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Discrete and Computational Geometry, Graphs, and Games

On the Number of k-Proper Connected Edge and Vertex Colorings of Graphs

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Abstract An edge (resp. vertex) coloring of a graph using a palette of w colors is called a k-proper connected (resp. vertex-k-proper connected) w-coloring if and only if there exist at least k vertex disjoint paths between all pairs of vertices having no two adjacent edges (resp. vertices) of the same color. In this work, we characterize the hardness of counting and approximately counting k-proper connected and vertex-k-proper connected colorings of graphs under color palette cardinality, vertex degree, bipartiteness, and planarity restrictions. In particular, we show that the problem of counting k-proper connected $(w = 2)$-colorings and vertex-k-proper connected $(w = 2)$-colorings of bipartite graphs is $\#P$-complete $\forall k \in \mathbb{N}_{>0}$, and that these results hold for subcubic bipartite planar graphs in the $k = 1$ case. With regard to approximate counting, among other findings we show that a Fully Polynomial-time Randomized Approximation Scheme (FPRAS) for counting k-proper connected $(w = 2)$-colorings and vertex-k-proper connected $(w = 2)$-colorings of bipartite graphs, for any $k \in \mathbb{N}_{>0}$, implies an FPRAS for counting strong orientations and spanning connected subgraphs, respectively, of arbitrary undirected simple graphs.

MSC: 05C15; 05C30; 05C31; 05C38; 05C40; 68Q17

Keywords: proper connected coloring; edge coloring; vertex coloring; Tutte polynomial; unilateral connectivity; strong connectivity; $\#P$; $\#P$-complete; counting hardness; FPRAS

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1. Introduction

Following Borozan et. al. [1], and letting G be a simple undirected graph having at least two vertices and an edge coloring $\mathcal{C}$, we call a simple path in G a *proper path* if and only if no two consecutive edges along the path have the same coloration. Here, if there exist k vertex disjoint proper paths between every pair of vertices in G, and if $\mathcal{C}$ is a decomposition of the edges of G into w distinct color classes, then we call G *k-proper connected* and refer to $\mathcal{C}$ as a *k-proper connected w-coloring*. In this context we can

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define $pc_k(G) = w$ as the *k-proper connection number* of G, where w corresponds to the minimum cardinality set of edge colors sufficient to allow for G to be k-proper connected.

We can observe that Borozan et. al.'s [1] concept of proper connectivity – independently conceptualized and developed by Andrews et. al. [2] – moderates Chartrand et. al.'s [3, 4] concept of *rainbow connectivity*, defined in terms of *rainbow paths* where all edges are required to have distinct colors. Furthermore, just as there is a natural variation on rainbow connectivity defined in terms of vertex colorings [5], there is likewise the concept of vertex-k-proper connectivity defined in terms of vertex proper paths where no two adjacent vertices along the path are permitted the same coloration [6].

To date, there have been a handful of complexity theoretic results pertaining to k-proper connected and vertex-k-proper connected colorings of graphs, though many such results remain unpublished. As one notable example, it is stated without further detail on "pg. 138" of the book *"Properly Colored Connectivity of Graphs"* by Li, Magnant, & Qin [7] that *"...Haggkvist (personal communication) showed that the problem of checking whether an edge coloring can make a graph G 2-proper connected is NP-complete..."*. However, it remains unclear to us how many distinct edge colorations Haggkvist considered in proving this result. Li et. al. [7] also reports personnel communications from Edmonds & Manoussakis and Ozeki that it is polynomial time tractable to decide if a given edge coloring makes a graph k-proper connected in the case where $k = 2$ and $k \geq 3$, respectively. The first published complexity theoretic results we are aware of were given in a circa 2017 paper by Ducoffe, Marinescu-Ghemeci, & Popa [8], wherein the authors proved that deciding if the $(k = 1)$-proper connection number for a graph G is ≤ 2 (i.e., if $pc_1(G) \leq 2$) was polynomial time tractable on bipartite and bounded treewidth graphs, and made the observation that the corresponding problem of deciding the vertex-$(k = 1)$-proper connection number was trivial as a consequence of every connected graph having a (necessarily bipartite) spanning tree. The authors additionally considered variants of proper edge and vertex colorings on digraphs due to Magnant et. al. [9], wherein one replaces proper paths with *directed proper paths*. They subsequently proved the NP-completeness of deciding if the directed $(k = 1)$-proper connection number is ≤ 2 and deciding if the directed vertex-$(k = 1)$-proper connection number is ≤ 2.

In this work, we examine for the first time the computational complexity of counting and approximately counting k-proper connected w-colorings and vertex-k-proper connected w-colorings of graphs under color palette cardinality, vertex degree, bipartiteness, and planarity constraints. In particular, we show that counting $(k = 1)$-proper connected $(w = 2)$-colorings of subcubic bipartite planar graphs is $\#P$-complete under Turing reductions via reduction from evaluating the Tutte polynomial at the point $T_G(0, 2)$ (Theorem 3.2 and Corollary 3.5). We then extend this result to show that counting k-proper connected $(w = 2)$-colorings of general bipartite graphs is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$ (Theorem 3.6 and Corollary 3.7). Next, we show that counting vertex-$(k = 1)$-proper connected $(w = 2)$-colorings of subcubic bipartite planar graphs is $\#P$-complete under Turing reductions via reduction from evaluating the Tutte polynomial at the point $T_G(1, 2)$ (Theorem 3.8), and extend this result to show that counting vertex-k-proper connected $(w = 2)$-colorings of general bipartite graphs is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$ (Theorem 3.9). With regard to approximate counting, we also prove a set of conditional results establishing, among other findings, that the existence of a Fully Polynomial-time Randomized Approximation Scheme (FPRAS) for counting k-proper connected $(w = 2)$-colorings and counting vertex-k-proper connected

($w = 2$)-colorings on bipartite graphs, for any $k \in \mathbb{N}_{>0}$, necessarily implies the existence of an FPRAS for counting the strong orientations (i.e., evaluating $T_G(0, 2)$ [10]) and spanning connected subgraphs (i.e., evaluating $T_G(1, 2)$ [11]), respectively, of an arbitrary undirected graph (Theorem 3.10).

2. Preliminaries

2.1. Graph Theoretic Terminology

Concerning basic graph theoretic terminology, we will generally follow Diestel [12], or where appropriate, Bondy & Murty [13]. However, for some brief clarifications, we define the degree of a vertex in a graph or digraph to be the number of edges or arcs that it is adjacent to (regardless of arc orientations), refer to a graph or digraph as being *cubic* (equiv. 3-regular or 3-valent) if and only if every vertex has degree 3, and refer to a graph or digraph as being *subcubic* if its maximum vertex degree is ≤ 3. To designate the minimum and maximum vertex degrees for a graph or digraph H, we write $\delta(H)$ and $\Delta(H)$, respectively. All graphs and digraphs in this work should be assumed to be simple.

2.2. Connectivity of Graphs and Digraphs

Let G be a simple undirected graph and D be a simple digraph. By Menger's theorem [14], we have that G is k-connected (resp. k-edge connected) if and only if there are at least k vertex disjoint paths (resp. at least k edge disjoint paths) between all pairs of vertices in the graph. If D is the product of orienting the edges of an undirected graph that is k-connected (resp. k-edge connected) we may call D weakly-k-connected (resp. weakly-k-edge connected).

We call D *strongly connected* if and only if, for all pairs of vertices v_a and v_b, there exists a directed path connecting v_a to v_b and a directed path connecting v_b to v_a. More generally, we say that D is *k-arc connected* if and only if at least k directed edges (equiv. arcs) need to be removed to destroy the strong connectivity of the digraph. Moderating the notion of strong connectivity, we call G *unilateral connected* if and only if, for all pairs of vertices v_a and v_b, there exists a directed path connecting v_a to v_b or a directed path connecting v_b to v_a. If D corresponds to an orientation of all edges in an undirected graph G, we call D a *strong orientation* (resp. *unilateral orientation*) if and only if D is strongly connected (resp. unilateral connected). We remark that, by a theorem of Robbins [15], a strong orientation of a graph G exists if and only if G is 2-edge connected. We also briefly note here that we will sometimes refer to a set of directed paths as *unilaterally connecting* a pair of vertices, even if it is at least possible for this set of directed paths to strongly connect the given pair of vertices.

In this work, we generalize the notion of a digraph being k-arc connected to vertex connectivity, calling a digraph *k-strong connected* if and only if at least k vertices need to be removed to destroy the strong connectivity of the digraph. We also introduce (to the best of our knowledge) the notion of a digraph being *k-arc-unilateral connected* (resp. *k-unilateral connected*) if and only if at least k arcs (resp. vertices) need to be removed to destroy the property of the digraph being unilaterally connected. Here, if D corresponds to an orientation of all edges in an undirected graph G, we call D a *k-strong orientation* (resp. *k-unilateral orientation*) if and only if D is k-strong connected (resp. k-unilateral connected).

2.3. Correspondence between Edge 2-Colored Bipartite Graphs and Bipartite Digraphs

Let G be a bipartite graph with partite sets V_X and V_Y, let $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathcal{A}$ be the set of all edge 2-colorings of G, and let $\Lambda_1, \Lambda_2, \ldots \in \mathcal{B}$ be the set of all possible orientations of the edges of G. Let $\mathcal{C}_i(G)$ refer to the graph resulting from applying the edge coloration $\mathcal{C}_i$ to G, and let $\Lambda_i(G)$ refer to the digraph generated from G by applying the orientation Λ_i. Here, we can observe a bijective correspondence – referred to as the "BB-correspondence" by Bang-Jensen & Gutin [16] – between sets $\mathcal{A}$ and $\mathcal{B}$. Specifically, for any given edge 2-coloration using the set of colors $\{0, 1\}$, we can always obtain a unique assignment of orientations to the edges of G by orienting an edge assigned coloration 0 (resp. coloration 1) towards a vertex in partite set V_X (resp. partite set V_Y) of G. In the other direction, for any given assignment of orientations to the edges of G, we can always obtain a unique edge 2-coloration by assigning the color 0 (resp. coloration 1) to an edge oriented towards a vertex in partite set V_X (resp. partite set V_Y) of G. We say that a given edge 2-coloration of G, $\mathcal{C}_i$, and an orientation of the edges of G, Λ_i, are *complementary* if and only if they correspond in exactly this manner. Observe that a proper path between any pair of vertices v_a and v_b in $\mathcal{C}_i(G)$ corresponds to a directed path between v_a and v_b in $\Lambda_i(G)$, and vice versa.

In Figure 1 we provide an illustration of the "BB-correspondence", where, for a bipartite graph G with partite sets V_X (white vertices) and V_Y (black vertices), an edge (Figure 1(a)) is oriented towards a vertex in partite set V_X if and only if it is assigned coloration 0 (solid line; Figure 1(b)), and an edge is oriented towards a vertex in partite set V_Y (i.e., a black vertex) if and only if it is assigned coloration 1 (dashed line; Figure 1(c)). To provide an explicit example, in Figure 1(d) we show an instance of G corresponding to H_3. Next, in Figure 1(e), for a randomly generated set of edge orientations Λ_r for G, we show $\Lambda_r(G)$. Finally, in Figure 1(f), for a 2-coloration $\mathcal{C}_r$ of the edges in G complementary to Λ_r, we show $\mathcal{C}_r(G)$.

2.4. The Complexity Class $\#P$

The complexity class $\#P$ [17, 18] consists of the set of all integer counting problems wherein one is tasked with computing the cardinality of a witness set certifying membership of a string x in a language $L \in NP$. While completeness for the class $\#P$ was originally defined in terms of Turing reductions [17, 18], completeness can also be defined with respect to weaker *many-one counting* ("weakly parsimonious") reductions [19, 20]. Here, to reduce an integer counting problem $f : \Sigma^* \to \mathbb{N}$ to another integer counting problem $h : \Sigma^* \to \mathbb{N}$ via a many-one counting reduction, one requires two polynomial time compatible functions $R_1 : \Sigma^* \to \Sigma^*$ and $R_2 : \mathbb{N} \to \mathbb{N}$ where $f(x) = R_2(h(R_1(x)))$. If R_2 is the identity function, then we refer to the many-one counting reduction as a *parsiomious* reduction.

2.5. Fully Polynomial-time Randomized Approximation Scheme (FPRAS)

Following Karp & Luby [21], we define a Randomized Approximation Scheme (RAS) as a procedure with some *error rate* parameter $0 < \epsilon < 1$ and *accuracy parameter* $0 < \delta < 1$, which, provided some input x for a counting problem f, outputs a value $\hat{f}_{(\epsilon,\delta)}(x)$ such that $Pr\left[\left(\frac{|\hat{f}_{(\epsilon,\delta)}(x)-f(x)|}{f(x)}\right) > \epsilon\right] < \delta$. Here, if the RAS has a running time polynomially bounded by $|x|$, ϵ^{-1}, and $\ln\left(\delta^{-1}\right)$, then we may refer to the RAS as a Fully Polynomial-time Randomized Approximation Scheme (FPRAS).

2.6. Approximation Preserving (AP) Reductions

Following Dyer et. al. [22], we define an *Approximation Preserving reduction* (AP-reduction) from a counting problem $f : \Sigma^* \to \mathbb{N}$ to $h : \Sigma^* \to \mathbb{N}$ (denoted $f \leq_{AP} h$), as a probabilistic oracle Turing machine $\mathcal{M}$ taking as input a string $x \in \Sigma^*$ and an error parameter $0 < \epsilon < 1$, and satisfying the following three conditions: (1) we have that all calls to $\mathcal{M}$ specify an input of the form $\{w, \delta\}$, where w is an instance of h and $0 < \delta < 1$ is an error bound satisfying the requirement that δ^{-1} be polynomially bounded by $|x|$ and ϵ^{-1}; (2) we have that $\mathcal{M}$ is a RAS for f if the oracle is a RAS for h; (3) the time complexity for $\mathcal{M}$ is polynomially bounded by $|x|$ and ϵ^{-1}. Here, if $f \leq_{AP} h$ and $h \leq_{AP} f$, we call f and h *AP-interreducible* and write $f \equiv_{AP} h$. If we have a reduction $f \leq_{AP} h$, and if we have that h has an FPRAS, then we necessarily have that f has an FPRAS.

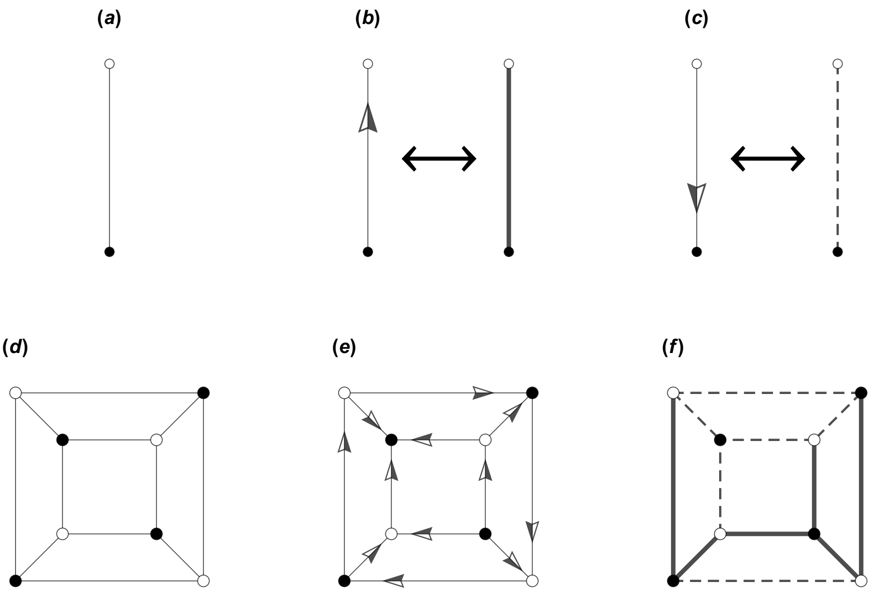
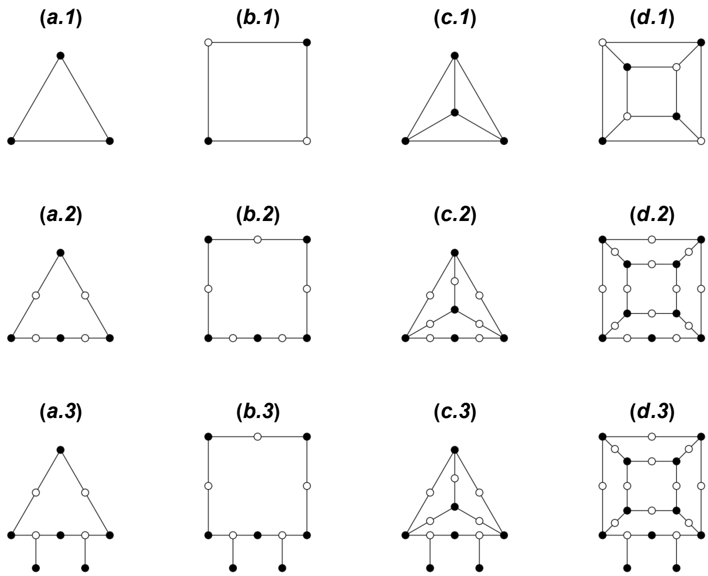


Figure 1. Illustration of the “BB-correspondence” [16] between edge 2-colorings of a bipartite graph G and an assignment of orientations to the edges of G; **(a)** an arbitrary edge of G where vertex coloration indicates partite set membership; **(b,c)** illustration of one of two possible manners of specifying a correspondence between edge 2-colorings (solid or dashed) and edge orientations (towards black or white vertices belonging to distinct partite sets); **(d)** an example where $G = H_3$; **(e)** $\Lambda_r(H_3)$, where Λ_r is a random assignment of orientations to the edges of H_3; **(f)** $\mathcal{C}_r(H_3)$, where $\mathcal{C}_r$ is an edge 2-coloring corresponding to Λ_r from (e).

3. Complexity of Counting Proper Connected (Edge, Vertex) Colorings of Graphs



(e)

Graph:	*#SC–Orientations = $T_G(0,2)$*	*#UC–Orientations*	*#EPCC(k=1, w=2)*
(a.1)	2	–	–
(b.1)	2	–	–
(c.1)	24	–	–
(d.1)	426	–	–
(a.2)	2	–	–
(b.2)	2	–	–
(c.2)	24	–	–
(d.2)	426	–	–
(a.3)	–	4	4
(b.3)	–	4	4
(c.3)	–	48	48
(d.3)	–	852	852

Figure 2. Illustrative examples of the reduction from *#SC-Orientations* to *#UC-Orientations* given in the Theorem 3.2 proof argument; **(a.1)** $K_3 = C_3$; **(b.1)** C_4; **(c.1)** K_4; **(d.1)** H_3; **(a.2)** – **(d.2)** bipartite graphs homeomorphic to the graphs shown in (a.1) – (d.1), where all edges are subdivided at least once and one edge is subdivided at least three times; **(a.3)** – **(d.3)** Theorem 3.2 bipartite reduction constructs of graphs (a.1) – (d.1); **(e)** table giving the number of strong orientations (*#SC-Orientations*) for the graphs (a.1) – (d.1) and (a.2) – (d.2), as well as the number of unilateral orientations (*#UC-Orientations*) and ($k = 1$)-proper connected ($w = 2$)-colorings ($\#EPCC_{(SBP)}(k = 1, w = 2)$) for the graphs shown in (a.3) – (d.3).

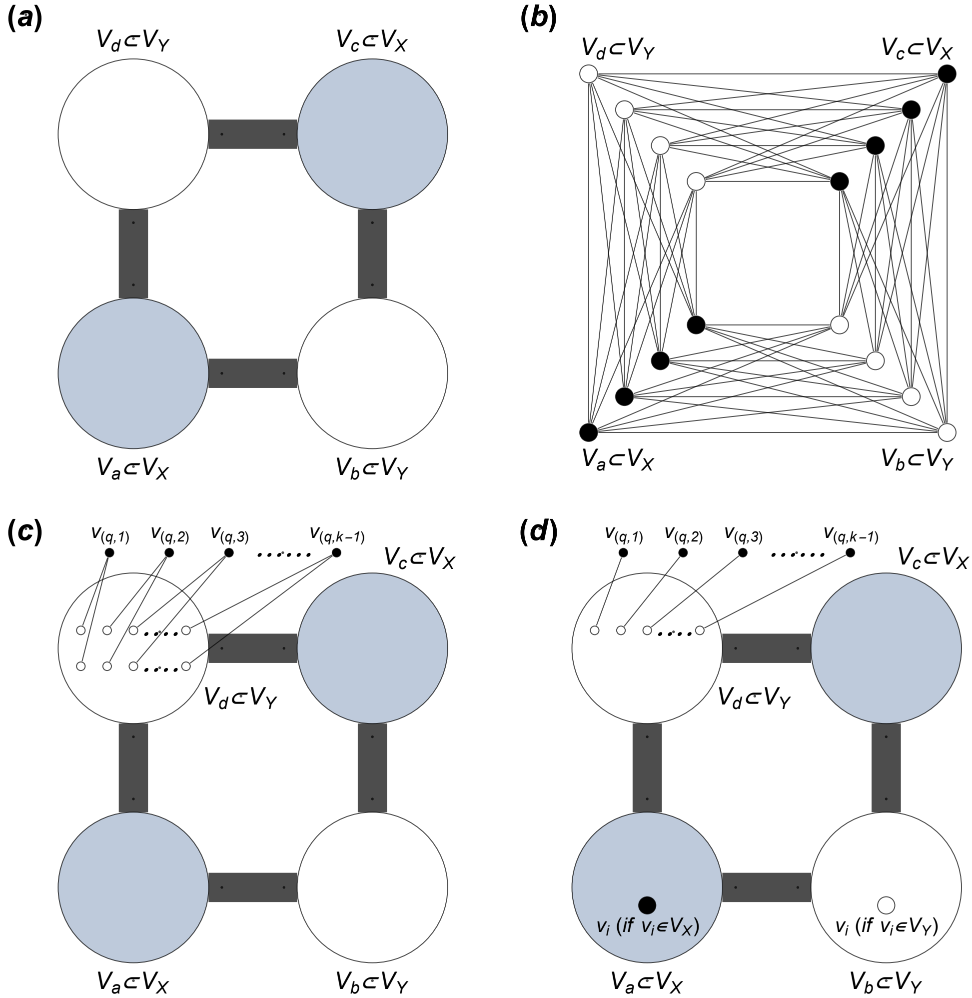


Figure 3. Illustrations of gadgets used for the Theorem 3.6 reduction from *#SC-Orientations* to *#k-UC-Orientations*$_{(B)}$; **(a)** abstract illustration of a bipartite ζ gadget, corresponding to the complete bipartite graph $K_{(4k,4k)}$, where $V_a \cup V_c$ and $V_b \cup V_d$ correspond to the two partite sets for the gadget, and (thick black) lines (e.g., between vertex sets V_a and V_b) indicate all pairs of vertices in the two sets are adjacent; **(b)** explicit example of (a), where $k = 2 \implies |V_a| = |V_b| = |V_c| = |V_d| = 4$; **(c)** gadget based on (a), having the indicated edges to gadget external vertices $v_{(q,1)}$ through $v_{(q,k-1)}$; **(d)** gadget based on (a) having the indicated edges to gadget external vertices $v_{(q,1)}$ through $v_{(q,k-1)}$, where a vertex $v_i \in V_H \setminus V_G$ is identified with a vertex in V_a or V_b if $v_i \in V_X$ or $v_i \in V_Y$, respectively.

Proposition 3.1. *The following connectivity properties can be determined in* $\mathcal{O}\left(|V|^3 \cdot |E|\right)$ *time: (1) k-arc connectivity; (2) k-arc-unilateral connectivity; (3) k-strong connectivity; (4) k-unilateral connectivity; (5) vertex-k-proper connectivity for a graph with a specified vertex coloring.*

Proof. For cases (1) through (4), let D be an arbitrary digraph. Recall that D is k-arc connected (resp. k-arc-unilateral connected) if and only if at least k directed edges must be removed to destroy the property of D being strongly connected (resp. unilaterally connected). Recall that D is k-strong connected (resp. k-unilateral connected) if and only if at least k vertices must be removed to destroy the property of D being strongly connected (resp. unilaterally connected). Now let $\mathcal{F}_D\left(v_s, v_t\right)$ correspond to the maximum flow between some pair of vertices v_s and v_t in D. In case (1), observe that D is k-arc connected if and only if, for all pairs of vertices, v_a and v_b, we have that $\mathcal{F}_D\left(v_a, v_b\right) \geq k$ and $\mathcal{F}_D\left(v_b, v_a\right) \geq k$. In case (2), we have that D is k-arc-unilateral connected if and only if, for all vertices v_a and v_b, we have that $\mathcal{F}_D\left(v_a, v_b\right) + \mathcal{F}_D\left(v_b, v_a\right) \geq k$. Finally, observe that we can reduce cases (3) and (4) to cases (1) and (2), respectively, by creating a graph D' where we split each vertex v_i in D into a pair of vertices $v_{(i,in)}$ and $v_{(i,out)}$, where all arcs oriented towards v_i (resp. away from v_i) are connected to and oriented towards $v_{(i,in)}$ (resp. connected to and oriented away from $v_{(i,out)}$), and an arc is added between $v_{(i,in)}$ and $v_{(i,out)}$ oriented towards $v_{(i,out)}$. Specifically, observe that the maximum flow between a pair of vertices $v_{(a,out)}$ and $v_{(b,in)}$ in D' corresponds to the number of vertex disjoint directed paths between v_a and v_b in D. Putting everything together, as the maximum flow between a pair of vertices v_s and v_t in an arbitrary digraph with vertex set V and edge set E can be computed in $\mathcal{O}\left(|V| \cdot |E|\right)$ time due to a circa 2013 result of Orlin [23], and as we need to compute the maximum flow at most $2 \cdot \binom{|V|}{2} = |V|^2 - |V|$ times in cases (1) through (4), we have a worst case time complexity of $\mathcal{O}\left(|V|^3 \cdot |E|\right)$ for computing each of the connectivity types in cases (1) through (4).

Finally, for case (5), let G be a graph having some vertex coloration $\mathcal{C}$. Recall that G is vertex-k-proper connected if and only if there exists k vertex disjoint vertex proper paths between all pairs of vertices. Create a digraph D by deleting all edges in G between vertices belonging to the same color class in $\mathcal{C}$, then replacing every remaining undirected edge with a pair of directed edges of opposite orientation (i.e., a pair of anti-parallel arcs). Now observe that D will be k-unilateral connected if and only if $\mathcal{C}$ makes G vertex-k-proper connected, and therefore, that there is an $\mathcal{O}\left(|E|\right)$ time reduction from case (5) to case (4). As we again have a worst case time complexity of $\mathcal{O}\left(|V|^3 \cdot |E|\right)$ in cases (1) through (4), this yields the proposition. ∎

Theorem 3.2. *The problem of counting unilateral orientations of subcubic bipartite planar graphs, #UC-Orientations*$_{(SBP)}$, *is #P-complete under Turing reductions.*

Proof. As unilateral connectivity can be determined in time polynomial in the size of an input digraph (see Proposition 3.1), it is straightforward to observe that the problem of counting unilateral orientations of subcubic bipartite planar graphs, which we denote *#UC-Orientations*$_{(SBP)}$, is in #P.

To show that *#UC-Orientations*$_{(SBP)}$ is #P-hard, we proceed via reduction from the problem of counting strong orientations of an arbitrary 2-edge connected subcubic planar graph of order at least 2, *#SC-Orientations*$_{(SP)}$. Here, the number of strong orientations of an arbitrary undirected graph can be computed by evaluating the Tutte polynomial at the point $T_G(0, 2)$ [10], which is $\#P$-complete under Turing reductions [24], and remains so even if the input graph is cubic and planar [25] or bipartite and planar [26].

To begin, provided a 2-edge connected subcubic planar graph G of order at least 2, we create a graph G' by subdividing all edges of G at least once, and subdividing one edge at least three times. We then create a graph Q from G' by adding a pendant vertex (i.e., a vertex of degree 1 after attachment) to two non-adjacent degree 2 vertices in G' arising from subdivision of the same edge in G. Observe that Q will be subcubic if G is subcubic, can always be made bipartite via a sufficient number of edge subdivision operations, and will be planar if and only if G is planar. In Figure 2 we provide examples of this surgery for G corresponding to $K_3 = C_3$, C_4, K_4, and H_3. In particular see Figure 2(a.1) through Figure 2(d.1) for the different instances of G, Figure 2(a.2) through Figure 2(d.2) for corresponding examples of G', and Figure 2(a.3) through Figure 2(d.3) for corresponding examples of Q.

To show that there necessarily exists a bijection between strong orientations of G and strong orientations of G', we observe the following lemma:

Lemma 3.3. *If G and H are undirected 2-edge connected homeomorphic graphs, then there is a bijective correspondence between strong orientations for G and strong orientations for H.*

Proof. Let G be an undirected 2-edge connected graph having vertex and edge sets V_G and E_G, respectively, and let H be a subdivision of G having vertex set V_H. Assign distinct labels to the vertices of G and maintain these labels during the surgery to create H. To slightly abuse notation, if a pair of vertices $v_i \in V_G$ and $v_i \in V_H$ have the same label, we will treat these vertices as identical and write $v_i \in V_H \cap V_G$.

Observe that, for every edge $v_a \leftrightarrow v_b \in E_G$, we will have an undirected path $p_{a,b} \in P$ of some length between v_a and v_b in H where v_a and v_b are the only vertices in $V_H \cap V_G$ covered by the path. Now observe that for any strong orientation of H, every path $p_{a,b} \in P$ must become a directed path from v_a to v_b or from v_b to v_a. Were this not the case, the orientation of some path $p_i \in P$ will have at least one sink or at least one source, and any strong orientation of a graph must be source- and sink-free. Accordingly, under the constraint that H has a strong orientation, we have a bijective correspondence between the orientation of an edge $v_a \leftrightarrow v_b \in E_G$ and the orientation of the corresponding path $p_{a,b}$ in H. Here, we say that an orientation of G *corresponds* to an orientation of H when we have that each edge in the orientation of G corresponds in such a manner to a directed path in the orientation of H.

We will now prove that this correspondence between orientations of edges in G and paths in H implies a bijection between strong orientations of G and H. Here, it suffices to show for an orientation $\Lambda_x(G)$ of G, and a corresponding orientation $\Lambda_y(H)$ of H, that $\Lambda_y(H)$ is strongly connected if and only if $\Lambda_x(G)$ is strongly connected. To establish for any pair of vertices $v_a, v_b \in V_H$ that there will be directed paths in $\Lambda_y(H)$ from v_a to v_b and v_b to v_a if $\Lambda_x(G)$ is a strong orientation, we need to consider the following three cases: (case 1) $v_a \in V_H \cap V_G$ and $v_b \in V_H \cap V_G$; (case 2) $v_a \in V_H \cap V_G$ and $v_b \in V_H \setminus V_G$; and (case 3) $v_a \in V_H \setminus V_G$ and $v_b \in V_H \setminus V_G$. For (case 1), observe that the aforementioned correspondence between directed paths in $\Lambda_y(H)$ and arcs in $\Lambda_x(G)$ will trivially guarantee at least one directed path from v_a to v_b and at least one directed path from v_b to v_a in $\Lambda_y(H)$ if $\Lambda_x(G)$ is a strong orientation. In (case 2), let $v_w \in V_H \cap V_G$ and $v_x \in V_H \cap V_G$ be the beginning and end vertices, respectively, of the directed path containing v_b and corresponding to a subdivided edge of G. Observe now that, as a consequence of the argument for (case 1), we have a directed cycle $v_w \to \ldots \to v_b \to \ldots \to v_x \to \ldots \to v_w$ in $\Lambda_y(H)$ (where any "..." section may have no vertices) if $\Lambda_x(G)$ is a strong orientation. Here, v_a will either correspond to v_w, correspond to v_x, or fall along any of the "..." sections, which trivially implies a directed path from v_a to v_b and a directed path from v_b to v_a. For (case 3), we can use the same argument as in (case 2), with the exception that v_a will necessarily fall along any of the "..." sections of the aforementioned directed cycle. Finally, observe that the argument for (case 1) establishes that $\Lambda_y(H)$ is strongly connected only if $\Lambda_x(G)$ is strongly connected.

Putting everything together, we have a bijective correspondence between strong orientations of G and H. This establishes the lemma. ∎

We will now establish that there exist exactly two unilateral orientations for Q per strong orientation of G'. To begin, let v_1 and v_2 be the unique pair of degree 1 vertices in Q, and let $\Lambda_{UC}(Q)$ be a unilateral orientation of Q. Observe that for v_2 to be reachable from v_1 or vice versa in $\Lambda_{UC}(Q)$, v_1 must be a unique source and v_2 must be a unique sink or vice versa. Let $v_a \leftrightarrow v_b$ be the edge in G that is subdivided at least three times in G', and where two non-adjacent subdivided vertices along this edge serve as the attachment points for v_1 and v_2 in creating the final reduction construct Q from G'. Additionally, let $v_a, v_b \in V_G$ retain their labels during the surgery to construct G' and Q. Here, we can observe that there must be a directed path in $\Lambda_{UC}(Q)$, corresponding to an orientation of the subdivided edge $v_a \leftrightarrow v_b$ in G, connecting either v_a to v_b or v_b to v_a. Letting this path be from v_a to v_b w.l.o.g., this implies that there must exist a directed path from v_b to v_a in order for there to exist at least one path from v_b to the unique sink corresponding to v_1 or v_2. Putting these observations together, we have that v_a and v_b, as well as all degree 2 vertices along the path from v_a to v_b corresponding to the edge $v_a \leftrightarrow v_b$ in G, must be part of the same maximal strongly connected component Γ_s in $\Lambda_{UC}(Q)$.

It remains to show that deleting v_1 and v_2 will make $\Lambda_{UC}(Q)$ a strong orientation. Here, we observe the following lemma:

Lemma 3.4. *If $\Gamma_1, \Gamma_2, \ldots$ correspond to the unique set of maximal strongly connected components of a graph, for any two such strongly connected components Γ_x and Γ_y, there cannot exist both a directed path beginning in Γ_x and ending in Γ_y and a directed path beginning in Γ_y and ending in Γ_x.*

Proof. Let Γ_x and Γ_y be two maximal strongly connected components of a graph. Imagine there exists a directed path $p_{a,c}$ from a vertex v_a in Γ_x to a vertex v_c in Γ_y, and a directed path $p_{d,b}$ from a vertex v_d in Γ_y to a vertex v_b in Γ_x (where we may have that $v_a = v_b$ and/or that $v_c = v_d$). Now let v_i and v_j be any pair of vertices in Γ_x and Γ_y, respectively. Observe that we can always extend or contract path $p_{a,c}$ to create a directed path from v_i to v_j, and extend or contract path $p_{d,b}$ to create a directed path from v_j to v_i. However, this is a contradiction since we earlier claimed that Γ_x and Γ_y were maximal strongly connected components, yielding the lemma. ∎

From our earlier arguments, we have that v_1 and v_2 – corresponding to a unique source and unique sink (or vice versa) for Q – are adjacent to the same maximal strongly connected component Γ_s in $\Lambda_{UC}(Q)$. Accordingly, letting v_i be an arbitrary vertex in Q belonging to some distinct maximal strongly connected component $\Gamma_i \neq \Gamma_s$, the unilateral connectivity of $\Lambda_{UC}(Q)$ implies that there must exist a directed path from v_i to any vertex in Γ_s as well as a directed path from any vertex in Γ_s to v_i. However, by Lemma 3.4 this is a contradiction. We therefore have that all vertices other that v_1 and v_2 in $\Lambda_{UC}(Q)$ must belong to the same maximal strongly connected component, meaning that $\Lambda_{UC}(Q)$ becomes a strong orientation for Q upon the deletion of vertices v_1 and v_2. Putting everything together, as there are exactly two manners in which to specify a source and sink for $\Lambda_{UC}(Q)$, we have that there are exactly two unilateral orientations of Q that uniquely correspond to a given strong orientation of G or G', and that $\#UC\text{-}Orientations_{(SBP)}$ is accordingly $\#P$-hard. Here, we refer the reader to Figure 2(e) for explicit examples of this correspondence for the four instances of the graph G.

Finally, as $\#UC\text{-}Orientations_{(SBP)} \in \#P$, we therefore have that the problem is $\#P$-complete under Turing reductions. ■

Corollary 3.5. *The problem of counting $(k = 1)$-proper connected $(w = 2)$-colorings of subcubic bipartite planar graphs, $\#EPCC_{(SBP)}(k = 1, w = 2)$, is $\#P$-complete under Turing reductions.*

Proof. This corollary follows from Theorem 3.2 and the “BB-correspondence” (see Preliminaries 2.3). ■

Theorem 3.6. *The problem of counting k-unilateral orientations of bipartite graphs, $\#k\text{-}UC\text{-}Orientations_{(B)}$, is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$.*

Proof. As k-unilateral connectivity can be determined in time polynomial in the size of an input digraph (see Proposition 3.1), it is straightforward to observe that the problem of counting k-unilateral orientations of bipartite graphs, which we denote $\#k\text{-}UC\text{-}Orientations_{(B)}$, is in $\#P$ for each $k \in \mathbb{N}_{>0}$.

Noting that Theorem 3.2 establishes the current theorem when $k = 1$, to show that $\#k\text{-}UC\text{-}Orientations_{(B)}$ is $\#P$-hard $\forall k \geq 2$, we will proceed in each case via reduction from the problem of counting strong orientations, $\#SC\text{-}Orientations$, of an arbitrary 2-edge connected graph of order at least k. Recall that counting strong orientations for an arbitrary graph corresponds to the evaluation of the Tutte polynomial at the point $T_G(0, 2)$ [10], which is $\#P$-complete under Turing reductions [24]. Observe furthermore that this $\#P$-completeness result holds in the case where we require a graph to be 2-edge connected, as an exactly 1-edge connected graph cannot admit a strong orientation.

To begin, let G be an undirected 2-edge connected graph of order at least k, and follow exactly the reduction given in the proof argument for Theorem 3.2 − with the exception that the initial graph is not required to be subcubic or planar − to transform G into a bipartite graph H where we have attached a pendant vertex (i.e., a vertex of degree 1 after attachment) to two non-adjacent degree 2 vertices arising from subdivision of the same edge in G. Following the Theorem 3.2 proof argument, we can observe that there are exactly two 1-unilateral orientations of H that uniquely correspond to a given strong orientation of G. Let V_G and V_H be the vertex sets for G and H, respectively, and recall that if a pair of vertices $v_i \in V_G$ and $v_i \in V_H$ have the same label, we will treat these vertices as being the same (e.g., we can write $v_i \in V_H \cap V_G$).

Following the construction shown in Figure 3(a), let a ζ gadget be an instance of the complete bipartite graph $K_{(4k,4k)}$, where $V_a \cup V_c$ and $V_b \cup V_d$ correspond to the two partite sets for the gadget, and $|V_a| = |V_b| = |V_c| = |V_d| = 2k$. See Figure 3(b) for an explicit example of this construction where $k = 2$ and $|V_a| = |V_b| = |V_c| = |V_d| = 4$. For every vertex $v_i \in V_H \cap V_G$, create $3k - 3$ instances of the ζ gadget, labeling these gadgets $\zeta_{(i,j,h)}$

for each pair (j,h) where $j \in [1,k-1]$ and $h \in [1,3]$. Correspondingly label the vertex sets V_a as $V_{(i,j,h,a)}$ for $\zeta_{(i,j,h)}$, V_b as $V_{(i,j+1,h,b)}$ for $\zeta_{(i,j+1,h)}$, and so forth. Additionally, for the two degree 1 vertices in $V_H \setminus V_G$, create instances of ζ gadgets ζ_β and ζ_γ, labeling the vertex set V_a as $V_{(\beta,a)}$ for ζ_β, V_a as $V_{(\gamma,a)}$ for ζ_γ, and so forth. Let $\mathcal{Z}$ be set of all the aforementioned gadgets constructed for the vertices $v_i \in V_H$, let V_X and V_Y be the two partite sets for a graph U corresponding to the union of H with the set of gadgets $\mathcal{Z}$, and to again slightly abuse notation, let V_H be the set of vertices in U disjoint from the gadgets in $\mathcal{Z}$.

Next, $\forall v_i \in V_H \cap V_G$, and for each $j \in [1,k-1]$, if $v_i \in V_X$ (resp. $v_i \in V_Y$) add edges between the following vertices: (step 1.1) v_i and an arbitrary vertex $v_{(s,i,j,1)} \in V_{(i,j,1,b)}$ (resp. $v_{(s,i,j,1)} \in V_{(i,j,1,a)}$) in $\zeta_{(i,j,1)}$; (step 1.2) v_i and an arbitrary vertex $v_{(s,i,j,3)} \in V_{(i,j,3,b)}$ (resp. $v_{(s,i,j,3)} \in V_{(i,j,3,a)}$) in $\zeta_{(i,j,3)}$; (step 1.3) $v_{(s,i,j,1)}$ and an arbitrary vertex $v_{(s,i,j,2)} \in V_{(i,j,2,a)}$ (resp. $v_{(s,i,j,2)} \in V_{(i,j,2,b)}$) in $\zeta_{(i,j,2)}$; and (step 1.4) $v_{(s,i,j,2)}$ and $v_{(s,i,j,3)}$. For the two vertices $v_i, v_k \in V_H \setminus V_G$, identify v_i with an arbitrary vertex in the set $V_{(\beta,a)}$ (resp. $V_{(\beta,b)}$) of the ζ_β gadget if $v_i \in V_X$ (resp. $v_i \in V_Y$), and identify v_k with an arbitrary vertex in the set $V_{(\gamma,a)}$ (resp. $V_{(\gamma,b)}$) of the ζ_γ gadget if $v_k \in V_X$ (resp. $v_k \in V_Y$). For any set $\{\zeta_{(i,j,1)}, \zeta_{(i,j,2)}, \zeta_{(i,j,3)}\}$, call the result of the immediately aforementioned (step 1.1) through (step 1.4) a $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadget if $v_i \in V_X \cap V_G$ (resp. $v_i \in V_Y \cap V_G$), and let $n_1 = |V_X \cap V_G| \cdot (k-1)$ (resp. $n_2 = |V_Y \cap V_G| \cdot (k-1)$) be the number of constructed copies of $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets. As a brief clarification, we remark that no vertex $v_i \in V_H \cap V_G$ is an element of the vertex set of any $\kappa_{(X,i,j)}$ or $\kappa_{(Y,i,j)}$ gadget.

To complete the construction, create a set of vertices $v_{(q,1)}, v_{(q,2)}, \ldots v_{(q,k-1)} \in V_Q$, where $|V_Q| = k-1$. For all ζ gadgets in the set $\mathcal{Z} \setminus \{\zeta_\beta, \zeta_\gamma\}$, as shown in Figure 3(c), attach each vertex $v_{(q,w)} \in V_Q$, $w \in [1,k-1]$, to two distinct vertices in each gadget vertex set $V_{(i,j,h,d)}$. For the remaining two gadgets $\zeta_\beta, \zeta_\gamma \in \mathcal{Z}$ – which we can assume to possess vertices identified with elements of the set $V_H \cap V_Y$ – as shown in Figure 3(d), attach each vertex $v_{(q,w)} \in V_Q$ to a single distinct vertex in each gadget vertex set $V_{(\beta,d)}$ and $V_{(\gamma,d)}$. Call the graph resulting from this reduction M, let V_M be its vertex set, and once again preserve vertex labels (e.g., in such a manner that we may refer to a vertex $v_i \in V_M \cap V_G$). Observe that M will be bipartite if and only if H is bipartite.

To proceed with our analysis, note that for M to be k-unilateral connected, if v_a and v_b are vertices in gadgets ζ_β and ζ_γ, respectively, there must exist $k-1$ vertex disjoint directed paths unilaterally connecting v_a and v_b traversing each of the vertices in $V_Q \subset V_M$, and there must exist an additional vertex disjoint directed path unilaterally connecting v_a and v_b that traverses the vertices $v_i, v_k \in V_H \setminus V_G$ while avoiding all of the vertices in V_Q. This implies that the edge connecting a vertex in $V_{(\beta,d)}$ to a vertex $v_{(q,w)} \in V_Q$ will be oriented towards $v_{(q,w)}$ if and only if the edge between $V_{(\gamma,d)}$ and $v_{(q,w)}$ is oriented away from $v_{(q,w)}$. Here, let Ψ_1 be the number of "legal" orientations of edges either with one end in ζ_β or ζ_γ and one end at a vertex $v_{(q,w)} \in V_Q$, or with both ends in ζ_β or ζ_γ, where we require that any such orientation – conditioned on the existence of a directed path between vertices $v_i, v_k \in V_H \setminus V_G$ disjoint from V_Q – allows for any pair of vertices in the union of the vertex sets for ζ_β or ζ_γ to be connected by k vertex disjoint directed paths.

To see that $\Psi_1 \geq 1$, first orient all edges in ζ_β connected to a vertex $v_{(q,w)} \in V_Q$ towards $v_{(q,w)}$, and orient all edges in ζ_γ connected to a vertex $v_{(q,w)} \in V_Q$ away from $v_{(q,w)}$. Now consider the orientation scheme $\Lambda_{cyc}(\zeta_r)$ for an arbitrary gadget $\zeta_r \in \mathcal{Z}$ where we orient all edges of the form $v_a \leftrightarrow v_b$ to point towards v_b if: $v_a \in V_{(r,a)}$ and $v_b \in V_{(r,b)}$; $v_a \in V_{(r,b)}$ and $v_b \in V_{(r,c)}$; $v_a \in V_{(r,c)}$ and $v_b \in V_{(r,d)}$; or $v_a \in V_{(r,d)}$ and $v_b \in V_{(r,a)}$. Observe that this will ensure the existence of $min\{|V_{(r,a)}|, |V_{(r,b)}|, |V_{(r,c)}|, |V_{(r,d)}|\} = 2k$ vertex disjoint directed paths unilaterally connecting any pair of vertices v_a and v_b internal to ζ_r, and furthermore ensure that the subgraph induced by the vertices internal to ζ_r is k-strong connected. Here, by adopting the orientation scheme $\Lambda_{cyc}(\zeta_\beta)$ and $\Lambda_{cyc}(\zeta_\gamma)$ for edges with both ends in ζ_β or both ends in ζ_γ, we can ensure the existence of k vertex disjoint directed paths unilaterally connecting any pair of vertices both internal to ζ_β or both internal to ζ_γ. Furthermore, under the assumption that the subgraph of M induced by V_H is unilateral connected, observe that this overall orientation scheme – again for edges with both endpoints in ζ_β or ζ_γ, or with one endpoint in ζ_β or ζ_γ and one endpoint at a vertex in V_Q – will ensure the existence of k vertex disjoint directed paths unilaterally connecting any vertex v_a internal to ζ_β to any vertex v_b internal to ζ_γ. In particular, $k-1$ of these paths will contain a single distinct vertex in the set V_Q, and the remaining path will include the vertices $v_i, v_k \in V_H \setminus V_G$ while avoiding all vertices in V_Q.

We next concern ourselves with the number of "legal" orientations of all edges having at least one endpoint in an instance of a $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadget, where we define a "legal" orientation as one which allows for M to be k-unilateral connected if the subgraph of M induced by V_H is unilateral connected. Here, we will establish: (property 1) that at least one such "legal" edge orientation exists for each $\kappa_{(X,i,j)}$ gadget and $\kappa_{(Y,i,j)}$ gadget if and only if the subgraph of M induced by V_H is unilateral connected; and (property 2) that the number of such "legal" edge orientations will necessarily be the same for each instance of a $\kappa_{(X,i,j)}$ gadget, as well as the same for each instance of a $\kappa_{(Y,i,j)}$ gadget.

Concerning (property 1), the "only if" direction is a straightforward consequence of the fact that $|V_Q| = k-1$, and accordingly, that there must exist at least one directed path connecting each pair of vertices in the subgraph of M induced by V_H. With regard to the "if" direction, consider an orientation scheme for an instance of a $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadget where: (step 2.1) we assume the orientation $\Lambda_{cyc}(\zeta_{(i,j,1)})$, $\Lambda_{cyc}(\zeta_{(i,j,2)})$, and $\Lambda_{cyc}(\zeta_{(i,j,3)})$ for edges with both endpoints interior to $\zeta_{(i,j,1)}$, both endpoints interior to $\zeta_{(i,j,2)}$, or both endpoints interior to $\zeta_{(i,j,3)}$; (step 2.2) for each ζ gadget in the set $\{\zeta_{(i,j,1)}, \zeta_{(i,j,2)}, \zeta_{(i,j,3)}\}$, we orient exactly one edge towards and one edge away from each vertex $v_{(q,w)} \in V_Q$; and (step 2.3) we orient the edges in the 4-cycle $\{v_i \leftrightarrow v_{(s,i,j,1)}, v_{(s,i,j,1)} \leftrightarrow v_{(s,i,j,2)}, v_{(s,i,j,2)} \leftrightarrow v_{(s,i,j,3)}, v_{(s,i,j,3)} \leftrightarrow v_i\}$ (constructed via (step 1.1) through (step 1.4)) to be a directed cycle (in either direction).

Now, recalling that every vertex $v_i \in (V_H \cap V_G) \cap V_X$ (resp. $v_i \in (V_H \cap V_G) \cap V_Y$) was made adjacent to $k-1$ instances of distinct $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets, and recalling that G was initially required to be of order at least k, it is straightforward to determine that these steps will ensure that any pair of vertices in the set V_M will be connected by at least k vertex disjoint directed paths. In particular, for every pair of vertices v_a and v_b internal to the same instance of a ζ gadget, it suffices to observe that (step 2.1) will ensure that the gadget is k-strong connected. For every pair of vertices v_a and v_b internal to distinct instances of a ζ gadget, (step 2.1) and (step 2.2) will ensure the existence of $k-1$ vertex disjoint directed paths unilaterally connecting v_a and v_b, one per each vertex in V_Q. In this context, there will then be an additional vertex disjoint directed path egressing and ingressing the gadgets via the directed cycles created in (step 2.3), the latter path avoiding vertices in V_Q and guaranteed to exist due to the assumption that the subgraph of M induced by V_H is unilateral connected. Finally, for any remaining pair of vertices in

M (including cases where $v_a \in V_Q$, $v_b \in V_Q$, or $v_a, v_b \in V_Q$), under the assumption that the subgraph of M induced by V_H is unilateral connected, it suffices to observe that (step 2.1) through (step 2.3) will ensure that every vertex in M has k vertex disjoint paths to vertices in k distinct instances of ζ gadgets, where we furthermore have that these paths include vertices in V_Q only as endpoints. These paths can then necessarily be truncated or extended as needed to yield k vertex disjoint directed paths unilateral connecting v_a and v_b. As this accounts for each possible pair of vertices v_a and v_b in M, we therefore have that (property 1) holds.

Concerning (property 2), note that every instance of a $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadget is equivalent, and thus, that the only possible factor influencing the orientations of the edges in these gadgets would be the particular vertex $v_i \in (V_H \cap V_G) \cap V_X$ (resp. $v_i \in (V_H \cap V_G) \cap V_Y$) in the 4-cycle created for the gadget via (step 1.1) through (step 1.4). Here, we can observe that any orientation scheme which allows for M to be k-unilateral connected will necessarily adopt the local edge orientation scheme described in (step 2.3). In particular, recall that there can exist only a single directed path from a vertex in $v_i \in V_G \cap V_H$ to a vertex internal to ζ_β to ζ_γ which avoids the vertices in V_Q, and observe that any such path from v_i to a vertex internal to ζ_β will initiate at v_i if and only if any such path from v_i to a vertex internal to ζ_γ does not initiate at v_i. Therefore, if the edges in $\{v_i \leftrightarrow v_{(s,i,j,1)}, v_{(s,i,j,1)} \leftrightarrow v_{(s,i,j,2)}, v_{(s,i,j,2)} \leftrightarrow v_{(s,i,j,3)}, v_{(s,i,j,3)} \leftrightarrow v_i\}$ are not oriented to create a directed 4-cycle, it will be impossible for at least one vertex in the host $\kappa_{(X,i,j)}$ or $\kappa_{(Y,i,j)}$ gadget to have k vertex disjoint directed paths to vertices in both ζ_β and ζ_γ. This yields that, for any k-unilateral orientation of M, each vertex $v_i \in (V_H \cap V_G) \cap V_X$ (resp. $v_i \in (V_H \cap V_G) \cap V_Y$) will be part of $k-1$ directed 4-cycles corresponding to distinct $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets, and thus, any directed path with an endpoint at another cycle vertex can be extended or contracted to be a directed with with an endpoint at v_i. Accordingly, a guarantee that there are k vertex disjoint paths connecting any pair of vertices internal to the same or distinct instances of $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets becomes a guarantee that every vertex $v_i \in (V_H \cap V_G) \cap V_X$ (resp. $v_i \in (V_H \cap V_G) \cap V_Y$) has $k-1$ vertex disjoint paths, with only endpoints in the subgraph of M induced by V_H, to any other vertex in M. Correspondingly, any orientation of the edges in one $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadget can be adopted by all $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets while still ensuring the k unilateral connectivity of M under the assumption that the subgraph of M induced by V_H is unilateral connected.

Having established (property 1) and (property 2), and letting $\Psi_{(2,X)}$ (resp. $\Psi_{(2,Y)}$) correspond to the aforementioned number of "legal" edge orientations for instances of $\kappa_{(X,i,j)}$ (resp. $\kappa_{(Y,i,j)}$) gadgets, it is now possible to see that for every strong orientation of G (resp. every unilateral orientation of H), there will be exactly $2 \cdot \Psi_1 \cdot \left(\Psi_{(2,X)}\right)^{n_1} \cdot \left(\Psi_{(2,Y)}\right)^{n_2}$ (resp. $\Psi_1 \cdot \left(\Psi_{(2,X)}\right)^{n_1} \cdot \left(\Psi_{(2,Y)}\right)^{n_2}$) k-unilateral orientations of the reduction construct M. Here, simply observe that the orientations counted by Ψ_1, $\left(\Psi_{(2,X)}\right)^{n_1}$, and $\left(\Psi_{(2,Y)}\right)^{n_2}$ cover all edges added to H to create the reduction construct M, and that by the proof argument for Theorem 3.2, there will be twice the number of unilateral orientations of H as strong orientations of G.

Putting everything together, as we can count the number of k-unilateral orientations of the reduction construct M per strong orientation of G via a procedure which is fixed-parameter tractable for parameter k, and as we have that this count will necessarily correspond to a fixed non-zero integer, in combination with Theorem 3.2, we have that counting k-unilateral orientations of bipartite graphs, $\#k\text{-}UC\text{-}Orientations_{(B)}$, is $\#P$-hard under Turing reductions $\forall k \in \mathbb{N}_{>0}$. Finally, since $\#k\text{-}UC\text{-}Orientations_{(B)}$ is in $\#P$ $\forall k \in \mathbb{N}_{>0}$, we therefore have that $\#k\text{-}UC\text{-}Orientations_{(B)}$ is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$. ∎

Corollary 3.7. *The problem of counting k-proper connected $(w = 2)$-colorings of bipartite graphs, $\#EPCC_{(B)}(k, w = 2)$, is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$.*

Proof. This corollary follows from Theorem 3.6, the definition of a k-unilateral connected graph (see Preliminaries 2.2), and the "BB-correspondence" (see Preliminaries 2.3). ∎

Theorem 3.8. *The problem of counting vertex-$(k = 1)$-proper connected $(w = 2)$-colorings of subcubic bipartite planar graphs, $\#VPCC_{(SBP)}(k = 1, w = 2)$, is $\#P$-complete under Turing reductions.*

Proof. As it is possible to determine if a given input graph having a vertex coloring using at most $(w = 2)$ colors is vertex-$(k = 1)$-proper connected in time polynomial in the graph size (see Proposition 3.1), it is straightforward to observe that the problem of counting vertex-$(k = 1)$-proper connected $(w = 2)$-colorings of subcubic bipartite planar graphs, which we denote $\#VPCC_{(SBP)}(k = 1, w = 2)$, is in $\#P$.

To show that $\#VPCC_{(SBP)}(k = 1, w = 2)$ is $\#P$-hard, we proceed via reduction from the problem of evaluating the Tutte polynomial at the point $T_G(1, 2)$. This latter problem is equivalent to counting the number of spanning connected subgraphs of a simple undirected graph G [11], $\#SC\text{-}Subgraphs$, and is $\#P$-complete under Turing reductions [24], even if the input graph is cubic and planar [25] or bipartite and planar [26].

To begin, let G be an arbitrary undirected cubic planar graph with vertex set V_G, where each vertex is assigned a unique label, and create a graph H by subdividing every edge of G exactly once. Let V_H be the vertex set for H, $V_S \subset V_H$ be the set of degree 2 vertices generated by the subdivision operations, and let $V_R \subset V_H$ be the set of degree 3 vertices retaining their unique labels from G. Note that H is subcubic, planar, and necessarily bipartite as a result of the number of vertices in every face being doubled as a result of the edge subdivision operations.

We will now show that every spanning connected subgraph of G is uniquely associated with two vertex-$(k = 1)$-proper connected colorings of H. Here, observe that a vertex-$(k = 1)$-proper connected coloring for H exists if and only if there is a vertex-$(k = 1)$-proper connected spanning tree for H. Observe that, for any such spanning tree, the degree 3 vertices $v_i \in V_R$ must be uniformly assigned the same color, which we will call c_1. Observe further that, for an arbitrary pair of vertices $v_a, v_b \in V_G$, the decision to assign a color c_2 or c_1 to the unique degree 2 vertex $v_i \in V_S$ adjacent to vertices with the same labels as $v_a, v_b \in V_G$, is equivalent to the decision to include or not include the edge between v_a and v_b, respectively, in a spanning connected subgraph for G. Lastly, observe that we may recolor all vertices in any vertex coloring of H such that $c_1 \to c_2$ and $c_2 \to c_1$ without consequence to our argument. Accordingly, we have that each spanning connected subgraph for G is associated with a unique pair of vertex-$(k = 1)$-proper connected colorings for H, which implies that $\#VPCC_{(SBP)}(k = 1, w = 2)$ is $\#P$-hard.

Putting everything together, since $\#VPCC_{(SBP)}(k = 1, w = 2) \in \#P$, we therefore have that the problem is $\#P$-complete under Turing reductions. ∎

Theorem 3.9. *The problem of counting vertex-k-proper connected* $(w = 2)$*-colorings of bipartite graphs,* $\#VPCC_B(k, w = 2)$*, is #P-complete under Turing reductions* $\forall k \in \mathbb{N}_{>0}$.

Proof. As it is possible to determine if a given input graph having a vertex coloring using at most $(w = 2)$ colors is vertex-k-proper connected in time polynomial in the graph size (see Proposition 3.1), it is straightforward to observe that the problem of counting vertex-k-proper connected $(w = 2)$-colorings of bipartite graphs, which we denote $\#VPCC_B(k, w = 2)$, is in $\#P$ for each $k \in \mathbb{N}_{>0}$.

Next, recall from Theorem 3.8 that $\#VPCC_{(SBP)}(k = 1, w = 2)$ is $\#P$-complete under Turing reductions. Here, to show that $\#VPCC_B(k, w = 2)$ is $\#P$-hard $\forall k \geq 2$, we proceed in each case by providing a parsimonious reduction from $\#VPCC_{(SBP)}(k = 1, w = 2)$. To begin, let G be a subcubic bipartite planar graph generated from a cubic planar graph via the reduction given in the Theorem 3.8 proof argument, where we assume each vertex partite set is of cardinality $\geq k$. Generate a graph G' by connecting a vertex v_1 to an arbitrary vertex in G, connecting a vertex v_2 to v_1 such that v_1 has degree 2 and v_2 has degree 1, and observe that G and G' will necessarily have the same number of spanning connected subgraphs. Now let V_K and V_L be the two vertex partite sets for G'. Generate a graph H by adding sets of vertices Q and M to G', where $|Q| = k - 1$ and $|M| = |V_L| \cdot (k - 1)$, connecting every vertex in V_K to every vertex in Q, connecting each vertex in V_L to a distinct set of $k - 1$ vertices in M such that no vertex in M is ever connected to more than one vertex in V_L, and finally connecting every vertex in M to every vertex in Q. Let V_H be the vertex set for H and let $V_X \subset V_H$ and $V_Y \subset V_H$ be the two vertex partite sets for H, where we have that $|V_X| = |V_K| + |V_L| \cdot (k - 1)$ and $|V_Y| = |V_L| + (k - 1)$.

As a consequence of vertices v_1 and v_2 being adjacent and having at most one vertex proper path to any other vertex in G', in order for H to be vertex-k-proper connected all vertices in Q must share the same coloration, all vertices in M must share the same coloration (e.g., where it would otherwise be impossible for a pair of vertices in M attached to a common vertex in G' to be connected by k vertex disjoint vertex proper paths), and the coloration of the vertices in sets Q and M must be distinct. Accordingly, we have that every vertex-$(k = 1)$-proper connected $(w = 2)$-coloring of G induces a single unique vertex 2-coloring of H. Furthermore, as there can be at most $k - 1$ vertex disjoint paths between a pair of vertices $v_a \in V_H \setminus (Q \cup M)$ and $v_b \in V_H \setminus (Q \cup M)$ containing a vertex $v_i \in Q$ or $v_j \in M$, a vertex 2-coloring of G must be at least vertex-$(k = 1)$-proper connected to induce a k-proper connected $(w = 2)$-coloring of H.

It now suffices to show that any vertex 2-coloring of H induced by a vertex-$(k = 1)$-proper connected $(w = 2)$-coloring of G ensures that H is vertex-k-proper connected. To do so, we need to show that there are k vertex disjoint vertex proper paths between v_a and v_b in the following cases: (case 1) $v_a \in V_H \setminus (Q \cup M)$ and $v_b \in V_H \setminus (Q \cup M)$; (case 2a) $v_a \in V_X \setminus (Q \cup M)$ and $v_b \in Q$; (case 2b) $v_a \in V_Y \setminus (Q \cup M)$ and $v_b \in Q$; (case 3a) $v_a \in V_X \setminus (Q \cup M)$ and $v_b \in M$; (case 3b) $v_a \in V_Y \setminus (Q \cup M)$ and $v_b \in M$; (case 4) $v_a \in Q$ and $v_b \in Q$; (case 5) $v_a \in Q$ and $v_b \in M$; and (case 6) $v_a \in M$ and $v_b \in M$;

Here, in (case 1), on account of G being vertex-$(k = 1)$-proper connected, we are guaranteed one vertex proper path between v_a and v_b where all vertices along the path are disjoint from $Q \cup M$. We also have $k - 1$ additional vertex disjoint vertex proper paths flowing through each of the distinct vertices in Q, which will be of length 2 if $v_a, v_b \in V_X$, length 3 if v_a and v_b belong to distinct partite sets, or length 4 if $v_a, v_b \in V_Y$.

In (case 2a), we have a single vertex proper path of length 1 between v_a and v_b, $k-2$ additional vertex disjoint vertex proper paths of length 3 between v_a and v_b traversing only vertices in the set $\{v_a\} \cup Q \cup M$, and a vertex proper path between v_a and some vertex $v_c \in V_X$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge to give the last required vertex disjoint vertex proper path between v_a and v_b. In (case 2b), we have $k-1$ vertex disjoint vertex proper paths of length 2 between v_a and v_b traversing only vertices in the set $\{v_a\} \cup Q \cup M$, and a vertex proper path between v_a and some vertex $v_c \in V_X$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge to give the last required vertex disjoint vertex proper path between v_a and v_b.

In (case 3a), we have $k-1$ vertex disjoint vertex proper paths of length 2 between v_a and v_b traversing only vertices in the set $\{v_a\} \cup Q \cup M$, and a vertex proper path between v_a and some vertex $v_c \in V_Y$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge to give the last required vertex disjoint vertex proper path between v_a and v_b. In (case 3b), we have $k-1$ vertex disjoint vertex proper paths between v_a and v_b traversing only vertices in the set $\{v_a\} \cup Q \cup M$ (where one path can be of length 1 and the rest of length 3, or all paths can be of length 3), and a vertex proper path between v_a and some vertex $v_c \in V_Y$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge to give the last required vertex disjoint vertex proper path between v_a and v_b.

In (case 4), as we originally assumed that the cardinality of the two partite sets for G had cardinality $\geq k$, we trivially have at least k vertex disjoint vertex proper paths of length 2 between v_a and v_b. In (case 5) we have a single vertex proper path of length 1 between v_a and v_b, $k-2$ additional vertex disjoint vertex proper paths of length 3 between v_a and v_b traversing only vertices in the set $Q \cup M$, and a vertex proper path between some pair of vertices $v_c \in V_X$ and $v_d \in V_Y$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge in either direction to give the last required vertex disjoint vertex proper path between v_a and v_b. Finally, in (case 6) we have $k-1$ vertex disjoint vertex proper paths of length 2 between v_a and v_b traversing only vertices in the set $Q \cup M$, and a vertex proper path between some pair of vertices $v_c, v_d \in V_Y$ consisting of a set of vertices disjoint from $Q \cup M$, which can be extended by a single edge in either direction to give the last required vertex disjoint vertex proper path between v_a and v_b.

As there exist at least k vertex disjoint vertex proper paths between v_a and v_b in each of the aforementioned cases, we therefore have that any vertex 2-coloring of H induced by the vertex-$(k=1)$-proper connected $(w=2)$-coloring of G ensures that H is vertex-k-proper connected. Putting everything together, since $\#VPCC_B(k, w=2) \in \#P$ for each $k \in \mathbb{N}_{>0}$, we therefore have that the problem is $\#P$-complete under Turing reductions $\forall k \in \mathbb{N}_{>0}$. ■

Theorem 3.10. *The following claims are true:*

• *(Claim 1) An FPRAS for counting $(k=1)$-proper connected $(w=2)$-colorings of graphs belonging to any bipartite superclass Ω of subcubic bipartite planar graphs, $\#EPCC_{(\Omega)}(k=1, w=2)$, implies an FPRAS for counting unilateral orientations of graphs in Ω, $\#UC$-$Orientations_{(\Omega)}$, and vice versa;*

• *(Claim 2) An FPRAS for counting k-proper connected* $(w = 2)$*-colorings of bipartite graphs,* $\#EPCC_{(B)}(k, w = 2)$*, implies an FPRAS for counting k-unilateral orientations of bipartite graphs,* $\#k\text{-}UC\text{-}Orientations_{(B)}$*, and vice versa;*
• *(Claim 3) An FPRAS for counting k-unilateral orientations of bipartite graphs,* $\#k\text{-}UC\text{-}Orientations_{(B)}$*, for any* $k \in \mathbb{N}_{>0}$*, implies an FPRAS for counting strong orientations of arbitrary graphs,* $\#SC\text{-}Orientations$*;*
• *(Claim 4) An FPRAS for counting unilateral orientations of subcubic bipartite graphs,* $\#UC\text{-}Orientations_{(SB)}$*, implies an FPRAS for counting strong orientations of subcubic graphs,* $\#SC\text{-}Orientations_{(S)}$*;*
• *(Claim 5) An FPRAS for counting unilateral orientations of bipartite planar graphs,* $\#UC\text{-}Orientations_{(BP)}$*, implies an FPRAS for counting strong orientations of planar graphs,* $\#SC\text{-}Orientations_{(P)}$*;*
• *(Claim 6) An FPRAS for counting unilateral orientations of subcubic bipartite planar graphs,* $\#UC\text{-}Orientations_{(SBP)}$*, implies an FPRAS for counting strong orientations of subcubic planar graphs,* $\#SC\text{-}Orientations_{(SP)}$*;*
• *(Claim 7) An FPRAS for counting vertex-k-proper connected* $(w = 2)$*-colorings of bipartite graphs,* $\#VPCC_B(k, w = 2)$*, for any* $k \in \mathbb{N}_{>0}$*, implies an FPRAS for counting spanning connected subgraphs of arbitrary graphs,* $\#SC\text{-}Subgraphs$*;*
• *(Claim 8) An FPRAS for counting vertex-*$(k = 1)$*-proper connected* $(w = 2)$*-colorings of subcubic bipartite graphs,* $\#VPCC_{SB}(k = 1, w = 2)$*, implies an FPRAS for counting spanning connected subgraphs of subcubic graphs,* $\#SC\text{-}Subgraphs_{(S)}$*;*
• *(Claim 9) An FPRAS for counting vertex-*$(k = 1)$*-proper connected* $(w = 2)$*-colorings of bipartite planar graphs,* $\#VPCC_{BP}(k = 1, w = 2)$*, implies an FPRAS for counting spanning connected subgraphs of planar graphs,* $\#SC\text{-}Subgraphs_{(P)}$*;*
• *(Claim 10) An FPRAS for counting vertex-*$(k = 1)$*-proper connected* $(w = 2)$*-colorings of subcubic bipartite planar graphs,* $\#VPCC_{SBP}(k = 1, w = 2)$*, implies an FPRAS for counting spanning connected subgraphs of subcubic planar graphs,* $\#SC\text{-}Subgraphs_{(SP)}$*.*

Proof. Concerning (Claim 1) and (Claim 2), these statements follow directly from the "BB-correspondence" (see Preliminaries 2.3). Concerning (Claim 3) in the case where $k = 1$, observe that the reduction given in the proof argument for Theorem 3.2 allows us to take an arbitrary connected graph G for which we would like to compute the number of strong orientations, $\#SC\text{-}Orientations$, and in polynomial time construct a bipartite graph G' having a number of unilateral orientations, $\#UC\text{-}Orientations$, equal to twice the number of strong orientations of G. Concerning (Claim 3) in the case where $k \geq 2$, observe that the reduction given in the proof argument for Theorem 3.6 similarly allows us to take an arbitrary connected graph G for which we would like to compute the number of strong orientations, and in polynomial time construct a bipartite graph G' where we have that the number of k-unilateral orientations of G' is equal to the number of strong orientations of G multiplied by a factor at most exponential in the cardinality of its vertex set, and which is also fixed-parameter tractable to compute as a function of k. Accordingly, we have that both the Theorem 3.2 and Theorem 3.6 reductions are AP-reductions (see Preliminaries 2.6 for an elaboration), and therefore that (Claim 3) is true in all cases where $k \geq 2$.

Concerning (Claim 4) through (Claim 6), observe that the aforementioned Theorem 3.2 reduction will work for arbitrary input graphs, preserve the property of an input graph being subcubic, preserve planarity, and generate a bipartite graph regardless of whether the input graph is bipartite. Finally, with regard to (Claim 7) through (Claim 10), we can establish the truth of these statements by simply following along the lines of the earlier arguments for (Claim 3) through (Claim 6) while exchanging Theorem 3.2 for Theorem 3.8 and Theorem 3.6 for Theorem 3.9, respectively. ■

4. Concluding Remarks and Open Problems

We remark that every $\#P$-completeness result in this work is proven via a many-one counting reduction from the problem of evaluating the Tutte polynomial at the point $T_G(0,2)$ (Theorem 3.2 & Theorem 3.6) or at the point $T_G(1,2)$ (Theorem 3.8 & Theorem 3.9). However, evaluating the Tutte polynomial at these points is only known to be $\#P$-complete under Turing reductions [24–26]. While we strongly suspect that our findings can be strengthened to show completeness under weaker many-one counting reductions, it is for this reason that either a different approach or a strengthening of the results in Jaeger et. al. [24], Vertigan [25], and Vertigan & Welsh [26] will be required.

Finally, concerning Theorem 3.10 of this work, recall that we established that the existence of an FPRAS for counting k-proper connected $(w = 2)$-colorings and counting vertex-k-proper connected $(w = 2)$-colorings of bipartite graphs, for any $k \in \mathbb{N}_{>0}$, implies the existence of an FPRAS for evaluating the Tutte polynomial at the points $T_G(0,2)$ (which again counts strong orientations) and $T_G(1,2)$ (which again counts spanning connected subgraphs), respectively. While the approximability of the Tutte polynomial at points $(x, y) \in \mathbb{N}_0^2$ remains largely uncharacterized [27], we conjecture that there exists an FPRAS for evaluating the polynomial at these two specific points.

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