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Multifold Tiles of Polyominoes and Convex Lattice Polygons

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Abstract A planar shape S is a k-fold tile if there is an indexed family $\mathcal{T}$ of planar shapes congruent to S that is a k-fold tiling: any point in $\mathbb{R}^2$ that is not on the boundary of any shape in $\mathcal{T}$ is covered by exactly k shapes in $\mathcal{T}$. Since a 1-fold tile is clearly a k-fold tile for any positive integer k, the subjects of our research are *nontrivial k-fold tiles*, that is, plane shapes with property "not a 1-fold tile, but a $k(\geq 2)$-fold tile." In this paper, we prove some interesting properties about nontrivial k-fold tiles. First, we show that, for any integer $k \geq 2$, there exists a polyomino with property "not an h-fold tile for any positive integer $h < k$, but a k-fold tile." We also find, for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones that are nontrivial k-fold tiles. Next, we prove that, for any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k, and for $k = 2$ and $k = 3$, no such convex unit-lattice polygon exists.

MSC: 52C20; 05B45; 05B50
Keywords: multiple tilings; k-fold tiles; polyominoes; convex lattice polygons

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1. Introduction

1.1. What are k-Fold Tilings?

An infinite family of plane shapes $\mathcal{T}$ is called a *tiling* if the shapes in $\mathcal{T}$ cover the whole plane without gaps or overlaps. A tiling $\mathcal{T}$ is called *monohedral* [5] if any two shapes belonging to $\mathcal{T}$ are congruent. If a tiling $\mathcal{T}$ is monohedral, the unique shape in the tiling $\mathcal{T}$ (up to congruence) is called a *tile*. In this paper, we consider only monohedral tilings, which have a great deal of classic and recent research (see Section 1.2).

As extensions of the conventional tilings and tiles, we study *k-fold tilings*, which cover the plane with multiplicity k, and *k-fold tiles* belonging to it. Although the strict definition of a k-fold tile will be described later, intuitively it means that an infinitely large family of copies of the shape (translations, rotations, and reflections are allowed) covers the whole plane such that the shapes overlap exactly k times at almost every point in it. As an example, consider the regular hexagon with a chipped right triangle as shown in Fig. 1. This shape is an 11-fold tile because, by overlapping 12 appropriately rotated or reflexed copies of it, we can obtain the regular hexagon with multiplicity 11, and by arranging them in the hexagonal tiling shown in Fig. 2, we obtain an 11-fold tiling.

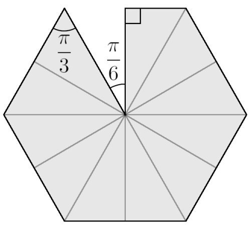

Figure 1. An 11-fold tile of the chipped regular hexagon.

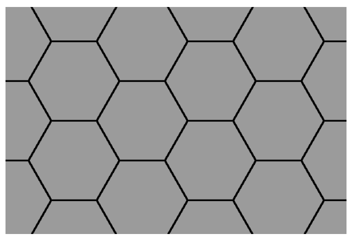

Figure 2. An 11-fold tiling with the regular hexagons with multiplicity 11.

Note that this shape is not a tile: we cannot fill the $\pi/6$ deficit angle in the chip in any way. Since we can obtain a k-fold tiling by piling up k sheets of a tiling, it is trivial to consider constructing a k-fold tiling with (1-fold) tiles. Hence we are interested in plane shapes with the property "not a tile, but a $k(\geq 2)$-fold tile." We call a plane shape having this property a *nontrivial k-fold tile*. That is, the chipped regular hexagon in Fig. 1 is a nontrivial 11-fold tile.

1.2. Background on Tilings

We recall some known results on simple ($k = 1$) tiling as follows:

Theorem 1.1 ([4]). *Any triangle and any quadrilateral is a tile.*

Theorem 1.2 ([17]). *For any integer $n \geq 7$, any convex n-gon is not a tile.*

Theorem 1.3 ([9, 21]). *Let the lengths of the sides of a convex hexagon H be denoted by a, b, $\ldots$, f, consecutively, and its angles between a and b, b and c, $\ldots$, f and a by*

A, B, ..., F, respectively. H is a tile if and only if it is at least one of the following three types.

$$\begin{array}{ll} \text{(i)} & A+B+C=2\pi,\ a=d, \\ \text{(ii)} & A+B+D=2\pi,\ a=d,\ c=e, \\ \text{(iii)} & A=C=E=\frac{2}{3}\pi,\ a=b,\ c=d,\ e=f. \end{array}$$

Theorem 1.4 ([9, 14, 21, 22, 24])**.** *Let the lengths of the sides of a convex pentagon P be denoted by a, b, ..., e, consecutively, and its angles between a and b, b and c, ..., e and a by A, B, ..., E, respectively. P is a tile if it is at least one of the following 15 types.*

$$\begin{array}{ll} \text{(i)} & A+B+C=2\pi, \\ \text{(ii)} & A+B+D=2\pi,\ a=d, \\ \text{(iii)} & A=C=D=\frac{2}{3}\pi,\ a=b,\ d=c+e, \\ \text{(iv)} & A=C=\frac{\pi}{2},\ a=b,\ c=d, \\ \text{(v)} & A=\frac{\pi}{3},\ C=\frac{2}{3}\pi,\ a=b,\ c=d, \\ \text{(vi)} & A+B+D=2\pi,\ A=2C,\ a=b=e,\ c=d, \\ \text{(vii)} & 2B+C=2D+A=2\pi,\ a=b=c=d, \\ \text{(viii)} & 2A+B=2D+C=2\pi,\ a=b=c=d, \\ \text{(ix)} & 2E+B=2D+C=2\pi,\ a=b=c=d, \\ \text{(x)} & E=\frac{\pi}{2},\ A+D=2B-D=\pi,\ 2C+D=2\pi,\ a=e=b+d, \\ \text{(xi)} & A=\frac{\pi}{2},\ C+E=\pi,\ 2B+C=2\pi,\ d=e=2a+c, \\ \text{(xii)} & A=\frac{\pi}{2},\ C+E=\pi,\ 2B+C=2\pi,\ 2a=d=c+e, \\ \text{(xiii)} & A=C=\frac{\pi}{2},\ 2B+D=2E+D=2\pi,\ e=2c=2d, \\ \text{(xiv)} & A=\frac{\pi}{2},\ 2B+C=2\pi,\ C+E=\pi,\ 2a=2c=d=e, \\ \text{(xv)} & A=\frac{\pi}{3},\ B=\frac{3}{4}\pi,\ C=\frac{7}{12}\pi,\ D=\frac{\pi}{2},\ E=\frac{5}{6}\pi,\ a=2b=2d=2e. \end{array}$$

In 2017, Rao [20] claimed that Theorem 1.4 is true even if "if" is replaced with "only if," that is, there are only the 15 types of tiles of convex pentagons mentioned in Theorem 1.4. This was shown by using a computer, and it seems that it has not been fully verified at this time.

On polyominoes, the following facts are known.

Theorem 1.5 ([4])**.** *For any positive integer $n \le 6$, any n-omino is a tile.*

Theorem 1.6 ([4])**.** *A heptomino is a tile if and only if it is not any of those four listed in Fig. 3.*

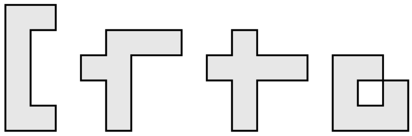

FIGURE 3. The heptominoes that are not tiles.

1.3. Previous Results on k-Fold Tilings

If all shapes in a k-fold tiling $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ are translates of T_1, then $\mathcal{T}$ is called a *k-fold translative tiling*, and T_1 is called a *k-fold translative tile*. In particular, if the translative vectors of T_i form a lattice $\Lambda = \{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 \mid a_1, a_2 \in \mathbb{Z}\}$ in $\mathbb{R}^2$ (where $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ are linearly independent vectors), then $\mathcal{T}$ is called a *k-fold lattice tiling*, and T_1 is called a *k-fold lattice tile*. These terms are defined in the references [26, 30].

The origin of the study of multiple tilings is the one by Furtwängler [3] in 1936. He considered trivial multiple lattice tilings in the Euclidean space as a generalization of what is called Minkowski's conjecture (see Zong's survey [28]). As far as we know, Marley [15, 16] first did the study focusing on nontrivial multiple translative tilings in $\mathbb{R}^2$. He discovered nontrivial 5-, 6-, and 35-fold (lattice) tiles of the convex 8-, 10-, and 12-gons, respectively. Recently, Yang and Zong [27] gave a characterization of all convex k-fold translative tiles in $\mathbb{R}^2$ for any $k = 2, 3, 4, 5$. Specifically, for any $k = 2, 3, 4$, they are classified as either parallelograms or centrally symmetric hexagons (this is also true for $k = 1$ [8]), and for $k = 5$, they are classified as either parallelograms, centrally symmetric hexagons, two classes of octagons, or one class of decagons.

Although there is various research on multiple translative tilings in the Euclidean space other than those mentioned above (for example, [1, 6, 7, 10, 12, 13, 26]), there seems to be no research on multiple tilings that also allow rotations and reflections. The subjects of our research are such nontrivial multiple tilings.

1.4. Our Contribution

In this paper, we mainly consider polyominoes and convex unit-lattice polygons as basic plane shapes and present the following results. First we show that, for any integer $k \geq 2$, there is a polyomino whose minimum tile-fold number is k. Second we find, for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones that are nontrivial k-fold tiles. Last, we prove that, for any integer $k = 5$ or $k \geq 7$, there is a convex unit-lattice polygon (see Definition 27) that is a nontrivial k-fold tile whose area is k, and for $k = 2$ and $k = 3$, there is no such convex unit-lattice polygon. We also find that, for $k = 4$, such a convex unit-lattice polygon must be a certain pentagon, if any.

2. Preliminaries

Let $\mathbb{N}^+$ be the set of positive integers and $\mathbb{N}_0^+$ be the set of nonnegative integers.

2.1. k-Fold Tiles

We give the formal definition of k-fold tiles.

Definition 2.1. Let $\mathcal{T}$ be an indexed family of T_i: $\{T_i \mid i \in \mathbb{N}^+\}$ where T_i is a closed and bounded set on the Euclidean plane $\mathbb{R}^2$, and for any $i, j \in \mathbb{N}^+$, T_i and T_j are congruent. $\mathcal{T}$ is a *k-fold tiling* if for any point $(x, y) \in \mathbb{R}^2$ that is not included in the boundary of any T_i, there exist exactly $k \in \mathbb{N}^+$ distinct i such that $(x, y) \in T_i$. A 1-fold tiling may be simply called a *tiling*. In a tilling, each copy of the tile is sometimes referred as a *piece*.

Definition 2.2. The shape belonging to a k-fold tiling is called a *k-fold tile*. A 1-fold tile may be simply called a *tile*.

Hereafter, we refer to the Euclidean plane as the plane.

Definition 2.3. If a plane shape P is a $k(\in \mathbb{N}^+)$-fold tile, then k is a *tile-fold number* of P. The set of tile-fold numbers of P is denoted by $\mathrm{TFN}(P)$. If an integer k satisfies that $k \in \mathrm{TFN}(P)$ and $h \notin \mathrm{TFN}(P)$ for every positive integer $h < k$, then we call k the *minimum tile-fold number* of P, and it is denoted by $\tau^\bullet(P)$ [30].

The following facts are trivial.

Observation 2.4. *For any plane shape P, if $h, k \in \mathrm{TFN}(P)$, then $h + k \in \mathrm{TFN}(P)$.*

Observation 2.5. *For any plane shape P, if $k \in \mathrm{TFN}(P)$, then for any $\ell \in \mathbb{N}^+$, $k\ell \in \mathrm{TFN}(P)$.*

The following lemma holds as an extension of the above observations.

Lemma 2.6. *For any plane shape P and any coprime integers $h, k \geq 2$, if $h, k \in \mathrm{TFN}(P)$, then for any integer $\ell \geq (h-1)(k-1)$, $\ell \in \mathrm{TFN}(P)$.*

Lemma 2.6 is derived from the following lemma.

Lemma 2.7 ([23])**.** *For any coprime $a, b \in \mathbb{N}^+$ and any integer $n \geq (a-1)(b-1)$, there exists $x, y \in \mathbb{N}_0^+$ such that $n = ax + by$.*

Proof of Lemma 2.6. Clear from Observations 2.4, 2.5, and Lemma 2.7. ∎

For a plane shape P, let $\tau(P)$ and $\tau^*(P)$ be the minimum integer k such that P is a k-fold translative tile and is a k-fold lattice tile, respectively, as with $\tau^\bullet(P)$. For convenience, we define $\tau^\bullet(P) = \infty$, $\tau(P) = \infty$, and $\tau^*(P) = \infty$ if P is not any multiple, multiple translative, and multiple lattice tile, respectively. For any plane shape P, the following inequality clearly holds:

$$\tau^\bullet(P) \leq \tau(P) \leq \tau^*(P). \tag{2.1}$$

These definitions and Inequality (2.1) are given in [30].

2.2. Polyominoes

A *polyomino* is a plane shape formed by joining one or more congruent squares edge to edge. For example, the four shapes listed in Fig. 4 are all polyominoes. Each square is called a *cell*. A polyomino with exactly n cells is called an *n-omino*. See Chapter 14 of the reference [5] for the strict definition.

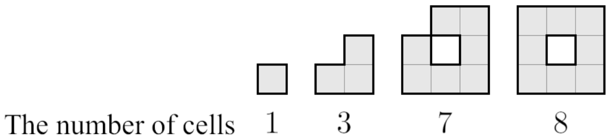

Figure 4. Examples of polyominoes.

We can prove the following lemma.

Lemma 2.8. *For any positive integer k, any k-omino is a k-fold lattice tile.*

Proof. Let us divide the plane into the cells by the unit grid. Along them, we arbitrarily arrange a k-omino P on the plane, and consider a family of k-ominoes $\{P + z \mid z \in \mathbb{Z}^2\}$. Then, each of the k cells composing the k-omino overlaps one of the cells on the plane exactly once. Since this happens in any cell on the plane, k-ominoes cover the plane with multiplicity k. ∎

2.3. Fold Bands

We introduce the following notion of *fold bands*, which will be useful later in some discussions.

Definition 2.9. Consider $n \in \mathbb{N}^+$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}_0^+$. An infinite horizontal strip of width 1 and multiplicity m_1 as shown in Fig. 5 is called an *m_1-fold band.* A series of m_1, m_2, ..., m_n-fold bands as shown in Fig. 6 is called an *$(m_1, m_2, \ldots, m_n)$-fold band.* If $m_1 = m_2 = \cdots = m_n = m$, then an $(m_1, m_2, \ldots, m_n)$-fold band may be also called an *$n \times m$-fold band.* If n is clear from the context, an $n \times m$-fold band may be called an *m-fold band* for simplicity.

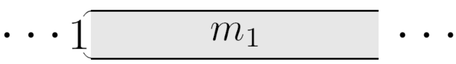

Figure 5. An m_1-fold band.

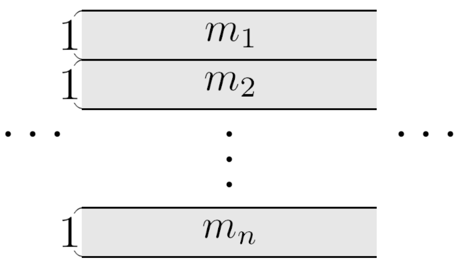

Figure 6. An $(m_1, m_2, \ldots, m_n)$-fold band.

If we cut the fold band in Fig. 6 with the vertical line, the cross-section of it looks as Fig. 7 from the side. (We add an explanation for this representation by using an example as follows. Fig. 8 is a 3D-mode illustration of a $(4, 2, 5, 1)$-fold band. Looking at the cross-section of Fig. 8 from the left, we have the view as Fig. 9. We will represent Fig. 9 as Fig. 10.) We call such a figure (like Fig. 7) a *cross-section representation of a fold band.*

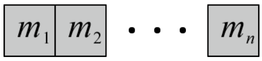

Figure 7. The cross-section representation of the $(m_1, m_2, \dots, m_n)$-fold band.

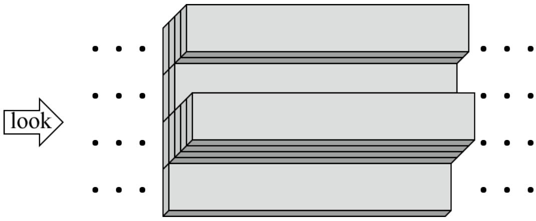

Figure 8. A 3D-mode illustration of a $(4, 2, 5, 1)$-fold band.

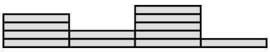

Figure 9. View of Fig. 8 from the left.

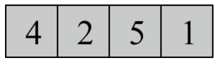

Figure 10. The cross-section representation of the $(4, 2, 5, 1)$-fold band.

3. Multifold Tiles on Polyominoes

In this chapter, we consider nontrivial k-fold tiles on polyominoes.

3.1. The Minimum Tile-Fold Number of Polyominoes

A plane shape shown in Fig. 11 is called a *holed-p-I* (an analogous shape is presented in [16], although only multiple translative tilings are considered there), where p is an integer greater than or equal to 2. (The name comes from that the p-omino consisting of p cells arranged straightly is called p-I.) A closed curve composed of a rectilinear polygonal line AB protruding from the left side and a line segment AB (and its interior) is called a *bump part*, and a closed curve composed of a rectilinear polygonal line CD congruent to AB and a line segment CD is called a *hole part*. We assume that the bump part (and the hole part) does not have any line or rotational symmetry. We also assume that the polygonal line AB can be exactly overlapped with the polygonal line CD by translating it in the horizontal direction. As long as these conditions are all satisfied, the shape of the bump part (and the hole part) can be arbitrary. Note that if the length of every edge is rational, we can regard it as a polyomino (by changing the unit of length).

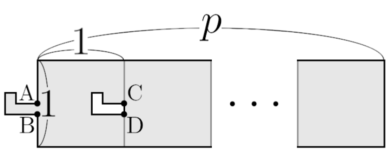

Figure 11. A holed-p-I.

Theorem 3.1. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^{\bullet}(P) = k$.*

Note that if we consider only multiple *translative* tilings, it is not difficult to show that a holed-p-I with any shape of the bump (and hole) part is a p-fold tile. It is shown as follows.

Lemma 3.2. *For any holed-p-I P, $\tau^{\bullet}(P) \leq \tau(P) = \tau^{*}(P) = p$.*

Proof. For any holed-p-I, a p-fold lattice tiling is easily obtained by using the method used in the proof of Lemma 2.8 as follows. We consider a tilling such that the hole part of each piece is completely filled with the bump part of another piece. Then, these (infinite number of) pieces make a p-fold band (see Fig. 12). By arranging copies of this band, we can obtain a $\infty \times p$-fold band, which is a p-fold tiling (see Fig. 13). Since this is a p-fold lattice folding and it is clear that there is no way to decrease the fold-number from p for any multifold translative folding, $\tau^{\bullet}(P) \leq \tau(P) = \tau^{*}(P) = p$ holds for any hold-p-I P. ■

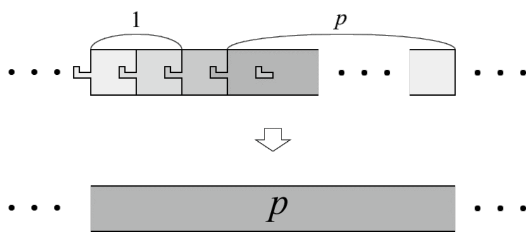

FIGURE 12. A p-fold band with the holed-p-I.

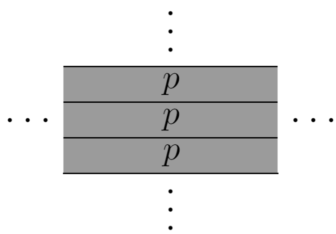

FIGURE 13. A p-fold tiling with p-fold bands.

We call this tiling a *regular lattice tiling* with holed-p-Is. From Lemma 3.2, the upper bound for Theorem 3.1 ($\tau^{\bullet}(P) \leq p$) is clear. However, it is not easy to show that a given holed-p-I is not a p'-fold tile for any $p' < p$. We will show that there is a holed-p-I having this condition, i.e., the holed-p-I shown in Fig. 14 is not a p'-fold tile for any $p' < p$. We prepare some lemmas before showing the proof. Note that $a = 2^{2p+1} - 2$. From here to the end of this section, let p be an arbitrary integer greater than or equal to 2.

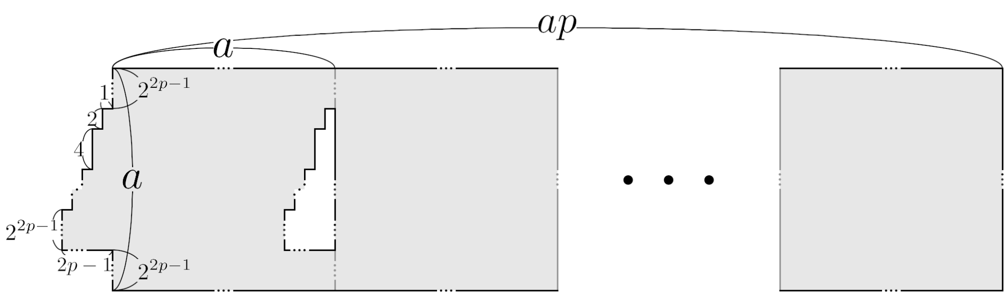

FIGURE 14. The holed-p-I in the proof of Theorem 3.1.

Fact 3.3. *Consider any cell edge e on the boundary of the holed-p-I. Then there is a square with the side length of 2^{2p-1} that is contained in the holed-p-I and one of whose sides includes e.*

Proof. Clear from Fig. 14. ■

We positively orient the boundary of the holed-p-I as illustrated by the arrows in Fig. 15, i.e., if you walk along the boundary following the orientation, you can see the inside (resp., outside) of the holed-p-I on your left (resp., right) side.

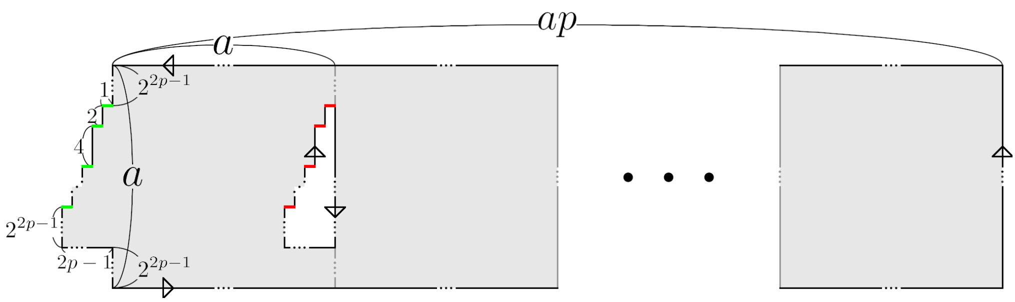

FIGURE 15. The oriented boundary and red edges.

Notice 3.4. *We orient every piece in a tiling according to the above rule, i.e., even for a piece that is a reflection of the original holed-p-I, the orientation of the boundary is made by the way mentioned above, i.e., if you walk along the boundary following the orientation, you can see the inside (resp., outside) of the (reflected) holed-p-I on your left (resp., right) side (see Fig. 16).*

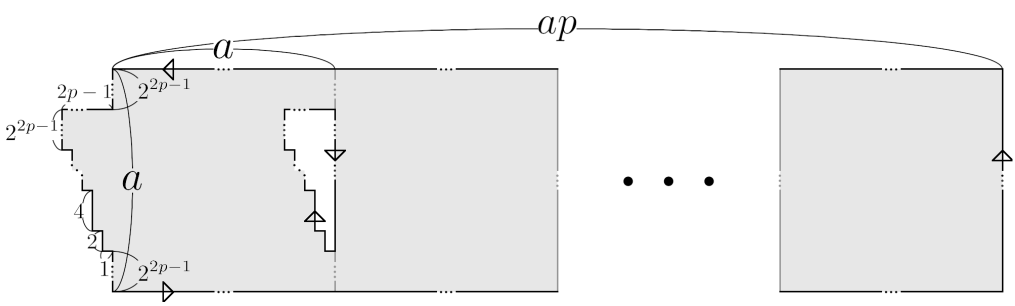

FIGURE 16. The reflection of Fig. 15 and the orientation of the boundary.

The red-colored edges shown in Fig. 15 are denoted by $e_1, e_2, \ldots, e_{2p-1}$ in order from the top, and we call them *red edges*. For any red edge e_i ($1 \leq i \leq 2p-1$), there is a cell attaching e_i from the left-hand side (i.e., above in Fig. 15) but there is no cell attaching e_i from the right-hand side (i.e., below in Fig. 15), and hence there is a difference in the multiplicity 1 between the left and right of e_i. We call this a *gap* caused by (a red edge) e_i.

Let $\mathcal{T}$ be an arbitrary p-fold tiling with the holed-p-Is. Consider a $T \in \mathcal{T}$ and e that is one of the red edges of T. In the tiling $\mathcal{T}$, the gap caused by e must be filled up by one or more tiles in $\mathcal{T}$. We call this situation that the gap is *eliminated* by the tile or tiles, and the tiles used for eliminating the gap is called the *supplements* of the gap. The set of supplements of the gap caused by e is denoted by $\sup(e)$. Let c_e be a cell (in T) one of whose edge is e. For a $T' \in \sup(e)$, there is at least one edge f (of a cell, say c_f) in T' such that f is on the boundary of T', c_e and c_f touch each other with e and f, and the interior of c_e and the interior of c_f never overlap each other (and hence the direction of e and f are opposite, see Notice 3.4). See Fig. 17 for examples. Note that f may not be on the boundary of the bump part. The set of such fs is denoted by $\mathrm{supedge}(e)$.

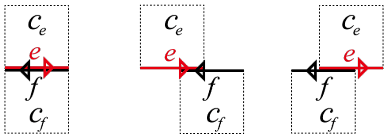

FIGURE 17. Examples of the relation between e and f: Broken-line boxes are cells c_e and c_f.

Lemma 3.5. *Let $\mathcal{T}$ be an m-fold tiling with the holed-p-Is for an integer $m \geq 1$ and let T be a tile in $\mathcal{T}$. Consider two arbitrary distinct red edges, e_i and e_j, of T. If there is a tile T' such that $T' \in \sup(e_i) \cap \sup(e_j)$, then $T' \in \sup(e_h)$ for every red edge e_h of T.*

Proof. The positional relationship of arbitrarily chosen two distinct red edges e_i and e_j $(1 \leq i < j \leq 2p-1)$ is as shown in Fig. 18, i.e., the vertical distance is $2^j - 2^i$ and the horizontal distance is $j - i$. For any pair of $f_i \in \mathrm{supedge}(e_i)$ and $f_j \in \mathrm{supedge}(e_j)$, the positional relationship of them must be as shown in Fig. 19, i.e., the vertical distance is $2^j - 2^i$ and the horizontal distance is greater than $j - i - 1$ and shorter than $j - i + 1$. By seeing Figs. 15 and 16 (the latter is the reflection of Fig. 15), we can observe that there is no pair of edges with the positional relationship in Fig. 19 except for the $2p - 1$ green-colored edges shown in Fig. 15 (note that $2^j - 2^i$ is even and $2 \leq 2^j - 2^i \leq 2^{2p-1} - 2$).

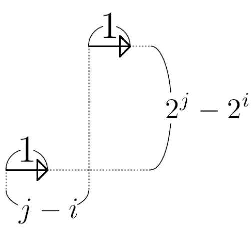

FIGURE 18. The positional relationship of two red edges.

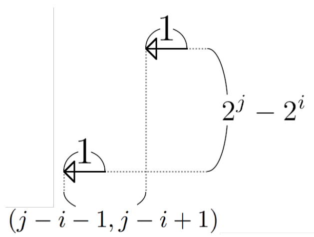

FIGURE 19. The positional relationship of two edges that eliminate gaps caused by the edges in Fig. 18.

Now only the $2p - 1$ green-colored edges in Fig. 15 leave the possibility. Let them be denoted by $e'_1, e'_2, \ldots, e'_{2p-1}$ in the same way as the red edges, and we call them *green* *edges.* Thus we assume that $f_i = e'_{i'}$ and $f_j = e'_{j'}$, where $1 \le i' < j' \le 2p-1$. The following equation must hold:

$$2^j - 2^i = 2^{j'} - 2^{i'}. \tag{3.1}$$

If we assume that $i < i'$, then since $j-i$, $j'-i$, and $i'-i$ are all positive integers, $2^i(2^{j-i} - 2^{j'-i} + 2^{i'-i} - 1) \neq 0$, and this contradicts Equation (3.1). From symmetry, if we assume that $i' < i$, then it also contradicts Equation (3.1), and hence $i = i'$. From Equation (3.1) it follows that $2^j = 2^{j'}$ and clearly $j = j'$. Therefore, e'_i and e'_j must be arranged correspondingly to e_i and e_j, respectively, which proves the statement. ∎

Lemma 3.6. *If an $m(\in \mathbb{N}^+)$-fold tiling $\mathcal{T}$ by the holed-p-I includes a piece whose hole part is not completely filled with a bump part of any piece, then there is a region that is covered by at least p pieces.*

Proof. For $T \in \mathcal{T}$, if $\bigcap_{i=1}^{2p-1} \sup(e_i) \neq \emptyset$, where $e_1, \ldots, e_{2p-1}$ are red edges of T, then T is called *semi-translatively covered.* First, we assume that all $T \in \mathcal{T}$ are semi-translatively covered. In this case, the tiling is the regular lattice tiling or a tiling very close to the regular lattice tiling. Even for the latter case, it is clear that most of points (concretely, inner points of any hold-p-I that are at least p distance far from the bump part and the hole part of the holed-p-I) are covered by at least p tiles.

Second, we assume that there is a $T_0 \in \mathcal{T}$ such that $\bigcap_{i=1}^{2p-1} \sup(e_i) = \emptyset$, where $e_1, \ldots, e_{2p-1}$ are red edges of T_0. From Lemma 3.5, every $T' \in \mathcal{T}$ is included in at most one of $\sup(e_i)$ $(i \in \{1, \ldots, 2p-1\})$. For $i \in \{1, \ldots, 2p-1\}$, $\sup(e_i) = \{T_1^i, \ldots, T_{n_i}^i\}$, where $n_i = |\sup(e_i)|$. Assume that T_0 is placed on the plane as shown in Fig. 20. Let c_0 and c_1 be the yellow-colored and blue-colored cells shown in Fig. 20, respectively. We show that at least p pieces overlap at c_0 or c_1.

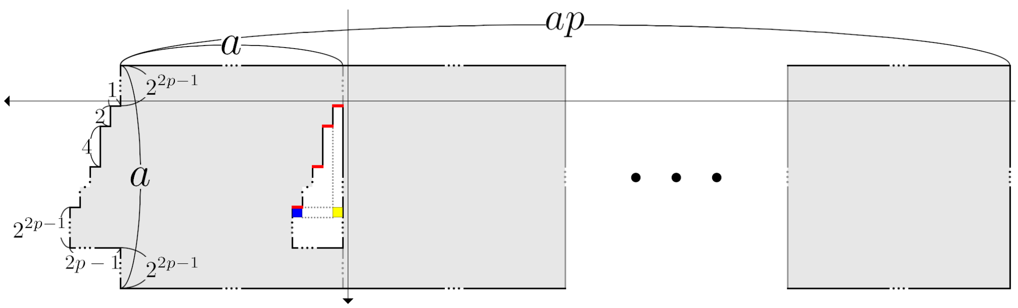

FIGURE 20. T_0, c_0, and c_1.

Based on T_0, we set the 2-dimensional Cartesian coordinate system with the x-axis in the horizontally leftward direction and the y-axis in the vertically downward direction such that the center of the cell with the red edge e_1 as the upper horizontal side is located at $(1, 1)$ as shown in Fig. 20. Let a cell whose center coordinates are $(i, j) \in \mathbb{Z}^2$ be denoted by *cell* (i, j). According to this system, c_0 and c_1 are represented by $(1, 2^{2p-1} - 1)$ and $(2p-1, 2^{2p-1} - 1)$, respectively. To construct an m-fold tiling that includes T_0, it is necessary to eliminate all gaps caused by the $2p-1$ red edges, $e_1, \ldots, e_{2p-1}$, of T_0. From Fact 3.3, each $\sup(e_i) = \{T_1^i, \ldots, T_{n_i}^i\}$ clearly covers at least one of c_0 or c_1. Since T_0 has $2p-1$ red edges, c_0 or c_1 is covered by at least p pieces. ∎

Proof of Theorem 3.1. Consider the holed-p-I shown in Fig. 14 and an $m(\in \mathbb{N}^+)$-fold tiling with it. If there is a piece whose hole part is not completely filled with a bump part of any piece, then $m \geq p$ from Lemma 3.6. Thus we assume that the hole part of any piece is completely filled with the bump part of another piece. In this case clearly the tiling must be the regular lattice tiling, and thus $m \geq p$. From the above discussion it follows that $\tau^\bullet(P) \geq p$ for the holed-p-I P. Considering Lemma 3.2, we obtain $\tau^\bullet(P) = p$. ■

Corollary 3.7. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^\bullet(P) = \tau(P) = \tau^*(P) = k$.*

Proof. Clear from Inequality (2.1), Theorem 3.1, and that the way of the multiple tiling shown in the proof of Theorem 3.1 is a multiple lattice tiling. ■

Note that one can show Theorem 3.1 by using a polyomino with no hole. For example, let us construct the "indented"-p-I by cutting the $(2p-1)\times 2^{2p-1}$ rectangle under the hole part out of the holed-p-I and attaching it to under the bump part as shown in Fig. 21. This change does not affect the proof of Theorem 3.1.

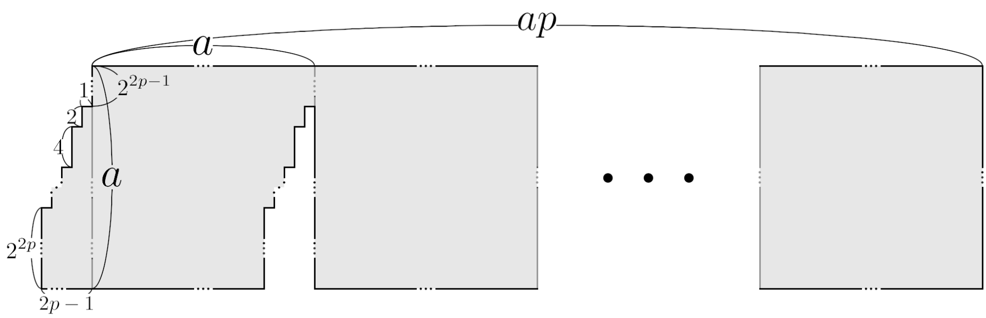

FIGURE 21. The "indented"-p-I.

3.2. The Lower Bound of the Number of Cells

Next, we focus our attention on the number of cells of a polyomino.

Definition 3.8. If an $n(\in \mathbb{N}^+)$-omino P is a nontrivial $k(\in \mathbb{N}^+)$-fold tile and there is no n'-omino that is a nontrivial k-fold tile for any positive integer $n' < n$, then n is called the *minimum size of nontrivial k-fold-tile polyomino* and P is called a *minimum-sized nontrivial k-fold-tile polyomino.*

We show the following theorem.

Theorem 3.9. *For any integer $k \geq 2$, the minimum size of nontrivial k-fold-tile polyomino is 7, and the heptominoes C7, F7, and X7 listed in Fig. 3 are all minimum-sized nontrivial k-fold-tile polyominoes. Furthermore, the heptomino G7 in Fig. 3 is also a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$.*

Note that it is only open whether G7 is a k-fold tile for $k = 3, 5$. We show some lemmas used for proving this theorem. First, we consider overlapping a heptomino in a diagonal direction. We consider an operation of repeating the process to translate a heptomino in the rightward and downward (leftward and upward) directions by 1 and to overlap them (see Fig. 22). We call this operation *Operation I*. The multiplicity of each cell of Fig. 22 is shown in Fig. 23. Moreover, by rotating it 180 degrees, we obtain the arrangement shown in Fig. 24. We call each of these arrangements a *diagonal* $(1,2,1,1,1,1)$*-fold band*.

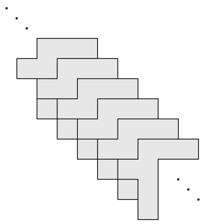

FIGURE 22. Operation I of F7.

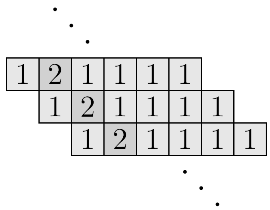

FIGURE 23. The multiplicity of each cell of Fig. 22.

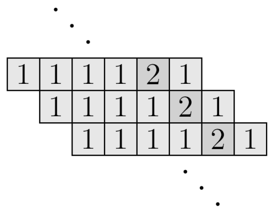

FIGURE 24. The 180 degree rotation of Fig. 23.

Such diagonal fold bands similarly construct a k-fold tiling as normal fold bands. Therefore, unless otherwise required, hereafter we also simply call them *fold bands*. In addition, by either or both of rotating F7 90 degrees clockwise and reflecting it in a vertical line before applying Operation I to it, we obtain a $(1,2,1,1,1,1)$-fold band again or a $(1,2,2,2)$-fold band. Similarly, by applying Operation I to C7, F7, X7, and G7, we obtain the (diagonal) fold bands as shown in Table 1. Note that we do not have to consider reflecting each of C7, X7, and G7 before applying the operation since they all have line symmetry.

TABLE 1. The fold bands obtained by Operation I.

Heptomino	Obtained fold bands
C7	$(1,2,1,1,1,1)$
F7	$(1,2,1,1,1,1),(1,2,2,2)$
X7	$(1,2,1,2,1),(2,2,1,2)$
G7	$(1,2,2,2),(1,2,1,2,1)$

In Operation I, we considered overlapping a heptomino with the slope -1. We next consider overlapping it with the slope $-1/2$, that is, an operation of repeating the process to translate a heptomino in the rightward and downward (leftward and upward) directions by 2 and 1, respectively, and to overlap them. We call this operation *Operation II*. We also call an arrangement obtained by this operation a *diagonal fold band* or simply a *fold band* as above. As with Operation I, by applying Operation II to C7, F7, X7, and G7, we obtain the (diagonal) fold bands as shown in Table 2.

TABLE 2. The fold bands obtained by Operation II.

Heptomino	Obtained fold bands
C7	$(1,1,1,0,1,0,1,0,1,1),(1,0,1,1,2,1,1)$
F7	$(1,0,1,1,1,0,1,1,1),(1,1,2,1,1,1),(1,0,1,0,2,2,1),(1,0,1,0,1,1,1,2)$
X7	$(1,0,1,1,1,1,2),(1,0,2,1,1,1,1)$
G7	$(1,1,2,0,2,1),(1,1,1,0,2,1,1)$

Lemma 3.10. *The four heptominoes listed in Fig. 3 are all 2-fold tiles.*

Proof. By applying Operation I to each of F7 and G7, we obtain a $(1,2,2,2)$-fold band, and by combining two pieces of it, we obtain a $(2,2,2,1+1,2,2,2)$-fold band. By applying Operation I to C7, we obtain a $(1,2,1,1,1,1)$-fold band, and by combining two pieces of it, we obtain a $(1,2,1+1,1+1,1+1,1+1,2,1)$-fold band, or a $(1,2,2,2,2,2,2,1)$-fold band. By applying Operation II to X7, we obtain a $(1,0,1,1,1,1,2)$-fold band, and by combining two pieces of it, we obtain a $(1,0+2,1+1,1+1,1+1,1+1,2+0,1)$-fold band, or a $(1,2,2,2,2,2,2,1)$-fold band. Clearly, for each of the four heptominoes listed in Fig. 3, a 2-fold tiling can be obtained from these fold bands. ■

Lemma 3.11. *The heptominoes C7, F7, and X7 listed in Fig. 3 are all 3-fold tiles.*

Proof. By applying Operation I to each of C7 and F7, we obtain a $(1,2,1,1,1,1)$-fold band, and by combining six pieces of it, we obtain a $(1,1,1+1,1+1,2+1,1+1+1,2+1,1+1+1,1+2,2+1,1+1+1,1+2,1+1+1,1+2,1+1,1+1,1,1)$-fold band, or a $(1,1,2,2,3,3,3,3,3,3,3,3,3,3,2,2,1,1)$-fold band. A 3-fold tiling can be obtained by combining an infinite number of copies of it. The cross-section of it is shown in Fig. 25.

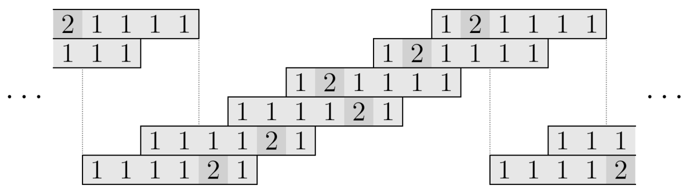

FIGURE 25. The cross-section representation of a 3-fold tiling with $(1,2,1,1,1,1)$-fold bands.

By applying Operation I to X7, we obtain both a $(2,2,1,2)$-fold band and a $(1,2,1,2,1)$-fold band, and by combining two pieces of the former and four pieces of the latter, we obtain a $(1,2+1,1+2,2+1,1+2,1+2,2+1,1+2,2+1,2+1,1+2,1+2,2+1,1+2,2)$-fold band, or a $(1,3,3,3,3,3,3,3,3,3,3,3,3,3,2)$-fold band. A 3-fold tiling can be obtained by combining an infinite number of copies of it. The cross-section of it is shown in Fig. 26.
■

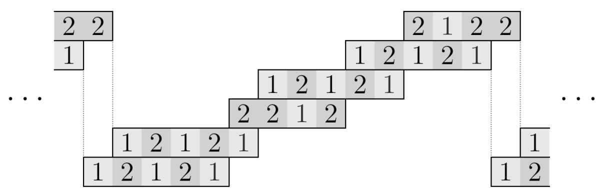

FIGURE 26. The cross-section representation of a 3-fold tiling with $(2,2,1,2)$-fold bands and $(1,2,1,2,1)$-fold bands.

Proof of Theorem 3.9. From Lemmas 3.10 and 3.11, it follows that C7, F7, and X7 are all 2-fold tiles and 3-fold tiles. Therefore, from Lemma 2.6, these heptominoes are all

k-fold tiles for any integer $k \geq (2-1)(3-1) = 2$. From this and Theorems 1.5 and 1.6, they are minimum-sized nontrivial k-fold-tile polyominoes for every $k \geq 2$. Similarly, from Lemmas 2.6, 2.8, and 3.10 and Theorems 1.5 and 1.6, it is also proven that G7 is a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$. ∎

4. Multifold Tiles on Convex Unit-Lattice Polygons

In this chapter, we consider nontrivial k-fold tiles on convex unit-lattice polygons.

Definition 4.1. A simple polygon whose all vertices lie in $\mathbb{Z}^2$ is called a *unit-lattice polygon.*

For example, the four shapes listed in Fig. 27 are all unit-lattice polygons.

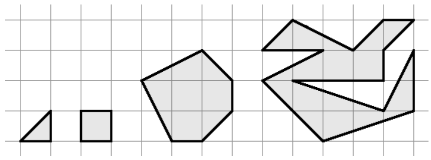

Figure 27. Examples of unit-lattice polygons.

In 2012, Gravin et al. [7] presented a convex unit-lattice octagon O_7 shown in Fig. 28 as a simple example of a nontrivial 7-fold (lattice) tile. This can be confirmed by considering $\{O_7 + z \mid z \in \mathbb{Z}^2\}$ as in the proof of Lemma 2.8. Since O_7 is centrally symmetric, each triangle that occurs from the division of O_7 by the unit grid can combine with another triangle by a translation to constitute a cell. From this and the fact that the area of O_7 is 7, they cover the plane with multiplicity 7. However, by Theorem 1.2, it is not a tile and is therefore a non-trivial 7-fold tile.

Theorem 4.2. *For any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon that is*

(i) *a nontrivial k-fold tile,*
(ii) *of area k, and*
(iii) *a hexagon if $k = 5$ or 8; an octagon otherwise.*

Definition 4.3 ([2]). A map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{p} \mapsto A\boldsymbol{p} + \boldsymbol{b}$ with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $\boldsymbol{b} \in \mathbb{Z}^2$ is called a *$\mathbb{Z}$-affine transformation.* Two convex unit-lattice polygons P and P' are said to be *equivalent* if there exists a $\mathbb{Z}$-affine transformation φ such that $\varphi(P) = P'$.

Theorem 4.4. *For $k = 2$ and $k = 3$, there does not exist any convex unit-lattice polygon that satisfies both conditions (i) and (ii) in Theorem 4.2. Furthermore, for $k = 4$, if there exists such a convex unit-lattice polygon, then it is equivalent to the pentagon (5,2,6)-b shown in Fig 40.*

Note that it is open whether there is a convex unit-lattice polygon that satisfies both conditions (i) and (ii) in Theorem 4.2 for $k = 4, 6$. First, we show some lemmas and prove Theorem 4.2.

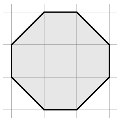

Figure 28. O_7.

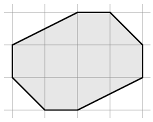

Figure 29. O_9.

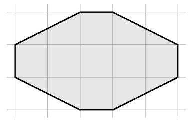

Figure 30. O_{11}.

Lemma 4.5. *For any integer $k = 7$ or $k \geq 9$, there exists a convex unit-lattice octagon of area k that is a nontrivial k-fold (lattice) tile.*

Proof. From Theorem 1.2, any convex octagon is not a tile. As with O_7, convex unit-lattice octagons O_9 and O_{11} shown in Figs. 29 and 30 are 9- and 11-fold lattice tiles, respectively. We now consider adding three cells to each of O_7, O_9, and O_{11} repeatedly while keeping the property of being a centrally symmetric convex unit-lattice octagon. Then it is clear that an octagon that was originally a k-fold lattice tile becomes a $(k+3)$-fold lattice tile after the addition as shown in Fig. 31. ∎

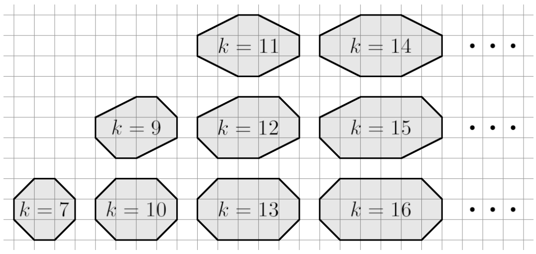

Figure 31. A nontrivial k-fold (lattice) tile of convex unit-lattice octagon for any integer $k = 7$ or $k \geq 9$.

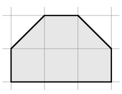

Figure 32. H_5.

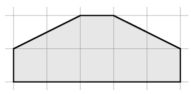

Figure 33. H_8.

Lemma 4.6. *For $k = 5$ or 8, there exists a convex unit-lattice hexagon of area k that is a nontrivial k-fold tile.*

Proof. It is clear from Theorem 1.3 that hexagons H_5 and H_8 are not tiles. Let us make the pair of H_5 by overlapping them as shown in Fig. 34. We also make the pair of H_8 in a similar way as shown in Fig. 35. Each number in Figs. 34 and 35 indicates the multiplicity.

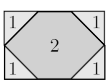

Figure 34. The pair of H_5.

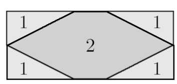

Figure 35. The pair of H_8.

By arranging pairs of H_5 on the plane in a manner of a tiling with hexagons, each of which is constituted by the 2-fold part of it, we obtain a $(\ldots, 3, 2, 3, 2, \ldots)$-fold band as shown in Fig. 36. By combining two $(\ldots, 3, 2, 3, 2, \ldots)$-fold bands, we obtain a $(\ldots, 3+2, 2+3, 3+2, 2+3, \ldots)$-fold band, or a 5-fold tiling. Similarly, by arranging pairs of H_8, we obtain a $(\ldots, 3, 3, 2, 3, 3, 2, \ldots)$-fold band as shown in Fig. 37. By combining three $(\ldots, 3, 3, 2, 3, 3, 2, \ldots)$-fold bands, we obtain a $(\ldots, 3+2+3, 3+3+2, 2+3+3, 3+2+3, 3+3+2, 2+3+3, \ldots)$-fold band, or an 8-fold tiling. The cross-section representation of it is shown in Fig. 38.

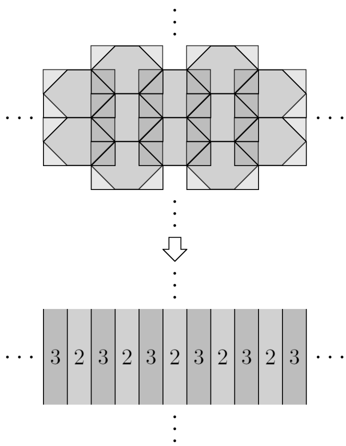

FIGURE 36. The arrangement of H_5.

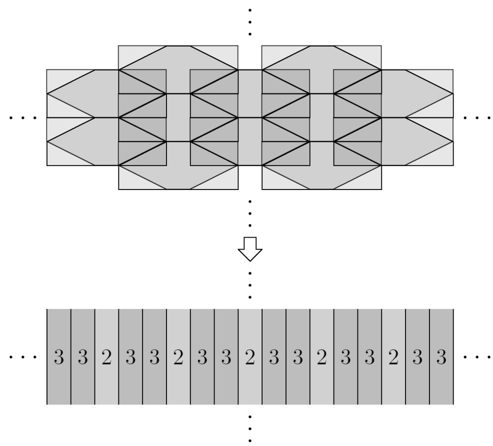

FIGURE 37. The arrangement of H_8.

3	2	3	3	2	3	3	2	3
2	3	3	2	3	3	2	3	3
3	3	2	3	3	2	3	3	2

FIGURE 38. The cross-section representation of an 8-fold tiling with $(\ldots, 3, 3, 2, 3, 3, 2, \ldots)$-fold bands.

Therefore, H_5 and H_8 are convex unit-lattice hexagons of area k that are nontrivial k-fold tiles for $k = 5$ and $k = 8$, respectively. ∎

Proof of Theorem 4.2. Clear from Lemmas 4.5 and 4.6. ∎

Next, we prove Theorem 4.4. Any $\mathbb{Z}$-affine transformation maps $\mathbb{Z}^2$ bijectively onto itself. We call points lying in $\mathbb{Z}^2$ *unit-lattice points*. For a unit-lattice polygon P, let $v = v(P)$, $b = b(P)$, $i = i(P)$, and $a = a(P)$ be the number of vertices of P, the number of unit-lattice points on the boundary of P, the number of interior unit-lattice points of P, and the area of P, respectively. It is known that $\mathbb{Z}$-affine transformations map convex unit-lattice polygons to convex unit-lattice polygons and preserve values v, b, i, and a. We show some known theorems on unit-lattice polygons and prove a lemma as a preparation.

Theorem 4.7 ([18]). *For any unit-lattice polygon,*

$$a = i + b/2 - 1. \tag{4.1}$$

Theorem 4.8 ([11, 19]). *Every convex unit-lattice polygon satisfying $i = 0$ is equivalent to one of the following polygons:*

(i) *A triangle with vertices $(0, 0)$, $(n, 0)$, and $(0, 1)$, where n is any positive integer.*
(ii) *A trapezoid with vertices $(0, 0)$, $(n, 0)$, $(m, 1)$, and $(0, 1)$, where n and m are any positive integers that satisfy $n \geq m$.*
(iii) *The triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.*

Theorem 4.9 ([11, 19]). *Every convex unit-lattice polygon satisfying both $i = 1$ and $v \geq 5$ is equivalent to exactly one of the four polygons shown in Fig 39.*

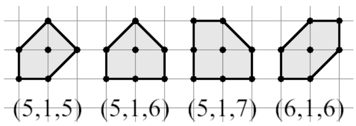

FIGURE 39. The four polygons in Theorem 4.9 (each polygon is named after (v, i, b)).

Theorem 4.10 ([25]). *Every convex unit-lattice polygon satisfying both $i = 2$ and $v \geq 5$ is equivalent to exactly one of the 21 polygons shown in Fig 40.*

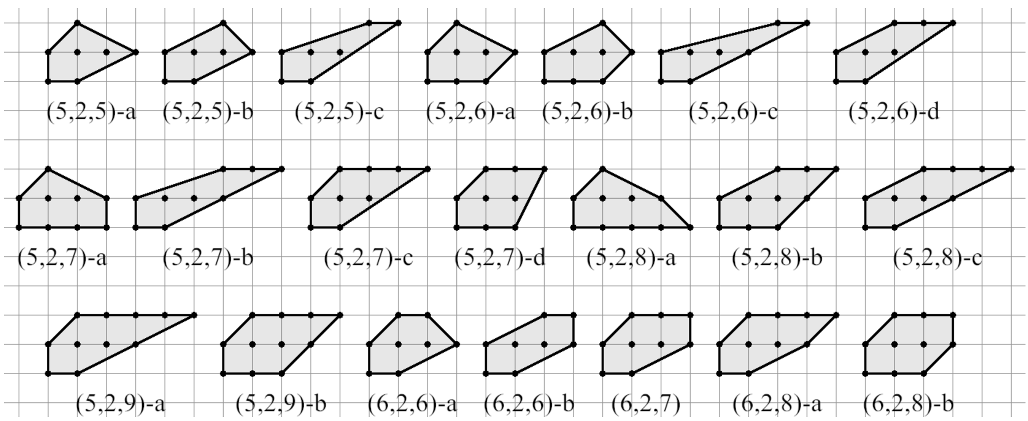

FIGURE 40. The 21 polygons in Theorem 4.10 (each polygon is named after (v, i, b)).

Lemma 4.11. *If P is a convex unit-lattice pentagon of Type (i) in Theorem 1.4, then for any $\mathbb{Z}$-affine transformation φ, $\varphi(P)$ is also a convex unit-lattice pentagon of Type (i) in Theorem 1.4. Additionally, if H is a convex unit-lattice hexagon of Type (i) in Theorem 1.3, then for any $\mathbb{Z}$-affine transformation φ, $\varphi(H)$ is also a convex unit-lattice hexagon of Type (i) in Theorem 1.3.*

Proof. A convex pentagon is Type (i) in Theorem 1.4 if and only if it has at least one pair of parallel sides. A convex hexagon is Type (i) in Theorem 1.3 if and only if it has at least one pair of parallel opposite sides of equal length. Since any non-singular affine transformation in $\mathbb{R}^2$ preserves both parallelism and the ratio of lengths of two parallel line segments, the statement is proved. ∎

Proof of Theorem 4.4. From Theorem 1.1, we can assume that $b \geq v \geq 5$. From Theorem 4.8, every convex unit-lattice polygon without interior unit-lattice points has three or four vertices. Hence we can also assume that $i \geq 1$. Then we can observe that there is no pair of integers (i, b) that satisfies Equation (4.1) with $a = 2$. From Equation (4.1) and $a = 3$ or 4, it follows that

$$(i, b) = \begin{cases} (1, 6), & (a = 3) \\ (1, 8), (2, 6). & (a = 4) \end{cases} \tag{4.2}$$

From Theorem 4.9, every convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (1, 6)$ is equivalent to either of the polygons (5,1,6) and (6,1,6) in Fig. 39, and there is no convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (1, 8)$. Moreover, from Theorem 4.10, every convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (2, 6)$ is equivalent to any one of the polygons (5,2,6)-a, (5,2,6)-b, (5,2,6)-c, (5,2,6)-d, (6,2,6)-a, and (6,2,6)-b in Fig. 40. The pentagons (5,1,6), (5,2,6)-a, (5,2,6)-c, and (5,2,6)-d are all Type (i) in Theorem 1.4, and the hexagons (6,1,6), (6,2,6)-a, and (6,2,6)-b are all Type (i) in Theorem 1.3. Hence from Lemma 4.11, any polygon excepting (5,2,6)-b appeared above is a tile. ∎

The pentagon (5,2,6)-b, which left the possibility in the above proof, is not any type of (i)–(xv) in Theorem 1.4. In fact, one can show that it is not a tile by examining local tilings with them thoroughly. However, we do not know whether any pentagon equivalent to (5,2,6)-b is not a tile. We also do not know whether the pentagon (5,2,6)-b or each pentagon equivalent to (5,2,6)-b is a 4-fold tile.

In a similar way, pairs of integers (i, b) are determined for $k = a = 6$ as follows:

$$(i, b) = (1, 12), (2, 10), (3, 8), (4, 6). \tag{4.3}$$

The assumption $v \geq 5$ and Theorems 4.9 and 4.10 eliminate the possibilities of $(i, b) = (1, 12), (2, 10)$. For $(i, b) = (3, 8), (4, 6)$, we can use Castryck's result [2]. For every $1 \leq g \leq 30$, he performed a computer calculation of all lattice polygons satisfying $i = g$ up to equivalence as a generalization of Theorems 4.9 and 4.10. The resulting data is available on his website. According to this data, up to equivalence, there are a total of 14 convex unit-lattice polygons satisfying both $v \geq 5$ and $(i, b) = (3, 8)$, and 21 convex unit-lattice polygons satisfying both $v \geq 5$ and $(i, b) = (4, 6)$. Moreover, we can use Lemma 4.11 for the twelve of the former and 14 of the latter. By examining the remaining nine polygons and polygons equivalent to them, it may be able to prove or disprove that there is a convex unit-lattice polygon of area 6 that is a nontrivial 6-fold tile.

5. Conclusion

In this paper, we studied nontrivial multiple tilings with polyominoes and convex unit-lattice polygons allowing translations, rotations, and reflections, and obtained four theorems. The first one (Theorem 3.1) claims that for any integer $k \geq 2$, there is a polyomino whose minimum tile-fold number is k. The second one (Theorem 3.9) claims that for any integer $k \geq 2$, the heptominoes C7, F7, and X7 listed in Fig. 3 are all minimum-sized nontrivial k-fold-tile polyominoes, and for any integer $k \geq 2$ except for $k = 3, 5$, the heptomino G7 listed in Fig. 3 is also minimum-sized nontrivial k-fold-tile polyomino. The third one (Theorem 4.2) claims that for any integer $k = 5$ or $k \geq 7$, there is a convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k. The last one (Theorem 4.4) claims that, for $k = 2$ and $k = 3$, there is no convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k, and for $k = 4$, such a convex unit-lattice polygon must be equivalent to a certain pentagon, if any.

As future work, we have some unsolved parts for minimum-sized nontrivial k-fold-tile polyominoes and nontrivial k-fold tiles of convex unit-lattice polygons whose area is k. For the former, although Theorem 3.9 presented such polyominoes for any integer $k \geq 2$, it is still open whether G7 is a 3- or 5-fold tile. For the latter, although Theorems 4.2 and 4.4 clarified whether there is such a polygon for any integer $k = 2, 3, 5$ or $k \geq 7$, it is still open for $k = 4$ and 6; the pentagon and the nine polygons (and shapes equivalent to them) leave the possibility, respectively. In particular, although one can show that the pentagon is not a tile, we do not know whether any pentagon equivalent to it is not a tile and whether each pentagon equivalent to it is a 4-fold tile. In addition, we can also consider the following problems.

Open Problem 5.1. For any integer $k \geq 2$, is there a polyomino whose minimum tile-fold number is k and whose size (the number of cells) is bounded by a polynomial function of k?

Open Problem 5.2. For any integer $k \geq 2$, is there a polygon P that satisfies $\mathrm{TFN}(P) = \{k\ell \mid \ell \in \mathbb{N}^+\}$?

Open Problem 5.3. For any $k = 2, 3$, or 4, is there a convex polygon that is a nontrivial k-fold tile?

Note that the size of the holed-k-I in the proof of Theorem 3.1 is $O(k \cdot 2^{4k})$, which is exponential. This polyomino may be a solution to Open Problem 5.2. Even if this is true, we should confirm that its tile-fold numbers are only multiples of k. Also note that a nontrivial 6-fold (lattice) tile of a convex polygon is already independently discovered by Marley [15, 16] (as mentioned in Section 1.2) and Zong [29].

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