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Wang Tiles: Connectivity when Tiling a Plane

Chao Yang

School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China
e-mail : sokoban2007@163.com

Abstract Inspired by the Nintendo game Zelda, the problem of tessellating the plane by translated copies from a set of Wang tiles is generalized to the connected tiling problem. We solve the connected tiling problem for the case that there is only one door color and one wall color in this paper.

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Keywords: Wang tiles; tiling; complexity

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1. Introduction

The motivation for this study comes from two sources. The first source is the original Domino Problem posed by Wang in 1961 [1, 2]. A *Wang tile* is a unit square with each edge assigned a color. Given a finite set of Wang tiles, we would like to tile the plane with translated copies of them such that the common edge of each pair of neighboring Wang tiles must have the same color. This is known as Wang's *Domino Problem*. This problem has been shown to be undecidable by Berger [3], and the proof of undecidability was later simplified by Robinson [4]. For a set of Wang tiles with a fixed small number of colors, the Wang's tiling problem can be decidable. Hu and Lin gave a complete classification of sets of Wang tiles with 2 colors which can tile the plane, all of them admit periodic tiling [5]. A complete classification of sets of Wang tiles with 3 colors which can tile the plane is obtained by Chen, Hu, Lai and Lin [6]. A recent result of Jeandel and Rao showed that there exists a set of Wang tiles with 4 colors that only tiles the plane non-periodically [7].

The second source is the video game Zelda published by Nintendo. One of the most successful titles of the Zelda series is *The Legend of Zelda: Link's Awakening*, which was released for the handheld console GameBoy in 1993 and for GameBoy Color in 1998. In 2019, the game was remade and published for the Switch console 26 years after its first release. Besides the huge improvement in graphics, the Switch version is almost identical to the GameBoy version in which the player walks in different 2D dungeons to solve puzzles. One major new feature added to the Switch version is a mini-game which allows

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gamers to create their own dungeons by putting together square rooms (see Figure 1 for a screenshot).

FIGURE 1. Screenshot of the game Zelda

If we regard the doors of the rooms as one color and the walls of the rooms as another color, the Zelda's dungeon-making mini-game is just a special case of Wang's problem. Zelda's dungeon-making puzzle and Wang's problem differ in two points: (1) Zelda's dungeon does not need to fill up the entire plane; (2) Zelda's dungeon must be connected, that is, the player must be able to walk from one room to any other rooms of the dungeon through doors.

This inspires us to propose the following problem, which incorporates connectivity into Wang's original problem.

Definition 1.1. (Connected Tiling Problem): Given two disjoint finite sets of colors, D (the set of *door* colors) and W (the set of *wall* colors), and a finite set of Wang tiles, with each side of a Wang tile assigned to one of the colors from either D or W, can we tile the whole plane with translated copies from this set of Wang tiles such that the tiling is *connected* (*i.e.* walk from one tile to any other tiles only through sides with colors in D)?

If $W = \emptyset$, then any tiling must be connected. Since in this case the problem is the same as Wang's original problem, the connected tiling problem includes Wang's original tiling problem as a special case. As a result, the general connected tiling problem must also be undecidable. But the problem can be solvable if we limit the number of colors. In this paper, we study the case that $|D| = |W| = 1$. We show that in this case the problem is solvable by giving a complete list of sets of Wang tiles which can tile the plane connectively.

Given $|D| = |W| = 1$, there are 16 possible Wang tiles, which are illustrated in Figure 2, with the color of D represented by blue lines. Labels are given to some of the tiles for reference.

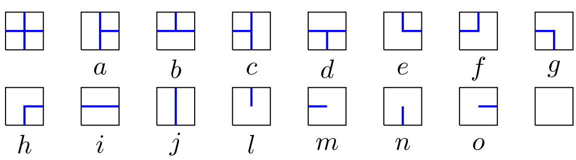


FIGURE 2. Wang tiles with $|D| = |W| = 1$

Using these 16 tiles, there are $2^{16}-1 = 65535$ different nonempty subsets. The following observation is obvious.

Proposition 1.2. *Let A be a set of Wang tiles which can tile the plane connectively, and let $B \supseteq A$ be a superset of A, then B can also tile the plane connectively.*

This suggests the following definition of a *minimal tiling set*.

Definition 1.3. A set of Wang tiles is called a *tiling set* if it can tile the plane connectively with translated copies. Moreover, it is called a *minimal tiling set* if any proper subset of it cannot tile the plane connectively.

So it suffices to classify all minimal tiling sets. We need a few more definitions before stating our main result. A tile with k ($k = 0, 1, 2, 3, 4$) edges assigned to colors in D is called a k-door tile. A pair of 3-door tiles is said to be *opposite* if the wall sides on each tile are opposite from one another (top and bottom or left and right). Otherwise, they are said to be *adjacent* because the sides assigned to a wall color are adjacent (see Figure 3).

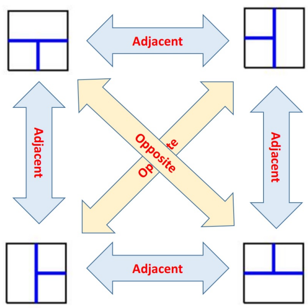


FIGURE 3. Relation between a pair of 3-door tiles: opposite or adjacent

The six 2-door tiles are divided into two groups. A 2-door tile is called *turning* if the two sides with door color are in adjacent directions. A 2-door tile is called *straight* if the two sides with door color are in opposite directions. There are four turning 2-door tiles and two straight 2-door tiles. The two straight 2-door tiles are called the *horizontal* 2-door tile and the *vertical* 2-door tile (see Figure 4).

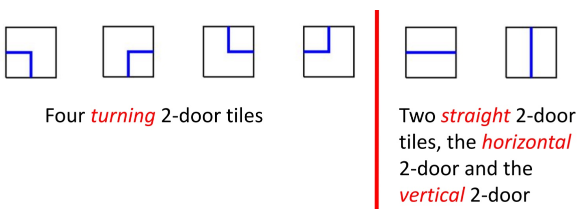


FIGURE 4. Turning and Straight 2-door tiles

We also define a relation between a pair of turning 2-door tiles. A pair of turning 2-door tiles is said to be *opposite* if there is no common direction of sides both assigned to a door color. Otherwise the pair is said to be *adjacent* (see Figure 5).

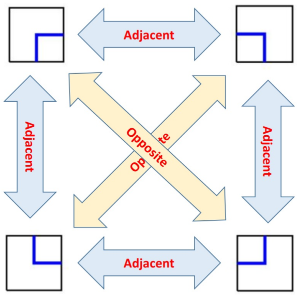


FIGURE 5. Relation between a pair of 2-door tiles: opposite or adjacent

Finally, let us define relations between tiles with different number of doors. A 2-door tile (or a 1-door tile) is said to be a *complement* of a 3-door tile if it contains a door in the direction that the 3-door tile is missing. Furthermore, a 1-door tile is said to be a 2-*step-complement* with regard to a 3-door tile and a complement turning 2-door tile, if its door is in one of the directions that the 2-door tile is missing and is perpendicular to the direction that the 3-door tile is missing (see Figure 6).

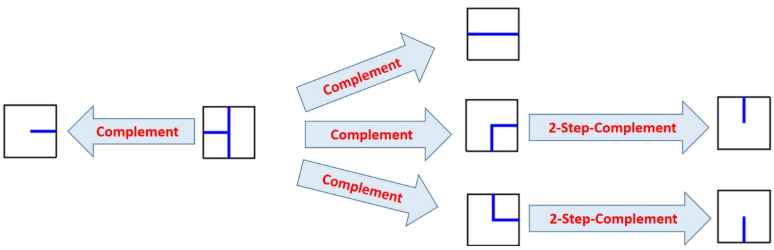


FIGURE 6. Complement to a 3-door tiles

A 1-door tile is said to be *parallel* with regard to a straight 2-door tile if it contains a door in the same direction as one of the doors of the straight 2-door tile (see Figure 7).

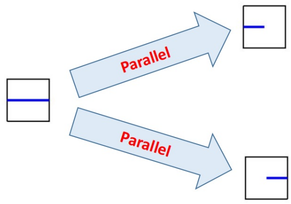


FIGURE 7. Parallel to a straight 2-door tiles

Our main result is the following theorem which gives a complete classification of the minimal tiling sets.

Theorem 1.4. *For the connected tiling problem with $|D| = |W| = 1$, a set of Wang tiles is a minimal tiling set if and only if it is one of the following* 11 *types:*

(1) the singleton set of the 4*-door tile;*
(2) sets consisting of a pair of opposite 3*-door tiles;*
(3) sets consisting of a pair of adjacent 3*-door tiles, and a complement (to both of the* 3*-door tiles) turning* 2*-door tiles;*
(4) sets consisting of a pair of adjacent 3*-door tiles, and the two straight* 2*-door tiles;*
(5) sets consisting of a pair of adjacent 3*-door tiles, a straight* 2*-door tile and a complement (to one of the* 3*-door tiles) parallel* 1*-door tile;*
(6) sets consisting of one 3*-door tile, a pair of complement turning* 2*-door tiles;*
(7) sets consisting of one 3*-door tile, a pair of opposite turning* 2*-door tile and a non-complement straight* 2*-door tile;*
(8) sets consisting of one 3*-door tile, a pair of opposite turning* 2*-door tile, a complement straight* 2*-door tile and either a complement* 1*-door tile or a* 2*-step-complement* 1*-door tile;*
(9) sets consisting of one 3*-door tile, a complement turning* 2*-door tile, two straight* 2*-door tiles and a* 2*-step-complement* 1*-door tile;*
(10) sets consisting of the four turning 2*-door tiles and one more straight* 2*-door tile;*
(11) sets consisting of the four turning 2*-door tiles and one* 1*-door tile.*

Similarly, we can define a maximal non-tiling set.

Definition 1.5. A set of Wang tiles is called a *non-tiling set* if it cannot tile the plane connectively with translated copies. Moreover, it is called a *maximal non-tiling set* if any proper superset of it can tile the plane connectively.

The rest of the paper is organized as follows: Section 2 shows each of the 11 types stated in Theorem 1.4 is a tiling set by giving detailed illustrations of tilings. Section 3 gives a complete list of maximal non-tiling sets. Section 4 completes the proof of our main theorem on the classification of minimal tiling sets, by combining the results in Section 2 and Section 3. Section 5 concludes the paper by making a few remarks.

2. Tiling Sets

In this section, we will show that all 11 types of sets listed in Theorem 1.4 are tiling sets. The proof for their minimality and completeness will be given in Section 4. To get an overall pathway of the proof of minimality and completeness, the reader may skip this section to read Lemma 3.1 in Section 3 for a list of 8 types of maximal non-tiling sets, and to read the proof of minimality and completeness in Section 4 first.

Proof of Theorem 1.4: all of the 11 *types are tiling sets.* To show they are tiling sets, it suffices to give a connected tiling to each of the 11 types. Let T be a tile set. A connected tiling of the plane can be represented by a mapping $\varphi : \mathbb{Z}^2 \to T$.

Type 1: the singleton set of the 4-door tile. Let φ be the constant mapping that maps every element in $\mathbb{Z}^2$ to the 4-door tile (see Figure 8).

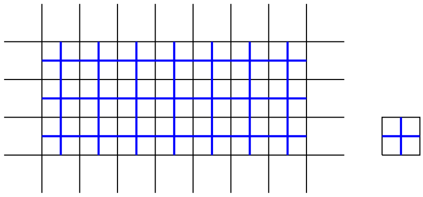


Figure 8. (Type 1) 4-door tile

Type 2: the sets consisting of a pair of opposite 3-door tiles. Because the four 3-door tiles form two pairs of opposite tiles, there are exactly 2 sets of this type. Without loss of generality, let $T = \{b, d\}$ be one of the two sets illustrated on the right of Figure 9. Recall that the labels of the tiles are defined in Figure 2. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} b, & \text{if } x+y \text{ is odd;} \\ d, & \text{if } x+y \text{ is even.} \end{cases}$$

The tiling obtained from this mapping is illustrated on the left of Figure 9. Note that the origin $(0, 0) \in \mathbb{Z}^2$ is illustrated by a square in light red.

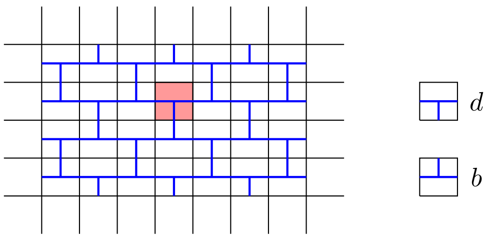


Figure 9. (Type 2) Two opposite 3-door tiles

Type 3: the sets consisting of a pair of adjacent 3-door tiles and a complement 2-door tile. Here, the 2-door tile is complement to both of the two adjacent 3-door tiles, hence it is uniquely determined by the pair of 3-door tiles. As a result, there are 4 different sets of this type as there are four ways to choose a pair of adjacent 3-door tiles. Without loss of generality, let $T = \{a, d, f\}$ be one of the sets of this type as illustrated on the right of Figure 10. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } x+y \equiv 0 \pmod 3; \\ d, & \text{if } x+y \equiv 1 \pmod 3; \\ f, & \text{if } x+y \equiv 2 \pmod 3. \end{cases}$$

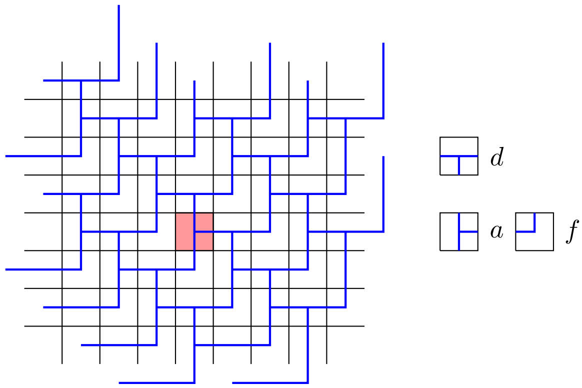


FIGURE 10. (Type 3) Two adjacent 3-door tiles

Type 4: the sets consisting of a pair of adjacent 3-door tiles and two straight 2-door tiles. There are four ways to choose a pair of adjacent 3-door tiles, so there are 4 different sets of this type. Without loss of generality, let $T = \{a, d, i, j\}$ be one of the sets of this type as illustrated on the right of Figure 11. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } x+y=0; \\ d, & \text{if } x+y=1; \\ i, & \text{if } x+y \geq 2; \\ j, & \text{if } x+y \leq -1. \end{cases}$$

The above tiling forms an infinite tree with stems (tiles a and d) and branches (tiles i and j).

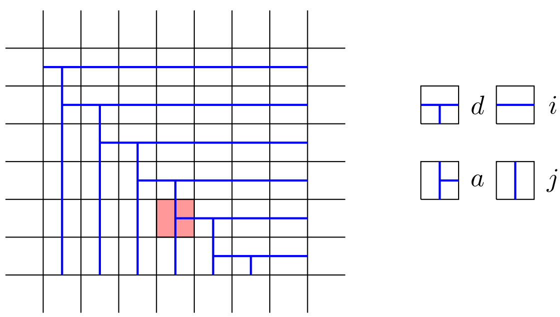


FIGURE 11. (Type 4) Two adjacent 3-door tiles

Type 5: the sets consisting of a pair of adjacent 3-door tiles, a straight 2-door tile, and a complement (to one of the 3-door tiles) parallel 1-door tile. The same as the previous two types, there are four ways to choose a pair of adjacent 3-door tiles. Additionally, there are two ways to choose one of the two straight 2-door tiles. But the 1-door tile is uniquely determined because it is parallel with regard to the 2-door tile and is complement to one of the two 3-door tiles. Therefore, there are $4 \times 2 = 8$ different sets of this type. Without loss of generality, let $T = \{a, d, i, m\}$ be one of the sets of this type as illustrated on the right of Figure 12. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } x + 2y = 0, -2, -4, -6, \cdots; \\ d, & \text{if } x + 2y = 2; \\ i, & \text{if } x + 2y = 1, 3, 4, 5, 6, \cdots; \\ m, & \text{if } x + 2y = -1, -3, -5, \cdots. \end{cases}$$

Similar to Type 4, the tiling of Type 5 forms an infinite tree. But the structure of the infinite tree formed by the tiling of Type 5 is slightly more complex, besides stems and branches (tiles a, d and i), there are also leaves (tiles m) that fill up the gaps between branches. For the cases from Type 6 to Type 9, we can all adopt this infinite tree viewpoint to get a overall understanding of the tilings.

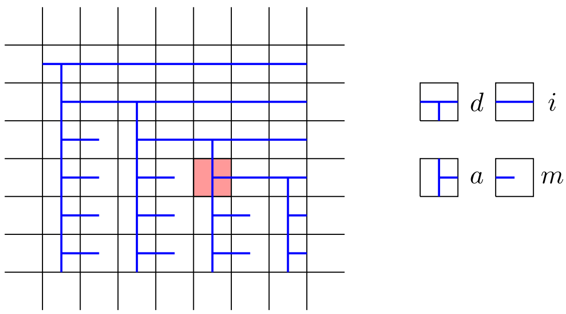


FIGURE 12. (Type 5) Two adjacent 3-door tiles

Type 6: the sets consisting of a 3-door tile and two complement turning 2-door tiles. Note that the 2-door tiles are uniquely determined because there are exactly two turning 2-door tiles which are complement to a given 3-door tile. Therefore, there are 4 different sets of this type as we have four ways to choose a 3-door tile. Without loss of generality, let $T = \{a, d, i, m\}$ be one of the sets of this type as illustrated on the right of Figure 13. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } (y \le 0 \wedge x \text{ is even}) \vee (y \ge 1 \wedge x \text{ is odd}); \\ g, & \text{if } (y \le 0 \wedge y \text{ is odd} \wedge x \text{ is odd}) \vee (y \ge 1 \wedge y \text{ is odd} \wedge x \text{ is even}); \\ f, & \text{if } (y \le 0 \wedge y \text{ is even} \wedge x \text{ is odd}) \vee (y \ge 1 \wedge y \text{ is even} \wedge x \text{ is even}). \end{cases}$$

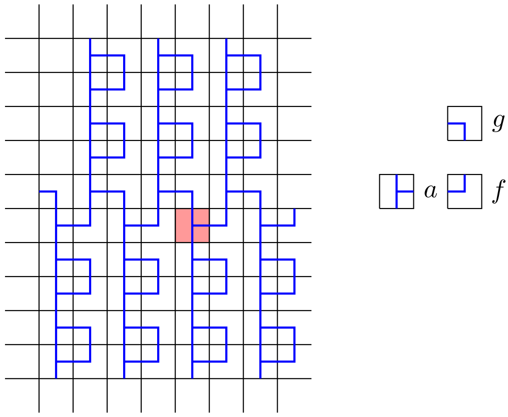


FIGURE 13. (Type 6) One 3-door tile

Type 7: the sets consisting of a 3-door tile, a pair of opposite turning 2-door tiles and a non-complement straight 2-door tile. Note the straight 2-door tile is uniquely determined with regard to the 3-door tile, but there are two ways to choose a pair of opposite turning 2-door tiles. Therefore, we have $4 \times 2 = 8$ different sets of this type. Without loss of generality, let $T = \{a, e, g, j\}$ be one of the sets of this type as illustrated on the right of Figure 14. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } 3x + y = 0, 2; \\ e, & \text{if } 3x + y = 4, 6, 8, 10, \cdots; \\ g, & \text{if } 3x + y = 3, 5, 7, 9, \cdots; \\ j, & \text{if } 3x + y = 1, -1, -2, -3, -4, \cdots. \end{cases}$$

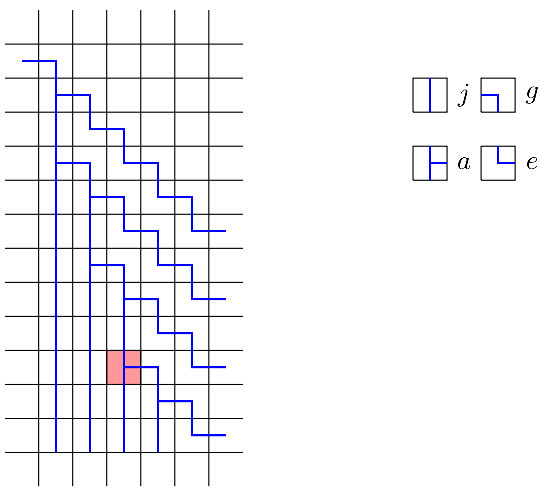


FIGURE 14. (Type 7) One 3-door tile

Type 8: the sets consisting of a 3-door tile, a pair of opposite turning 2-door tiles, a complement straight 2-door tile and a complement (or 2-step-complement) 1-door tile. We divide type 8 into 2 subtypes according to the 1-door tile. Sets that contain a complement 1-door tile are called type 8A, and sets that contain a 2-step-complement 1-door tile are called type 8B.

For type 8A, there are four ways to choose a 3-door tile, and two ways to choose a pair of opposite turning 2-door tiles. But the straight 2-door tile and the 1-door tile are uniquely determined with regard to the 3-door tile. Therefore, we have $4 \times 2 = 8$ different sets of the type. Without loss of generality, let $T = \{a, e, g, i, m\}$ be one of the sets of this type as illustrated on the right of Figure 15. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} a, & \text{if } x + y \leq 0 \wedge x \text{ is even;} \\ e, & \text{if } x + y > 0 \wedge y - x \equiv 0 \pmod 4; \\ g, & \text{if } x + y > 0 \wedge y - x \equiv 1 \pmod 4; \\ i, & \text{if } x + y \geq 0 \wedge y - x \equiv 2, 3 \pmod 4; \\ m, & \text{if } x + y < 0 \wedge x \text{ is odd.} \end{cases}$$

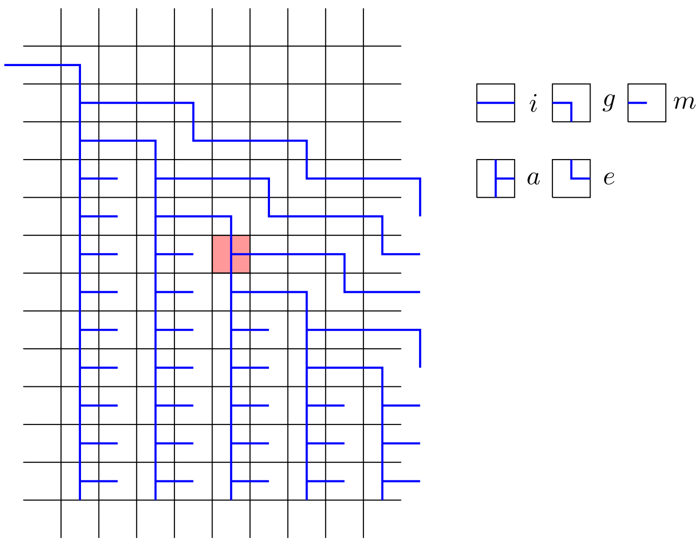


FIGURE 15. (Type 8A) One 3-door tile

For type 8B, no matter which pair of opposite turning 2-door tile are chosen, exactly one of them is complement to the 3-door tile, so the 2-step-complement 1-door tile is also uniquely determined. So the same as type 8A, type 8B also has 8 different sets. Without loss of generality, let $T = \{a, e, g, i, l\}$ be one of the sets. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y)=\begin{cases} a, & \text{if } (x+2y=-4,-2,0,2)\wedge x\equiv 0 \pmod 4;\\ e, & \text{if } (x+2y\le -4\ \vee x+2y=5,8,11,\cdots)\wedge x\not\equiv y \pmod 2;\\ g, & \text{if } (x+2y\le -5\ \vee x+2y=-3,-2,1,4,7,\cdots)\wedge x\equiv y \pmod 2;\\ i, & \text{if } ((x+2y=-3,0,1,4,7,10,\cdots)\wedge x\not\equiv y \pmod 2))\\ & \vee((x+2y=-1,2,5,8,11,\cdots)\wedge x\equiv y \pmod 2))\\ & \vee(x+2y=3,6,9,12,\cdots);\\ l, & \text{if } (x+2y=-1,-5)\wedge x\equiv 3 \pmod 4. \end{cases}$$

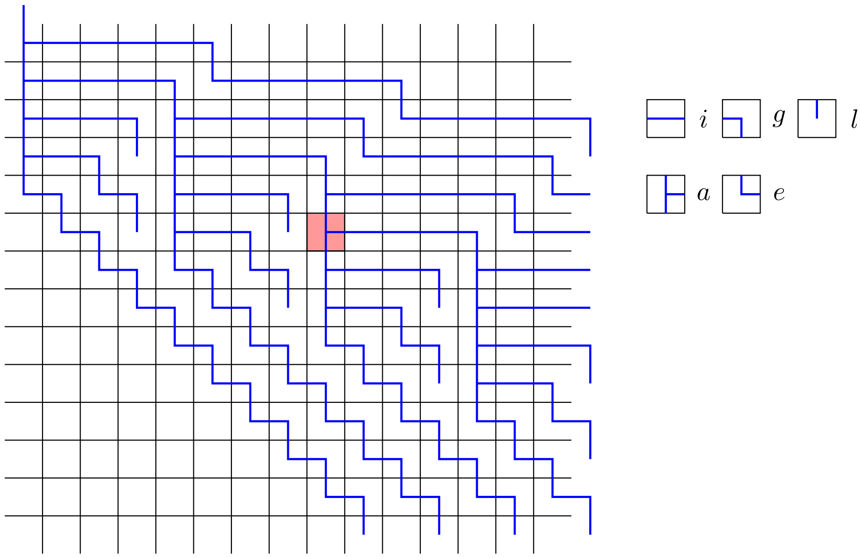


FIGURE 16. (Type 8B) One 3-door tile

Type 9: sets consisting of one 3-door tile, a complement turning 2-door tile, two straight 2-door tiles and a 2-step-complement 1-door tile. There are four ways to choose the 3-door tile, two ways to choose a complement 2-door tile. So there are $4\times 2=8$ different sets in this type. Without loss of generality, let $T=\{a,i,j,g,l\}$ be one of the sets of this type. For each element $(x,y)\in\mathbb{Z}^2$, let

$$\varphi(x,y)=\begin{cases} a, & \text{if } (x=-i)\wedge(i^2\le y<(i+1)^2) \text{ for } i=0,1,2,\cdots;\\ g, & \text{if } (x=y-i^2-i+1)\wedge(i^2\le y<(i+1)^2) \text{ for } i=0,1,2,\cdots;\\ i, & \text{if } (-i<x<y-i^2-i+1)\wedge(i^2\le y<(i+1)^2) \text{ for } i=1,2,\cdots;\\ j, & \text{if } ((y<x^2)\wedge(x,y)\ne(1,0))\\ & \vee((2-i<x\le i)\wedge(i^2<y<i^2+x+i-1) \text{ for } i=2,3,\cdots);\\ l, & \text{if } (1-i<x\le i)\wedge(y=i^2) \text{ for } i=1,2,\cdots. \end{cases}$$

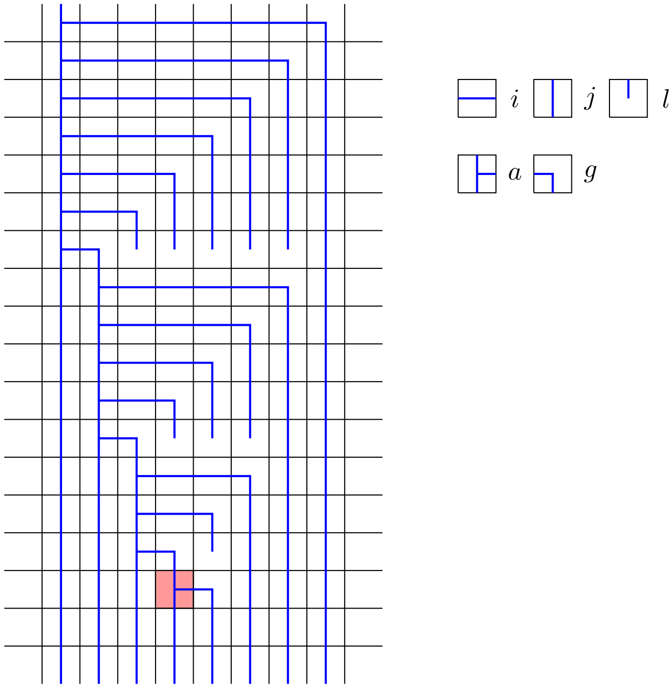


Figure 17. (Type 9) One 3-door tile

Type 10: the sets consisting of four turning 2-door tiles and one straight 2-door tile. Since we have two ways to choose the straight 2-door tile, there are 2 different sets of this type. Without loss of generality, let $T = \{e, f, g, h, i\}$ be one of the sets of this type as illustrated on the right of Figure 18. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} e, & \text{if } (x+2y = -1, 3, 7, \cdots \wedge 2y - x = -3, -7, \cdots) \\ & \vee(x+2y = -1, -5, -9, \cdots \wedge 2y - x = 1, 5, 9, \cdots); \\ f, & \text{if } (x+2y = -2, 2, 6, \cdots \wedge 2y - x = -2, -6, \cdots) \\ & \vee(x+2y = -2, -6, \cdots \wedge 2y - x = 2, 6, \cdots); \\ g, & \text{if } (x+2y = 0, 4, 8, \cdots \wedge 2y - x = 0, -4, -8, \cdots) \\ & \vee(x+2y = 0, -4, -8, \cdots \wedge 2y - x = 4, 8, \cdots); \\ h, & \text{if } (x+2y = 1, 5, \cdots \wedge 2y - x = -1, -5, \cdots) \\ & \vee(x+2y = 1, -3, -7, \cdots \wedge 2y - x = 3, 7, \cdots); \\ i, & \text{if } (x+2y \geq 2 \wedge 2y - x \geq 1) \vee (x+2y \leq -3 \wedge 2y - x \leq 0). \end{cases}$$

Because each tile has exactly two sides assigned to the door color, the tiling defined by the above mapping φ forms a two-way infinite path which has a central symmetry by a 180° rotation at the mid-point of the left side of the tile illustrated in red.

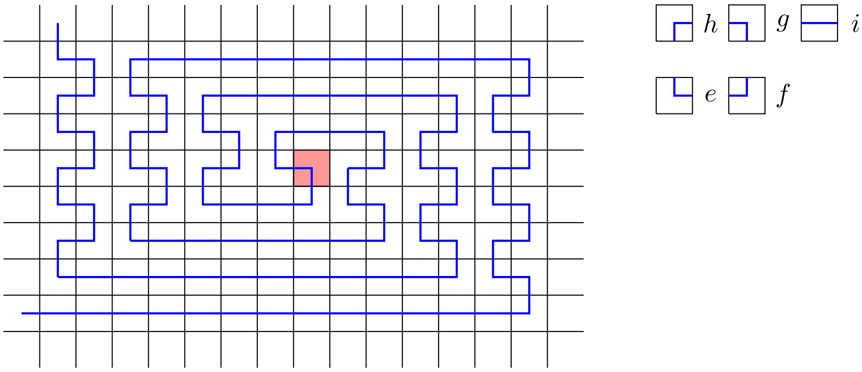


FIGURE 18. (Type 10) Four turning 2-door tiles

Type 11: the sets consisting of four turning 2-door tiles and one 1-door tile. Since we have four ways to choose the 1-door tile, there are 4 different sets of this type. Without loss of generality, let $T = \{e, f, g, h, n\}$ be one of the sets of this type as illustrated on the right of Figure 19. For each element $(x, y) \in \mathbb{Z}^2$, let

$$\varphi(x,y) = \begin{cases} e, & \text{if } (x > 0 \wedge y > 0 \wedge x \equiv y \pmod 2) \\ & \vee(x \le 0 \wedge y < 0 \wedge x \not\equiv y \pmod 2); \\ f, & \text{if } (x > 0 \wedge y \le 0 \wedge x \equiv y \pmod 2) \\ & \vee(x \le 0 \wedge y \ge 0 \wedge (x,y) \ne (0,0) \wedge x \not\equiv y \pmod 2)); \\ g, & \text{if } (x > 0 \wedge y > 0 \wedge x \not\equiv y \pmod 2) \\ & \vee(x \le 0 \wedge y < 0 \wedge x \equiv y \pmod 2); \\ h, & \text{if } (x > 0 \wedge y \le 0 \wedge x \not\equiv y \pmod 2) \\ & \vee(x \le 0 \wedge y \ge 0 \wedge (x,y) \ne (0,0) \wedge x \equiv y \pmod 2)); \\ n, & \text{if } (x,y) = (0,0). \end{cases}$$

The above tiling forms a one-way infinite path which starts at the center (the red tile) and extends spirally outward.

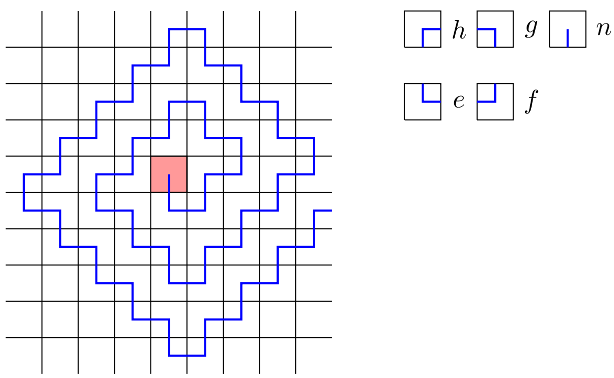


FIGURE 19. (Type 11) Four turning 2-door tiles

■

Remark 2.1 (Aperiodicity)**.** Note that some types of minimal tiling sets only admit non-periodic connected tilings. For example, any connected tilings of type 11 must make use of the 1-door tile, and the 1-door tile cannot be used for more than twice in such a tiling. It is obvious that in a tiling with translation symmetry, all tiles must appear infinite times. As a result, any connected tiling of type 11 has no translation symmetry.

3. Maximal Non-tiling Sets

We give a complete list of maximal non-tiling sets in this section. Note that every maximal non-tiling set must contain the 0-door tile, so we omit it for simplicity.

Lemma 3.1. *For the connected tiling problem with $|D| = |W| = 1$, the following are maximal non-tiling sets:*

(1) the set consisting of the four turning 2-door tiles;
(2) sets consisting of two straight 2-door tiles, three turning 2-door tiles and four 1-door tiles;
(3) sets consisting of one 3-door tile, a complement straight 2-door tile, two non-complement turning 2-door tiles, one complement turning 2-door tile, and two non-complement (neither complement nor 2-step-complement) 1-door tiles;
(4) sets consisting of one 3-door tile, two straight 2-door tiles, a pair of non-complement adjacent turning 2-door tiles, and four 1-door tiles;
(5) sets consisting of one 3-door tile, two straight 2-door tiles, a pair of adjacent turning 2-door tiles (one is complement to the 3-door tile and the other is non-complement to the 3-door tile), and all the 1-door tiles but the 2-step-complement one;
(6) sets consisting of one 3-door tile, one straight 2-door tiles, a pair of adjacent turning 2-door tiles (one is complement to the 3-door tile and the other is non-complement to the 3-door tile), and four 1-door tiles;
(7) sets consisting of a pair of adjacent 3-door tiles, three non-complement (to at least one of the 3-door tile) turning 2-door tiles, and four 1-door tiles;
(8) sets consisting of a pair of adjacent 3-door tiles, one straight 2-door tile, a pair of non-complement (to one of the 3-door tile) adjacent turning 2-door tiles, and all the 1-door tiles but the complement (to one of the 3-door tile) parallel one.

Proof. Given a tiling of a set T, i.e. a mapping $\varphi : \mathbb{Z}^2 \to T$, we can construct an *associated infinite graph* G as follows. The vertex set of G is $V(G) = \mathbb{Z}^2$. Two vertices (x, y) and (x', y') of G are connected by an edge in G if and only if they are adjacent (i.e. $x = x', |y - y'| = 1$ or $y = y', |x - x'| = 1$), and the common edge of $\varphi(x, y)$ and $\varphi(x', y')$ is in door color. So a tiling is connected if and only if the associated graph is connected. We will apply graph-theoretical arguments to show the impossibility of tiling the entire plane connectively mainly in type 1 and type 2.

Type 1: the set consisting of the four turning 2-door tiles. There is only one set of this type, which consists of only 2-door tiles, so the associated graph of any tiling of this set is a 2-regular graph. As a result, suppose by contradiction that this set can tile the plane connectively, then the associated graph must be a two-way infinite path. Without loss of generality, there must be 3 adjacent tiles in the connected tiling as illustrated on the left of Figure 20. This uniquely determines the fourth tile on the lower left corner (see the right of Figure 20).

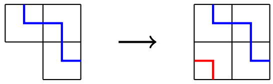
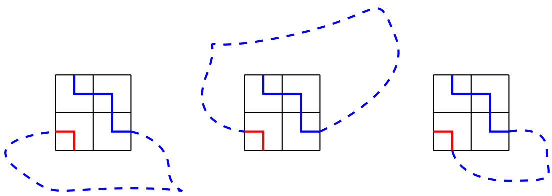


FIGURE 20. (Type 1) Four turning 2-door tiles

Because the overall associated graph must be a two-way infinite path, the two segments of path (the blue and the red segments) on the right of Figure 20 must be linked together. Again, without loss of generality, we assume that the red segment of the path is linked to the blue segment by extending the lower right tile from the door facing east. If we extend the lower right tile from the door facing east, and link to the lower left tile from the door facing west, then it must be one of two the cases (a) and (b) illustrated Figure 21, namely either go around from below or from above. But either way, it will forbid one end of the path from extending infinitely, which is a contradiction with the fact that the associated graph must be a two-way infinite path.



(a) (b) (c)

FIGURE 21. (Type 1) Four turning 2-door tiles

Therefore, if we extend the blue segment from the door facing east, then we can only link to the lower left tile from the door facing south (see (c) of Figure 21). But again this is impossible. To see this, we color the tiles in gray and white checkerboard pattern as illustrated in Figure 22. Because there is no straight 2-door tile in this tile set, the path formed by the tiling has to take a 90° turn at each tile. As a result, the extended path starting from the eastwards door of the lower right tile always enters a gray tile from east or west, and enters a white tile from north or south. The lower left tile is gray, so there is no way to enter it from south.

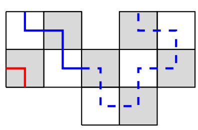


FIGURE 22. (Type 1) Four turning 2-door tiles

Type 2: sets consisting of two straight 2-door tiles, three turning 2-door tiles and four 1-door tiles. Because there are four ways to choose 3 turning 2-door tiles, we have 4 different sets of this type. Without loss of generality, let T be a set of this type whose 3 turning 2-door tiles are e, f and h. Every tile in T has either one door or two doors, so if T can tile the plane connectively, the associated graph must be either a two-way infinite path, or a one-way infinite path. Obviously, this cannot be done without using the turning 2-door tiles. If we only use two of the turning 2-door tiles f and h in the tiling, we can only get an infinite path running from southwest to northeast, but cannot occupy the entire plane (see Figure 23).

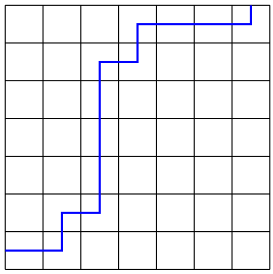


FIGURE 23. (Type 2) A path from southwest to northeast

Finally, consider the case that the third turning 2-door tile e is used in a tiling of T. Place a copy of e at any location of the plane. It is easy to see that by extending from this copy of e, with all the tiles we have in T, there is no way to obtain a path which can reach the southwest area with regard to this copy of e.

Type 3: sets consisting of one 3-door tile, a complement straight 2-door tile, two non-complement turning 2-door tiles, one complement turning 2-door tile, and three non-complement 1-door tiles. In this type, there are four ways to choose one 3-door tile. The complement (with regard to the 3-door tile we have chosen) straight 2-door tile is determined by the 3-door tile. The two non-complement turning 2-door tiles and three non-complement 1-door tiles are also determined by the choice of the 3-door tile, but we still have two ways to choose a complement turning 2-door tile. Therefore, we have $4 \times 2 = 8$ different sets of this type. Without loss of generality, let $T = \{a, i, e, h, f, l, o\}$. Obviously, if T can tile the plane connectively, the unique 3-door tile must be used in the connected tiling. Otherwise, it would be a subset of type 2 which we have just shown the impossibility.

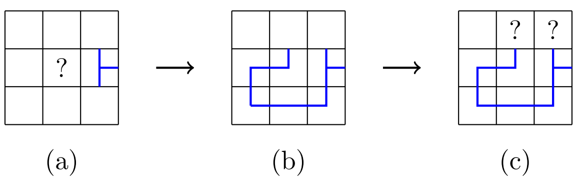


FIGURE 24. (Type 3) One 3-door tile

We show the impossibility of tiling the plane connectively for this type by applying arguments on the local placement of the tiles. First, place a copy of the 3-door tile a anywhere on the plane. Which tile should we place to the west of this 3-door tile to make the tiling connected (see Figure 24 (a))? The only solution is illustrated in Figure 24 (b), because the tile f is the only tile in the set T with a door facing west. But now there is no way to put tiles to the two locations with question marks in Figure 24 (c) to make them compatible with each other.

Type 4: sets consisting of one 3-door tile, two straight 2-door tiles, a pair of non-complement adjacent turning 2-door tiles, and four 1-door tiles. There are four ways to choose a 3-door tile, the two non-complement adjacent turning 2-door tiles are determined by the choice of the 3-door tile. So there are a total of 4 different sets of this type. Without loss of generality, let $T = \{a, i, j, e, h, l, m, n, o\}$. Similar to type 3, any connected tiling of type 4 must make use of the 3-door tile, otherwise it becomes a subset of type 2. Put the 3-door tile anywhere on the plane, then there is no way to connect the 3-door tile to its neighbor to the west by using the tiles available in T (see Figure 25).

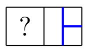


FIGURE 25. (Type 4) One 3-door tile

For the remaining four types, similar arguments as type 3 and type 4 about the local placement of tiles can be applied to show the impossibility of tiling the plane connectively.

Type 5: sets consisting of one 3-door tile, two straight 2-door tiles, a pair of adjacent turning 2-door tiles (one is complement to the 3-door tile and the other is non-complement to the 3-door tile), and all the 1-door tiles but the 2-step-complement one. There are four ways to choose the 3-door tile and two ways to choose a pair of turning 2-door tiles satisfying the requirement. So there are $4 \times 2 = 8$ different sets of this type.

Type 6: sets consisting of one 3-door tile, one straight 2-door tiles, a pair of adjacent turning 2-door tiles (one is complement to the 3-door tile and the other is non-complement to the 3-door tile), and four 1-door tiles. There are four ways to choose the 3-door tile, two ways to choose a straight 2-door tile, and two ways to choose a pair of turning 2-door tiles satisfying the requirement. So there are $4 \times 2 \times 2 = 16$ different sets of this type.

Type 7: sets consisting of a pair of adjacent 3-door tiles, three non-complement turning 2-door tiles, and four 1-door tiles. There are four ways to choose a pair of adjacent 3-door tiles. There are exactly three turning 2-door tiles that are not complement to at least one of the two 3-door tiles, so they are uniquely determined. In other words, among the four turning 2-door tiles, we just exclude the one which is complement to both of the two 3-door tiles. The 1-door tiles also have no choice as we include all of them in this type. So there are 4 different sets of this type.

Type 8: sets consisting of a pair of adjacent 3-door tiles, one straight 2-door tile, a pair of non-complement adjacent turning 2-door tiles, and all the 1-door tiles but the complement parallel one. There are four ways to choose a pair of 3-door tiles, two ways to choose one straight 2-door tile, and two ways to choose a pair of non-complement adjacent turning 2-door tiles (i.e. a pair of adjacent turning 2-door tiles that are both non-complement to one of the 3-door tile). The 1-door tiles are determined by the 3-door tiles, by excluding just the 1-door tile which is both complement to one of the 3-door tiles and parallel to the straight 2-door tile. Therefore, there are $4 \times 2 \times 2 = 16$ different sets of this type.

For any non-tiling set belonging to one of the above 8 types, if we add one more tile to that set, it is easy to check that it will become a superset of one of the 11 types of tiling set of Theorem 1.4. This implies that all of the 8 types of non-tiling sets are maximal. ∎

4. Proof of Main Results

Proof of Theorem 1.4: minimality and completeness. The minimality can now be easily checked in the following way. Let T be any set of one of the 11 types listed in Theorem 1.4. Let $x \in T$ be an arbitrary tile of T. Then we can check in all cases that $T \setminus \{x\}$ is a subset of one of the non-tiling sets listed in Lemma 3.1.

Finally, we show that our list of minimal tiling sets is complete, namely no tiling sets are missing from our list. In fact, the maximal non-tiling sets are obtained as follows. First, we generate the complete list of all sets of Wang tiles with $|D| = |W| = 1$. Then, we remove all the sets which are superset of one of the 11 types of Theorem 1.4 from the list. After that, in the remaining list of sets (which forms a partial order set under the subset relation '$\subseteq$'), we find all the maximal sets with regard to this partial order. The results are exactly the 8 types listed in Lemma 3.1. Since we have already shown that all of the 8 types are non-tiling sets, our list of 11 types of minimal tiling sets is complete. ∎

5. Conclusion

In this paper, we obtain a complete list of minimal tiling sets for the connected tiling problem with $|D| = |W| = 1$, which implies the problem is decidable in this case. It is worth noting that among the 11 types of minimal tiling sets, some only admit non-periodic tilings (see type 11), and some admit only semi-periodic tilings (i.e. with translation symmetry in only one direction, see type 7 or 8). This is in contrast with the situation of the original Wang's tiling problem. It is known that for Wang's tiling problem with 2 or 3 colors, all the minimal tiling sets have full periodic tilings [5, 6] (i.e. with translation symmetry in two independent directions). Our result indicates that even though the tiling sets are aperiodic, we can still show its decidability. This gives us hope to extend the decidability results to the cases with more colors for the Wang's original tiling problem and for the connected tiling problem. A natural next step is to solve the connected tiling problem for $|D| + |W| = 3$.

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References

[1] H. Wang, Proving theorems by pattern recognition - II, Bell System Technical Journal 40 (1) (1961) 1–41.

[2] H. Wang, Games, logic and computers, Scientific American 213 (5) (1965) 98–107.

[3] R. Berger, The undecidability of the domino problem, Memoirs of the American Mathematical Society 66 (1966) 1–72.

[4] R. Robinson, Undecidability and nonperiodicity for tilings of the plane, Inventiones Mathematicae 12 (1971) 177–209.

[5] W.-G. Hu, S.-S. Lin, Nonemptiness problems of plane square tiling with two colors, Proceedings of the American Mathematical Society 139 (2011) 1045–1059.

[6] H.-H. Chen, W.-G. Hu, D.-J. Lai, S.-S. Lin, Nonemptiness problems of Wang tiles with three colors, Theoretical Computer Science 547 (2014) 34–45.

[7] E. Jeandel, M. Rao, An aperiodic set of 11 Wang tiles, Advances in Combinatorics (2021) https://doi.org/10.19086/aic.18614.