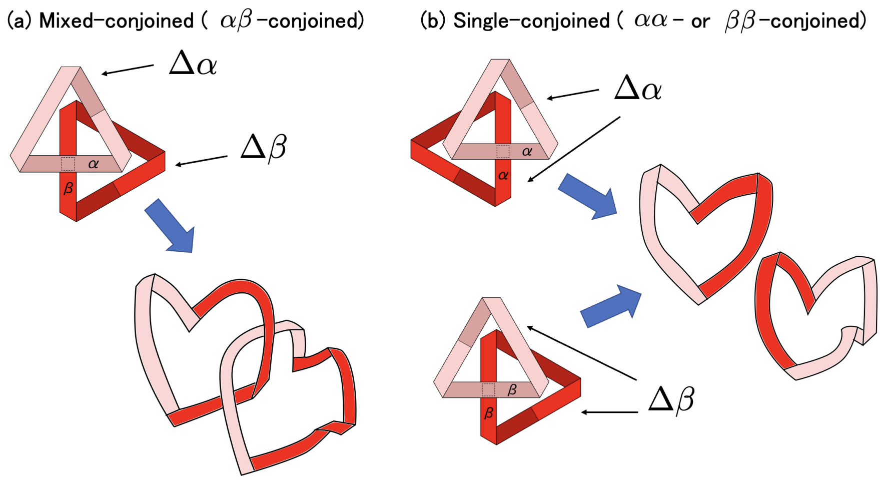


FIGURE 12. Möbius Love-Fate Theorem

Remark 3.2. From now on, we identify the closed ribbons (e.g. annulus, Möbius bands) or the collection of ribbons with the center of the ribbons, which are the knots or the links. By then, we ignore the twists of the ribbons.

Proof. **Case 1: Mixed-conjoined Möbius**

Collapse each Möbius band of a mixed-conjoined Möbius into Möbius delta. Without loss of generality, any conjoined Möbius can be turned into the conjoined Δs where the layer I of one delta and the layer III of the other delta are jointed (FIGURE 13(a)). The intersection between two bands of a conjoined Möbius forms a square. We divide it into four quadrants 1,2,3,4 labeled clockwise (FIGURE 13(a)). Bisect this conjoined Δs, and then one band (we name this a petal P_0) consists of two strips with a half width derived from quadrants 1 and 3, and the other band or petal P_1 consists of two strips with a half width derived from the quadrants 2 and 4 (FIGURE 13(b)(c)). We name the link of the two petals P_0 and P_1 the petal link (P_0, P_1). Next, give the direction to the link diagram of this petal link (P_0, P_1) as shown in FIGURE 13(d). Computing the linking number of (P_0, P_1), we have $\frac{1}{2}(-1+1+1+1) = 1$. By theorem 2.7, these two petals P_0 and P_1 are interlocked (FIGURE 13(d)).

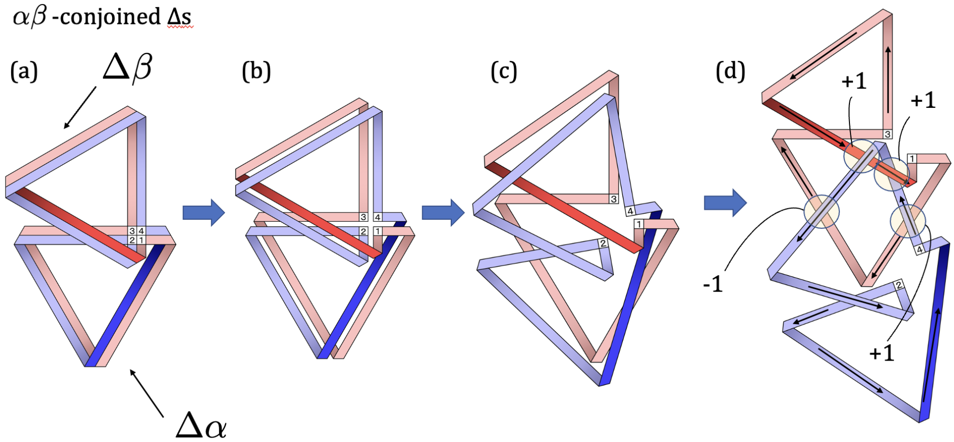


FIGURE 13. An illustrative proof for $\alpha\beta$-conjoined Δs

Case 2: Single-conjoined Möbius In the same manner as case 1, any single-conjoined Möbius can be turned into a conjoined deltas where the layer I of one and the layer III of the other are joined (FIGURE 14(a), FIGURE 15(a)). Bisect it and transform them in $\mathbb{R}_3$ as shown (FIGURE 14(b)-(d), FIGURE 15(b)-(d)). As you can see below in FIGURE14(d)(/FIGURE15(d)), these two petals P_0 and P_1 are separated.

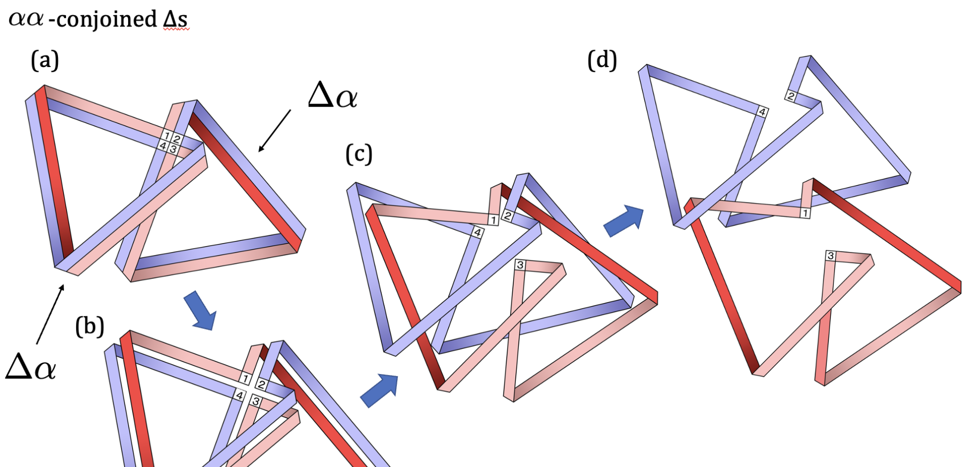


FIGURE 14. An illustrative proof for $\alpha\alpha$-conjoined Δs

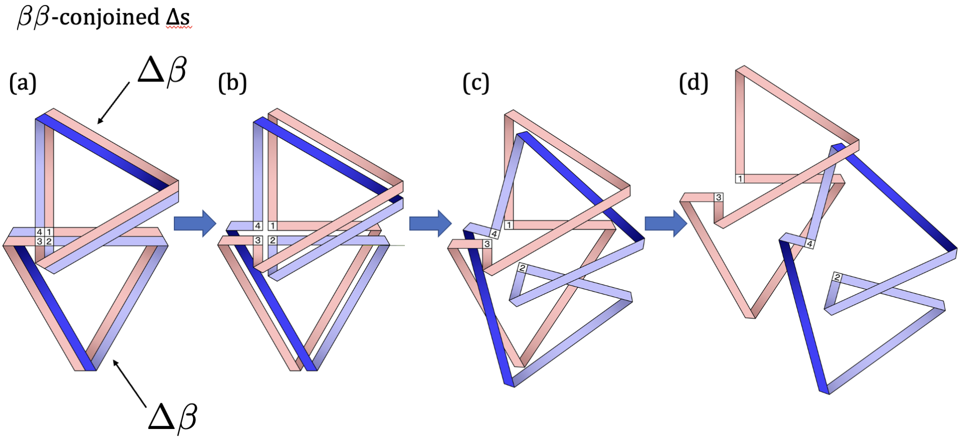


FIGURE 15. An illustrative proof for $\beta\beta$-conjoined Δs ■

4. Main Results

Next, consider bisecting N-flowers. Now, assume $N = 3$ and take a 3-star and a 3-flower as an example (FIGURE 16(a)), since the same thing holds for $N \geq 4$. Then, arrange N-rectangles $S_0, S_1, \ldots, S_{N-1}$ so that they meet at the center with each other as shown in (a). Then, glue the overlapped part so that you can get an N-star as shown in (b)-1. By connecting A_xB_x with $A_{x+N}B_{x+N}$ with -1 or +1 half twist to make the Möbius band M_x $(x = 0, 1, \ldots, N-1)$ whose type is α or β , as shown in (b)-2. If you bisect one M_x (one of the Möbius band) in (b)-2 along its centerline, it would be like (c)-2. Then, you can see the green knot with one self-crossing as shown in (c)-3, which we call the petal P_x (derived from S_{x-1} and S_x). Note that, we can get the link diagram of any petal P_x with one self-crossing and we can further see that for any $y \neq z$, there is a link diagram of the petal link (P_y, P_z) with 4 crossing points, as shown in FIGURE 16 (d). Here below, the subscripts are the numbers module N. For example, $S_0 = S_N$ and $P_{-1} = P_{N-1}$.

FIGURE 17 shows the part of the resulting objects of bisection of 4-flowers for $(\alpha\alpha\alpha\beta)$. In general, the following three theorems uniquely determine the resulting petal link with all the petals P_x $(x = 0, 1, \ldots, N-1)$ from a bisection of an N-flower.

In the following theorems, S_{x-1}, S_x and S_{x+1} are three clockwisely-consecutive strips in an N-star. And let $x-1, x$ and $x+1$ be the ith, jth and kth numbers in the gluing order of an N-flower with $p_0 \to p_1 \to \cdots \to p_{N-1}$, that is, $p_i = x-1, p_j = x, p_k = x+1$.

Theorem 4.1. *Suppose M_x is a Type α Möbius band (with -1 half twist) in an N-flower, which is created from a strip S_x in an N-star. The two petals P_x and P_{x+1} derived from M_x are interlocked if $i < j < k$ or $j < k < i$ or $k < i < j$. Otherwise, the two petals P_x and P_{x+1} are separated.*

Theorem 4.2. *Suppose M_x is a Type β Möbius band (with +1 half twist) in an N-flower, which is created from a strip S_x in an N-star. The two petals P_x and P_{x+1} derived from M_x are interlocked if $k < j < i$ or $j < i < k$ or $i < k < j$. Otherwise, the two petals P_x and P_{x+1} are separated.*

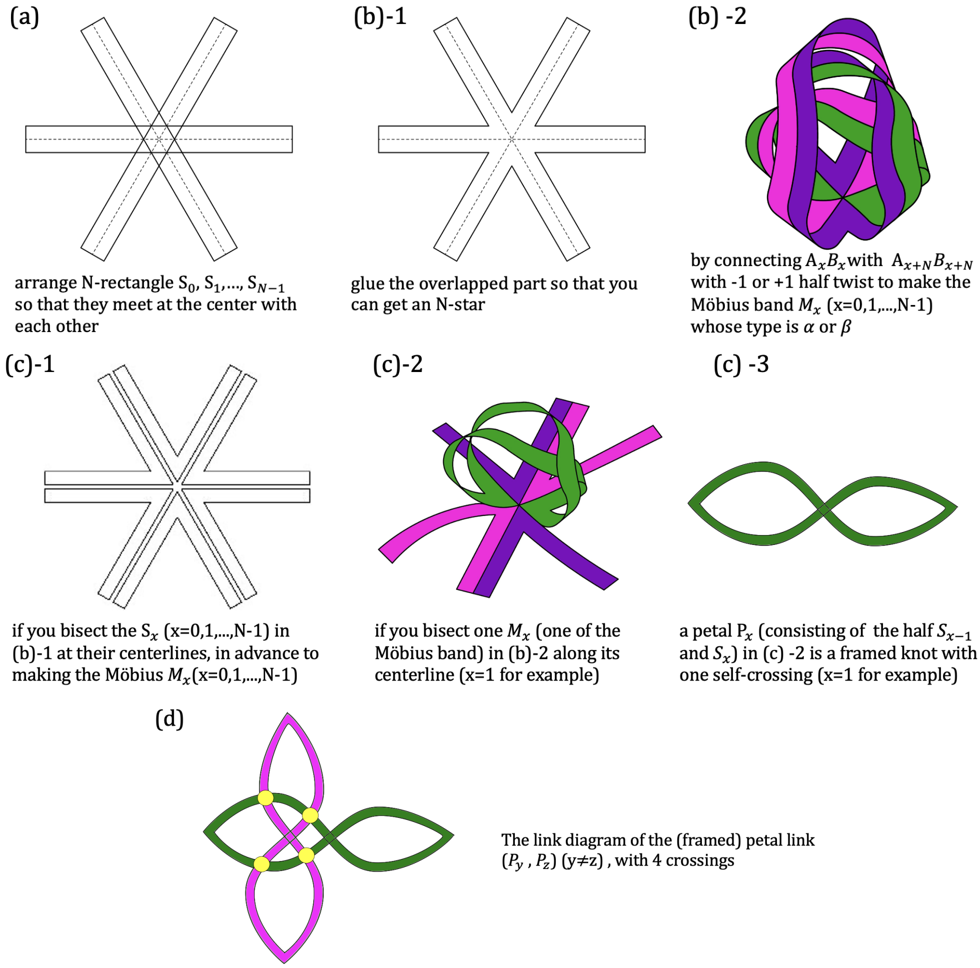


FIGURE 16. Petals obtained from a bisection of a 3-flower and the link diagram of the petal link (P_y, P_z) $(y \neq z)$

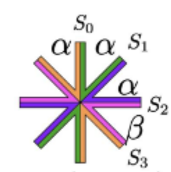
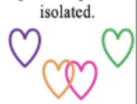
Gluing order / 4-star	0→1→2→3	0→1→3→2	0→2→1→3	0→2→3→1	0→3→1→2	0→3→2→1
 $\mathcal{S} = (\alpha\alpha\alpha\beta)$	 4 loops form a path, with green & purple are in the middle.	Pink and orange are interlocked.  All except pink form a cycle.	The purple loop is isolated  Other 3 loops form a path, green in the middle.	The green loop is isolated  Other 3 loops form a path, purple in the middle.	Orange and pink are interlocked.  All except orange form a cycle.	The purple & green loops are isolated.  Orange and pink are interlocked.

FIGURE 17. The part of resulting objects of bisected 4-flowers for $(\alpha\alpha\alpha\beta)$

Theorem 4.3. *Assume that the two petals P_y and P_z ($0 \leq y < z \leq N-1$) are not derived from a common Möbius band in an N-flower (i.e. $y+1 < z$, $(y,z) \neq (0, N-1)$). Let $y-1, y, z-1$ and z be i-th, j-th, k-th and l-th numbers in the gluing order of the N-flower with $p_0 \to p_1 \to p_2 \to \cdots \to p_{N-1}$, that is, $p_i = y-1, p_j = y, p_k = z-1, p_l = z$. The two petals P_y and P_z are interlocked if $i < k < j < l$ or $i < l < j < k$ or $j < k < i < l$ or $j < l < i < k$ or $k < i < l < j$ or $l < i < k < j$ or $k < j < l < i$ or $l < j < k < i$. That is, if you call the pair (i, j) as A and the pair $(k,\ l)$ as B, and if A and B show up alternatively when you put i, j, k, l in order, the two petals P_y and P_z are interlocked. Otherwise, the two petals P_y and P_z are separated.*

Proof of Theorem 4.1.

The petals P_x and P_{x+1} are formed using three clockwise-consecutive (half) strips S_{x-1}, S_x and S_{x+1} in an N-star. As shown in FIGURE 16, the diagram of the petal link (P_x(red), P_{x+1} (green)) can have four crossing points : C_{jj}, C_{ik}, C_{jk} and C_{ij}. Give orientations to the petal link (P_x, P_{x+1}) in the diagram as shown in FIGURE 18. In this oriented link diagram of the two petals P_x and P_{x+1}, note that A_{x+N} goes to A_x and B_x goes to B_{x+N}, as well as A_{x+1+N} goes to A_{x+1} and B_{x-1} goes to B_{x-1+N}.

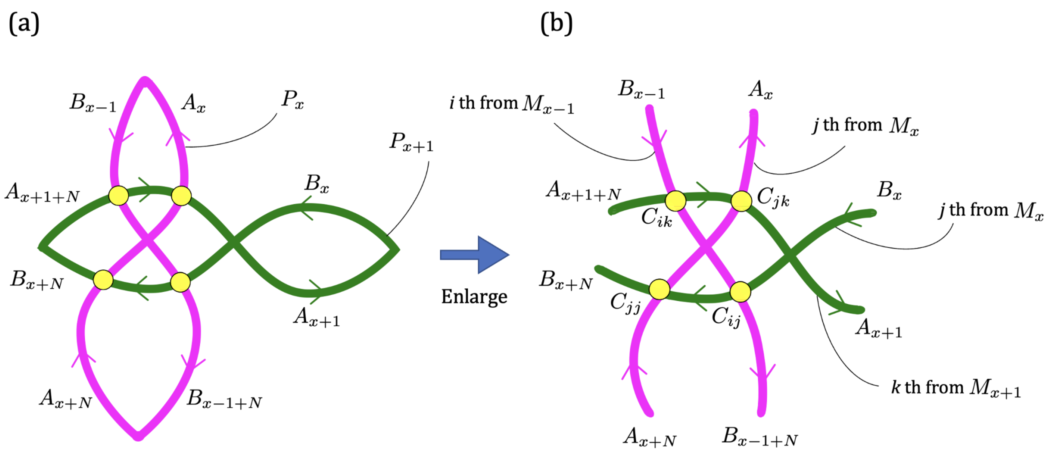


FIGURE 18. A link diagram of the petals P_x and P_{x+1}

Remark 4.4. Note that in FIGURE 19, we fix the vertices $A_x, B_x, A_{x+N}, B_{x+N}$ on the ground where S_x was originally laid, and glue -1 half twist on it to make the Type α Möbius strip M_x as shown in FIGURE 19. $A_x A_{x+N}$ lies over $B_x B_{x+N}$ when M_x is Type α.

The crossing point C_{jj} is on the bisected Type α Möbius strip M_x, so its sign is independent from the values i, j, and k (orders of making M_{x-1}, M_x, and M_{x+1}). However, the three other crossing points, C_{ik} (intersection of $A_{x+1+N}A_{x+1}$ in M_{x+1} and P_{x+1} and $B_{x-1}B_{x-1+N}$ in M_{x-1} and P_x), C_{jk} (intersection of $B_{x-1}B_{x-1+N}$ in M_{x-1} and P_x and $A_{x+1+N}A_{x+1}$ in M_{x+1} and P_{x+1}), and C_{ij} (intersection of $B_{x-1}B_{x-1+N}$ in M_{x-1} and P_x and $B_x B_{x+N}$ in M_x and P_{x+1}), all depend on the value of i, m and j.

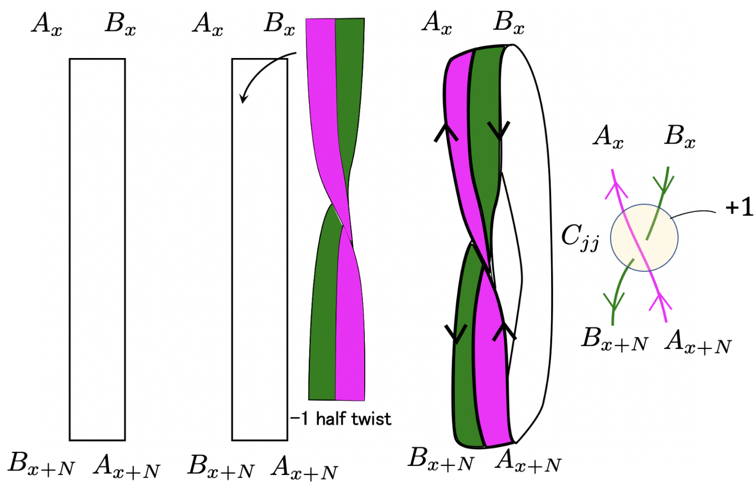


FIGURE 19. An α-type Möbius strip M_x

Back to the link diagram of the petal link P_x and P_{x+1}, and just focus on the crossing point C_{jj} as shown in FIGURE 18(b). When M_x is Type α, A_xA_{x+N} lies over B_xB_{x+N} and you can see that the sign of the crossing point C_{jj} is +1. It means that if, sign$(C_{jj}) =$ +1 regardless of the order of i, j, k when M_x is Type α .

Remark 4.5. If M_x is Type α (Möbius strip with -1 half twist), the sign of C_{jj} (j is the order of M_x in this generalized N-flower) is +1. Similarly, if M_x is Type β (Möbius strip with +1 half twist), the sign of C_{jj} is -1.

The signs of the other three crossing points C_{ik}, C_{jk} and C_{ij} depend on the relationships between i, j and k. For instance if $i < j$, then the ith band (from M_{x-1}) is under the jth band (from M_x), and so forth. Based on FIGURE 18, TABLE 1 summarizes the signs of each crossing point relative to the all possible scenarios between i, j and k.

TABLE 1.

	$i<j<k$	$j<k<i$	$k<i<j$	$i<k<j$	$k<j<i$	$j<i<k$
C_{ik}	-1	+1	+1	-1	+1	-1
C_{jk}	+1	+1	-1	-1	-1	+1
C_{ij}	+1	-1	+1	+1	-1	-1
sum of them	+1	+1	+1	-1	-1	-1

Another way to compute the signs of C_{ik}, C_{jk}, C_{ij} is as below: sign$(C_{ik}) = +1$ if and only if $B_{x-1}B_{x-1+N}$ is over $A_{x+1+N}A_{x+1}$, i.e. $i > k$ sign$(C_{jk}) = +1$ if and only if $A_{x+1+N}A_{x+1}$ is over $A_{x+N}A_x$, i.e. $k > j$ sign$(C_{ij}) = +1$ if and only if B_xB_{x+N} is over $B_{x-1}B_{x-1+N}$, i.e. $j > i$.

Therefore, the linking number of the petal link (P_x, P_{x+1}) is

$$\begin{aligned} lk(P_x, P_{x+1}) &= \frac{1}{2}(C_{jj} + C_{ik} + C_{jk} + C_{ij}) \\ &= \frac{1}{2}(1 + C_{ik} + C_{jk} + C_{ij}) \\ &= \begin{cases} 0 & \text{if } i < k < j \text{ or } k < j < i \text{ or } j < i < k \cdots (*), \text{ and} \\ 1 & \text{if } i < j < k \text{ or } k < i < j \text{ or } j < k < i \cdots (**). \end{cases} \end{aligned}$$

If the condition $(**)$ holds, the petals P_x and P_{x+1} are interlocked by Theorem 2.7. If the condition $(*)$ holds (i.e., their linking number is 0), for example, if $i < k < j$, the link diagram of the petal link (P_x, P_{x+1}) is as shown in FIGURE 20. Here, at C_{ik} and C_{ij}, the green petal P_{x+1} lies over the red petal P_x, while at C_{jj} and C_{jk}, the green petal P_{x+1} lies below the red petal P_x. Using Lemma 4.6 (or by the natural transformations of the space), you can show that the petals P_x and P_{x+1} are separated. The case $j < i < k$ is similarly shown. If $k < j < i$, the link diagram of the petal link P_x and P_{x+1} is as below. Here, the green petal P_{x+1} lies always below the red petal P_x, which means that the petals P_x and P_{x+1} are separated as shown in FIGURE 20. The other cases are also similarly shown.

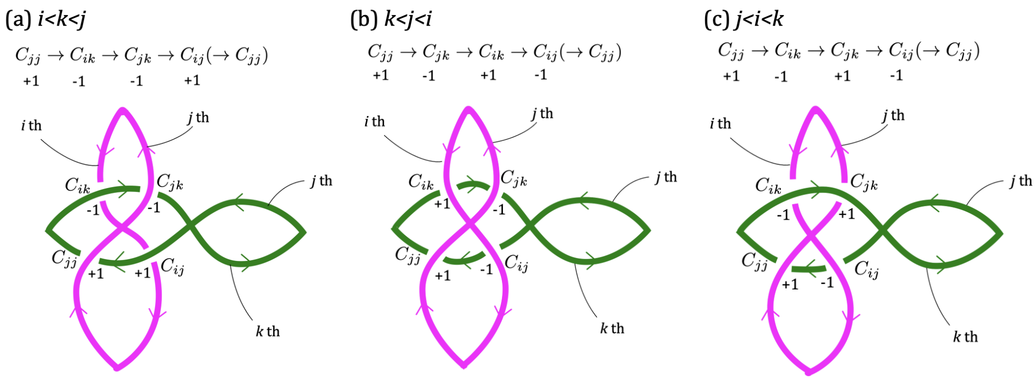


FIGURE 20. The separated two petals

Lemma 4.6. *If the following two conditions (i) (ii) hold in a link diagram of an oriented link (L_1, L_2), then L_1 and L_2 are separated.*

(i) Each L_i has at most one self-crossing.
(ii) L_1 crosses L_2 at four points, and when we trace L_1 in a direction, the signs +1 and -1 appear alternately at these four crossings.

(Proof is easy: If (ii) holds, then there is a pair of crossings between L_1 and L_2 with different signs that are also consecutive when we trace L_2 in a direction. Then these two crossings can be eliminated by Reidemeister moves.)

Proof of Theorem 4.2

In the same way as proof of Theorem 4.1, the petals P_x and P_{x+1} are formed using three clockwise-consecutive (half) strips S_{x-1}, S_x and S_{x+1} in an N-star. All the settings are the same except for the sign of C_{jj}, which is -1 when M_x is the Type β Möbius band (with +1 half twist). sign$(C_{jj}) = -1$, while the signs of the remaining three crossing points are the same as the ones listed in TABLE 1.

Therefore, the linking number $lk(P_x, P_{x+1})$ of the petals P_x and P_{x+1} is

$$\begin{aligned} lk(P_x, P_{x+1}) &= \frac{1}{2}(C_{jj} + C_{ik} + C_{jk} + C_{ij}) \\ &= \frac{1}{2}(-1 + C_{ik} + C_{jk} + C_{ij}) \\ &= \begin{cases} -1 & \text{if } i < k < j \text{ or } k < j < i \text{ or } j < i < k \cdots (*), \text{ and} \\ 0 & \text{if } i < j < k \text{ or } k < i < j \text{ or } j < k < i \cdots (**). \end{cases} \end{aligned}$$

If the condition $(*)$ holds, the petals P_x and P_{x+1} are interlocked by Theorem 2.7. If the condition $(**)$ holds (i.e., their linking number is 0), it is similarly shown that the petals P_x and P_{x+1} are separated as in Theorem 4.1.

Proof of Theorem 4.3.

In this case, P_y is formed using two adjacent strips S_{y-1} and S_y and P_z is formed using two adjacent strips S_{z-1} and S_z. In addition, we fix $y + 1 < z$, and $(y, z) \neq (0, N - 1)$ to ensure that P_y and P_z are not derived from the same strip.

Let $y - 1, y, z - 1, z$ be the ith, jth, kth and lth numbers in the gluing order of an N-flower with $p_0 \to p_1 \to \cdots \to p_{N-1}$, i.e. $p_i = y - 1$, $p_j = y$, $p_k = z - 1$, $p_l = z$. Draw a link diagram of the petal link P_y (red) and P_z (green), with the orientation given in FIGURE 21.

There are four crossing points C_ik (intersection of the line on the ith M_{y-1} and the line on the k th M_{z-1}) , C_{il} (intersection of the line on the ith M_{y-1} and the line on the lth M_z) C_{jk}(intersection of the line on the jth M_y and the line on the kth M_z) and C_{jl}(intersection of the line on the jth M_y and the line on the lth M_z).

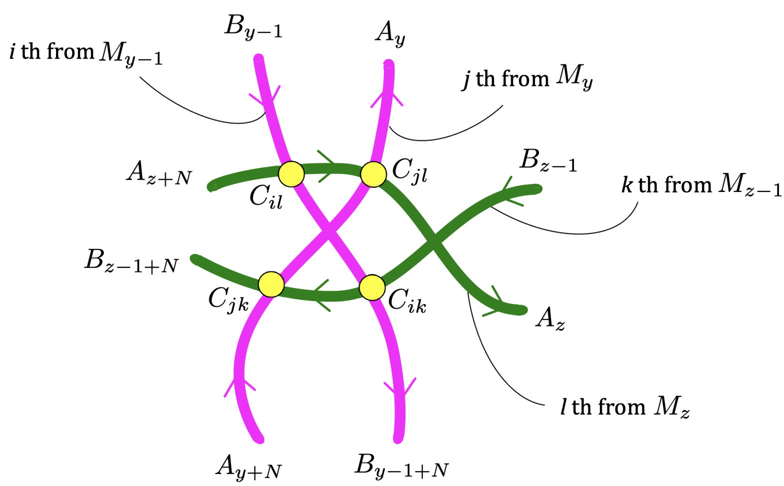


FIGURE 21. A link diagram of the petal link P_y(red) and P_z (green)

Based on FIGURE 21, the signs of C_{ik}, C_{il}, C_{jk} and C_{jl} under each possible scenario are given in TABLE 2 and TABLE 3.

TABLE 2.

	$i<k<l$	$i<l<k$	$l<i<k$	$l<k<i$	$k<i<l$	$k<l<i$
C_{ik}	$+1$	$+1$	$+1$	-1	-1	-1
C_{il}	-1	-1	$+1$	$+1$	-1	$+1$

TABLE 3.

	$j<k<l$	$j<l<k$	$l<j<k$	$l<k<j$	$k<j<l$	$k<l<j$
C_{jk}	-1	-1	-1	$+1$	$+1$	$+1$
C_{jl}	$+1$	$+1$	-1	-1	$+1$	-1

Another way to compute the signs of $C_{ik}, C_{il}, C_{jk}, C_{jl}$ is as below:

$$\begin{aligned}
\text{sign}(C_{ik}) &= +1 \text{ if and only if } i<k\\
\text{sign}(C_{il}) &= +1 \text{ if and only if } i>l\\
\text{sign}(C_{jk}) &= +1 \text{ if and only if } j>k\\
\text{sign}(C_{jl}) &= +1 \text{ if and only if } j<l
\end{aligned}$$

Therefore the linking number of two petals P_y and P_z is

$$lk(P_x, P_{x+1}) = \tfrac{1}{2}(C_{ik}+C_{il}+C_{jk}+C_{jl})$$

$$= \begin{cases} -1 & \text{if } i<l<j<k \text{ or } j<k<i<l \text{ or } k<i<l<j \text{ or } l<j<k<i \cdots (*),\\ 0 & \text{if } i<j<k<l \text{ or } j<k<l<i \text{ or } k<l<i<j \text{ or } l<i<j<k\\ & \text{or } i<j<l<k \text{ or } j<l<k<i \text{ or } k<i<j<l \text{ or } l<k<i<j\\ & \text{or } i<k<l<j \text{ or } j<i<k<l \text{ or } k<l<j<i \text{ or } l<j<i<k\\ & \text{or } i<l<k<j \text{ or } j<i<l<k \text{ or } k<j<i<l \text{ or } l<k<j<i \cdots (**), \text{ and}\\ +1 & \text{if } i<k<j<l \text{ or } j<l<i<k \text{ or } k<j<l<i \text{ or } l<i<k<j \cdots (***) \end{cases}$$

If the condition $(*)$ or $(***)$ holds, the petals P_y and P_z are interlocked by Theorem 2.7. If $(**)$ holds (i.e., their linking number is 0), you can see that four crossing points between the petal P_y and P_z in all cases can be eliminated by Reidemeister moves similarly in Theorem 4.1 and 4.2. Therefore, the petals P_y and P_z are separated.

We can summarize the procedure for finding the resulting object of a bisection of an arbitrary N-flower with $p_0 \to p_1 \to \cdots \to p_{N-1}$ for S into the following flow chart (FIGURE 22).

Flowchart: How to find whether or not $\dfrac{N(N-1)}{2}$ pairs of two petals from an N-flower are interlocked. Check all pairs of petals of a bisection of an N-flower.

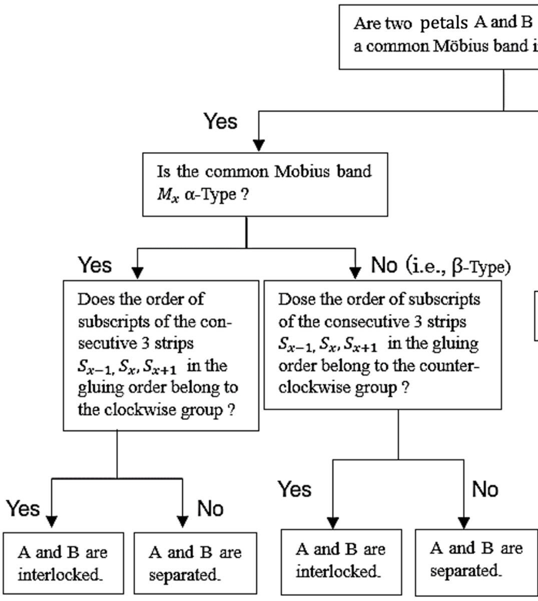


FIGURE 22. Flowchart

Example 4.7. A heart ring is obtained from an N-flower with $0 \to 1 \to \cdots \to N-1$ for $S = (\alpha\alpha\cdots\alpha)$ (FIGURE 23), while a heart chain is obtained from the flower with $0 \to 1 \to \cdots \to N-1$ for $S = (\alpha\alpha\cdots\alpha\beta)$. If the flower with $0 \to 1 \to \cdots \to N-1$ for $S = (\beta\beta\cdots\beta)$, the mutually-separated N hearts are obtained (FIGURE 24).

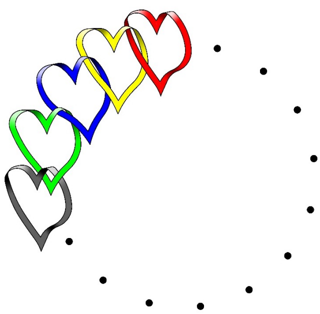


FIGURE 23. N hearts ring

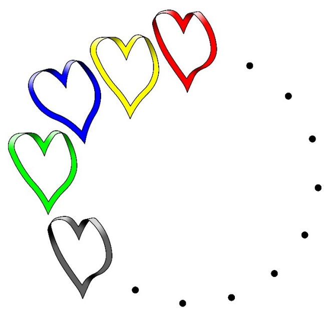


FIGURE 24. Separated N hearts

Example 4.8. A heart chain is obtained from a 5-flower with $0 \to 1 \to \cdots \to 4$ for $S = (\beta\alpha\alpha\alpha\alpha)$ as shown in FIGURE 25. All quadruplets of four distinct strips in this case satisfies the condition $(**)$ in Theorem 4.3.

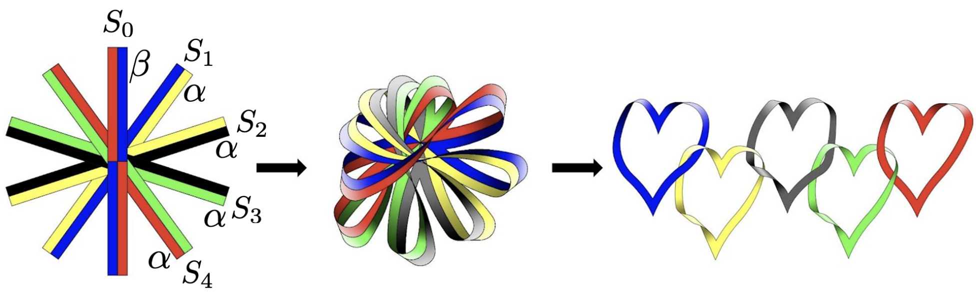


FIGURE 25. Five heart chain

Example 4.9. The resulting objects of a 5-flower with $0 \to 2 \to 4 \to 1 \to 3$, $0 \to 2 \to 1 \to 3 \to 4$ for $S = (\alpha\beta\alpha\beta\alpha)$ are as shown in FIGURE 26(a), (b), respectively. Some pairs of two loops (yellow & green, yellow & red, black & red and black & blue), which are derived from no common strips, are interlocked. All these quadruplets of strips satisfy the conditions $(*)$ or $(***)$ in Theorem 4.3.

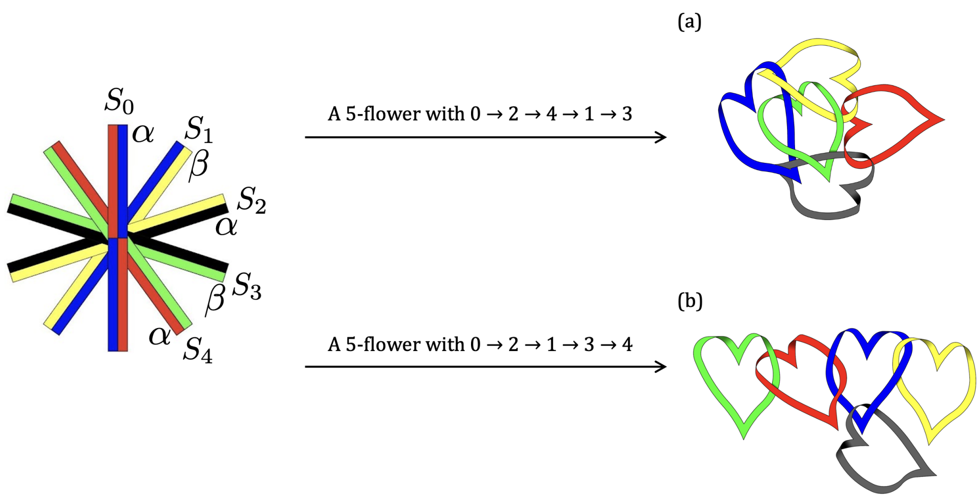


FIGURE 26. Two examples of bisections of 5-flowers

Remark 4.10. In the Examples above, the twists of the petals are not sometimes correctly shown.

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