Thai Journal of Mathematics
Volume 21 Number 4 (2023)
Pages 887–898

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Discrete and Computational Geometry, Graphs, and Games

On the Connectivity of Non-Commuting Graph of Finite Rings

Borworn Khuhirun, Khajee Jantarakhajorn* **and Wanida Maneerut**

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University
e-mail : borwornk@mathstat.sci.tu.ac.th (B. Khuhirun); Khajee@mathstat.sci.tu.ac.th (K. Jantarakhajorn); mneveryday@gmail.com (W. Maneerut)

Abstract The non-commuting graph of a non-commutative ring R, denoted by Γ_R, is a simple graph with vertex set of elements in R except for its center. Two distinct vertices x and y are adjacent if $xy \neq yx$. In this paper, we study the vertex-connectivity and edge-connectivity of a non-commuting graph associated with a finite non-commutative ring R and prove their lower bounds. We show that the edge-connectivity of Γ_R is equal to its minimum degree. The vertex-connectivity and edge-connectivity of Γ_R are determined when R is a non-commutative ring of order p^n where p is a prime number, and $n \in \{2, 3, 4, 5\}$.

MSC: 05C25; 05C40
Keywords: vertex-connectivity; edge-connectivity; minimum degree

Submission date: 29.01.2022 / Acceptance date: 07.02.2023

1. Introduction

Let R be a non-commutative ring and $Z(R)$ be the center of R. The *centralizer* of an element x in R is defined to be $C_R(x) = \{y \in R : xy = yx\}$ and a non-commutative ring R is called a *CC-ring* if every centralizer of non-central element in R is commutative. The *non-commuting graph* of R, denoted by Γ_R, is a graph whose vertex set is $R \backslash Z(R)$ and two distinct vertices x and y are adjacent if and only if $xy \neq yx$. This graph was introduced by Erfanian et al. [10]. The interplay between ring-theoretic properties and graph-theoretic properties has become a focus of research over the last decade. Many papers have assigned a group or a ring to a graph and investigated the properties of the associated graph,[1–4, 11, 12, 15, 16].

For a graph G, $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. The *degree* of vertex u in G, denoted by $\deg(u)$, is the number of edges incident with u. The *minimum degree* of G is the minimum degree among all vertices of G, denoted by $\delta(G)$. A $u-v$ *path* P in G is a sequence of distinct vertices, beginning with u and ending

*Corresponding author.

Published by The Mathematical Association of Thailand.

at v such that consecutive vertices in P are adjacent in G. The path P is denoted by $P: v_0, v_1, v_2, ..., v_k$ where $u = v_0$ and $v = v_k$. The number of edges encountered in P is the *length* of path P. A graph G is said to be *connected* if G contain a $u - v$ path for every pair u, v of distinct vertices of G. The *distance* between u and v is the smallest length of any $u - v$ path in G, denoted by $d(u, v)$. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and denoted by $\mathrm{diam}(G)$. A *complete graph* is a graph in which every two distinct vertices are adjacent. A graph G is a *$k-$partite graph* if $V(G)$ can be partitioned into k subsets $V(G) = V_1 \cup V_2 \cup V_3 \cup ... \cup V_k$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$, called partite sets, such that a adjacent to b if and only if a and b belong to different partite sets. A graph G is called a *complete $k-$partite graph* if G is $k-$partite and every two vertices in different partite sets are adjacent. The *vertex-connectivity* of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. The *edge-connectivity* of G, denoted by $\lambda(G)$, is the minimum number of edges whose removal from G results in a disconnected or a trivial graph.

Erfanian et al. [10] studied various graph theoretical properties of Γ_R such as completeness and planarity. They also determined the diameter, girth, domination number, chromatic number, and clique number of Γ_R.

The study of non-commuting graphs of rings was continued by Dutta and Basnet [8]. They proved that Γ_R is connected and determined the degree of vertices in Γ_R.

In this paper, we study the vertex-connectivity and edge-connectivity of the non-commuting graph associated with a finite non-commutative ring R. We prove a lower bound for $\kappa(\Gamma_R)$ and $\lambda(\Gamma_R)$. We show that the edge-connectivity of Γ_R is equal to $\delta(\Gamma_R)$, the minimum degree of Γ_R. In particular, we consider the relation between $\kappa(\Gamma_R)$, $\lambda(\Gamma_R)$ and $\delta(\Gamma_R)$. Finally, for a ring R of order p^n, we determine $\kappa(\Gamma_R)$ and $\lambda(\Gamma_R)$ where p is a prime number, and $n \in \{2, 3, 4, 5\}$.

2. Preliminaries

Throughout this paper, we let R be a finite non-commutative ring unless stated otherwise. We provide some useful results which will be used throughout this paper.

Theorem 2.1. [8, Proposition 2.1] *Let R be a finite ring. Then Γ_R is connected.*

Theorem 2.2. [10, Theorem 2.1] *Let R be a non-commutative ring. Then diam$(\Gamma_R) \leq 2$.*

Theorem 2.3. [13] *If G is a connected graph of diameter at most 2, then $\lambda(G) = \delta(G)$.*

Theorem 2.4. [5] *Let G be a graph of order n. If G is not a complete graph, then $\kappa(G) \geq 2\delta(G) + 2 - n$.*

Theorem 2.5. [10, Theorem 2.2] *Let R be a non-commutative ring. Then Γ_R is complete if and only if $|R| = 4$.*

Lemma 2.6. [9, p.512] *Let R be a finite ring with identity of order p^n, where p is a prime number. If $n < 3$, then R is commutative.*

Lemma 2.7. [16, Lemma 2.5] *Let p be a prime number and R be a non-commutative ring of order p^3 with identity. Then $|Z(R)| = p$.*

Lemma 2.8. [16, Lemma 2.2] *Let R be a finite non-commutative ring and $Z(R) \neq \{0\}$. Then $[R : Z(R)] = \frac{|R|}{|Z(R)|}$ is not prime.*

Theorem 2.9. [15, Theorem 2.1] *Let p be a prime number and R be a non-commutative ring of order p^4 with identity. Then $C_R(x)$ is a commutative ring for all $x \in R \backslash Z(R)$.*

Lemma 2.10. [14] *If R is a ring of prime order p, then R is commutative.*

Lemma 2.11. [6, p.567] *If G is a complete $k-$partite graph of order n whose largest partite set contains n_k vertices, then $\kappa(G) = \lambda(G) = \delta(G) = n - n_k$.*

Lemma 2.12. *Let R be a non-commutative ring. Then $|Z(R)| < |C_R(x)| < |R|$ for all $x \in R \backslash Z(R)$*

Proof. Let $x \in R \backslash Z(R)$. It obvious that $Z(R) \subseteq C_R(x) \subseteq R$. Since $x \notin Z(R)$, we have $C_R(x) \subsetneq R$. Also, $x \in C_R(x) \setminus Z(R)$. Hence $|Z(R)| < |C_R(x)| < |R|$. ∎

3. Main Results

3.1. Edge-Connectivity and Vertex-Connectivity

In this section, we study the edge-connectivity and the vertex-connectivity of the non-commuting graph for a finite non-commutative ring R. We prove that $\lambda(\Gamma_R) = \delta(\Gamma_R)$ and present a lower bound and an upper bound for the edge-connectivity of Γ_R. In particular, we develop an upper bound for $\lambda(\Gamma_R)$ when R is a non-commutative ring and R has a nilpotent element of degree n. Examples are also given to ensure that our bounds are sharp. Moreover, we give a lower bound for the vertex-connectivity of Γ_R . We begin this section with the following lemma:

Lemma 3.1. *Let R be a finite non-commutative ring. Then $\lambda(\Gamma_R) = \delta(\Gamma_R)$.*

Proof. Let R be a finite non-commutative ring. By Theorem 2.1 and Theorem 2.2, Γ_R is a connected graph of diameter at most 2 and so by Theorem 2.3, $\lambda(\Gamma_R) = \delta(\Gamma_R)$. ∎

Lemma 3.2. *Let R be a finite non-commutative ring. Then $\delta(\Gamma_R) \geq \frac{|R|}{2}$.*

Proof. Let $x \in V(\Gamma_R)$. Since R is a non-commutative ring and $C_R(x)$ is an additive subgroup of R, $|R| = m|C_R(x)|$ for some positive integer $m \geq 2$. Then $|C_R(x)| \leq \frac{|R|}{2}$ and so $|R| - |C_R(x)| \geq \frac{|R|}{2}$. Since $\deg(x) = |R| - |C_R(x)|$ for every $x \in V(\Gamma_R)$, we get $\delta(\Gamma_R) \geq \frac{|R|}{2}$. ∎

Lemma 3.3. *Let R be a finite non-commutative ring. Then $\delta(\Gamma_R) \leq |R| - 2$.*

Proof. For any $x \in R \backslash Z(R)$, it is clear that $0, x \in C_R(x)$. Thus $|C_R(x)| \geq 2$. Then $|R| - |C_R(x)| \leq |R| - 2$. Since $\deg(x) = |R| - |C_R(x)|$ for every $x \in V(\Gamma_R)$, we get $\delta(\Gamma_R) \leq |R| - 2$. ∎

As a consequence, we obtain a lower bound and an upper bound for both $\delta(\Gamma_R)$ and $\lambda(\Gamma_R)$.

Theorem 3.4. *Let R be a finite non-commutative ring. Then*

$$\frac{|R|}{2} \leq \delta(\Gamma_R) = \lambda(\Gamma_R) \leq |R| - 2.$$

The following example shows that the bounds given above are sharp.

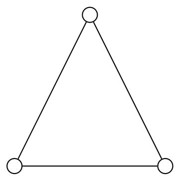
Example 3.5. Let $R = \{0, x, y, z\}$ be a non-commutative ring under the addition and multiplication given by Table 1. Then Γ_R is the graph as shown below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	y	z
y	0	0	0	0
z	0	x	y	z

TABLE 1. The addition and multiplication of $R = \{0, x, y, z\}$

Γ_R :



Thus, $\delta(\Gamma_R) = \lambda(\Gamma_R) = 2$, so the bounds in Theorem 3.4 are sharp.

As in the proof of Lemma 3.3, if R is a finite non-commutative ring with identity, we have $0, 1, x \in C_R(x)$ and so $|C_R(x)| \geq 3$. Then we obtain the following result:

Corollary 3.6. *Let R be a finite non-commutative ring with identity. Then $\lambda(\Gamma_R) \leq |R| - 3$.*

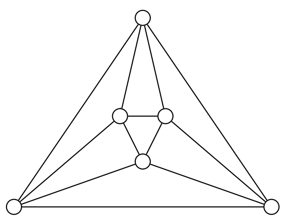
Furthermore, if $x \in R$ is a non-central nilpotent element of degree n, then $0, x, x^2, x^3, ...,$ $x^{n-1} \in C_R(x)$, so an upper bound for $\lambda(\Gamma_R)$ is obtained.

Corollary 3.7. *Let R be a non-commutative ring containing a non-central nilpotent element of degree n. Then $\lambda(\Gamma_R) \leq |R| - n$.*

The following examples show rings that satisfy Corollary 3.6 and Corollary 3.7, respectively.

Example 3.8. Let $R = T_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \ a,\ b,\ c \in \mathbb{Z}_2 \right\}$. Then R is a non-commutative ring with identity of order 8 and Γ_R is the graph below:

Γ_R :



Observe that $\lambda(\Gamma_R) = 4 < |R| - 3$.

Example 3.9. Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| \, a, b \in \mathbb{Z}_4 \right\}$. Then R is a non-commutative ring of order 16 and $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-central nilpotent element of degree $n = 3$. By letting

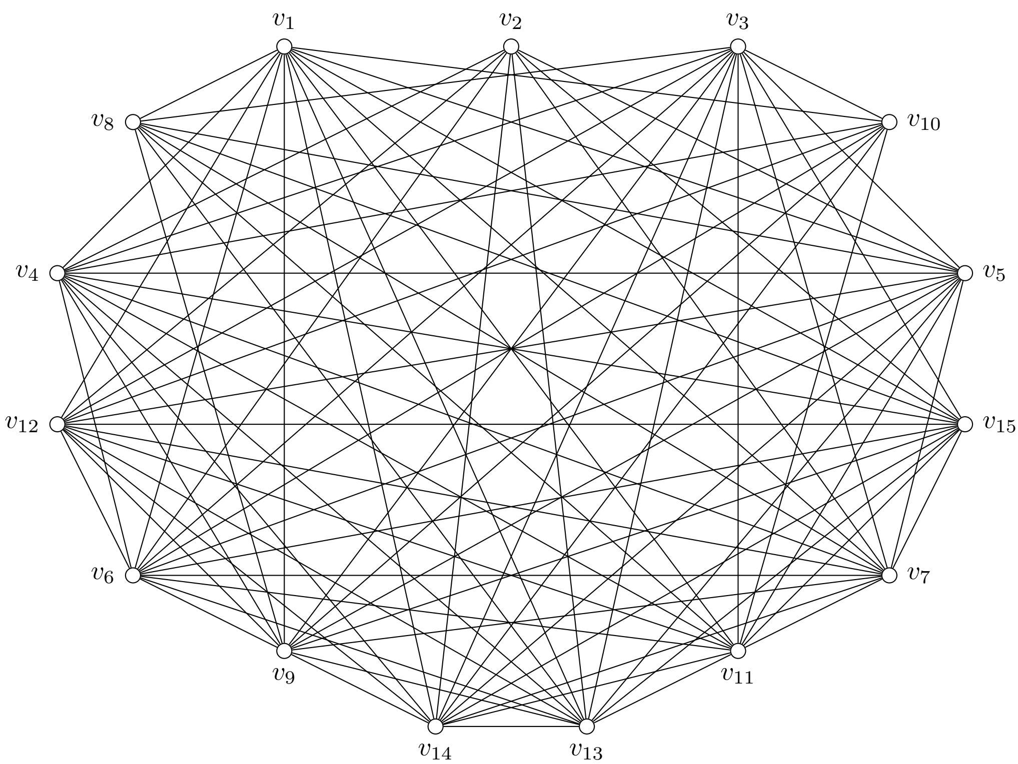
$$\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix},$$

$$v_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix},$$

$$v_8 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_9 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_{10} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_{11} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix},$$

$$v_{12} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}. \quad v_{13} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_{14} = \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_{15} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix},$$

Γ_R is the following graph:



Notice that $\lambda(\Gamma_R) = 8 < 16 - 3 = |R| - n$.

If R is a finite non-commutative ring, then a lower bound of both edge-connectivity and vertex-connectivity of Γ_R can be determined in the following propositions:

Proposition 3.10. *Let R be a finite non-commutative ring. Then $\lambda(\Gamma_R) \geq 2$.*

Proof. It follows from Theorem 3.4 and $|R| \geq 4$. ∎

Proposition 3.11. *Let R be a finite non-commutative ring. Then $\kappa(\Gamma_R) \geq 2$.*

Proof. Suppose, to the contrary, that there exists a finite non-commutative ring R such that $\kappa(\Gamma_R) = 1$. Then there is a vertex x such that $\Gamma_R - x$ is a disconnected graph or a trivial graph. Then we consider the next two cases:

Case 1: Assume that $\Gamma_R - x$ is a trivial graph. Then $\deg(x) = 1$. By Lemma 3.2 and $|R| \geq 4$, we get $\delta(\Gamma_R) \geq \dfrac{|R|}{2} \geq \dfrac{4}{2} = 2$, which contradicts $\deg(x) = 1$.

Case 2: Assume that $\Gamma_R - x$ is a disconnected graph. Then there is at least 2 components of $\Gamma_R - x$, say Γ_1 and Γ_2. Assume that y_1 is a vertex of Γ_1 such that y_1 adjacent with x and y_2 is a vertex of Γ_2 such that y_2 adjacent with x. Thus, $xy_1 \neq y_1x$, $xy_2 \neq y_2x$ and there is no $y_1 - y_2$ path in $\Gamma_R - x$. Then $y_1y_2 = y_2y_1$. Next, we consider $x + y_1$. Since $y_1(x + y_1) \neq (x + y_1)y_1$ and $y_2(x + y_1) \neq (x + y_1)y_2$, we have $x + y_1 \notin Z(R)$, so $x + y_1 \in V(\Gamma_R)$. Furthermore, $P : y_1,\ x + y_1,\ y_2$ is a $y_1 - y_2$ path in $\Gamma_R - x$, which is a contradiction. ∎

Moreover, if R is a non-commutative ring with $|R| > 4$, then the previous lower bound of vertex-connectivity of Γ_R can be improved as shown below:

Theorem 3.12. *Let R be a non-commutative ring with $|R| > 4$. Then $\kappa(\Gamma_R) \geq |Z(R)|+2$.*

Proof. Suppose that $|R| > 4$. By Theorem 2.5, we get Γ_R is not a complete graph. By Theorem 2.4, we have $\kappa(\Gamma_R) \geq 2\delta(\Gamma_R) + 2 - (|R| - |Z(R)|)$. Also, by Lemma 3.2, $\kappa(\Gamma_R) \geq 2(\frac{|R|}{2}) + 2 - (|R| - |Z(R)|)$. Therefore, $\kappa(\Gamma_R) \geq |Z(R)| + 2$. ∎

3.2. Edge-Connectivity and Vertex-Connectivity of a CC-Ring

In this section, we turn our attention to CC-rings and their properties, starting with the following lemmas.

Lemma 3.13. *Let R be a finite CC-ring. If $x, y \in R\backslash Z(R)$ and $xy \neq yx$, then $C_R(x) \cap C_R(y) = Z(R)$.*

Proof. Let $x, y \in R\backslash Z(R)$ be such that $xy \neq yx$. Since $Z(R)$ is a subring of $C_R(x) \cap C_R(y)$, we get $Z(R) \subseteq C_R(x) \cap C_R(y)$. Next, we will show that $C_R(x) \cap C_R(y) \subseteq Z(R)$. Suppose, to the contrary, that there exists $a \in (C_R(x) \cap C_R(y))\backslash Z(R)$. Then $xa = ax$ and $ya = ay$. Thus $x, y \in C_R(a)$. Since R is a CC-ring, $C_R(a)$ is commutative. Then $xy = yx$, a contradiction. Therefore, $C_R(x) \cap C_R(y) = Z(R)$. ∎

Lemma 3.14. *Let R be a finite CC-ring and $x, y \in R\backslash Z(R)$. Then $xy = yx$ if and only if $C_R(x) = C_R(y)$.*

Proof. Let R be a finite CC-ring. Suppose that $x, y \in R\backslash Z(R)$ with $xy = yx$. Then $y \in C_R(x)$. We will show that $C_R(x) \subseteq C_R(y)$. Let $a \in C_R(x)$. Since R is a CC-ring, $C_R(x)$ is commutative. It implies that $ya = ay$, and so $a \in C_R(y)$. Thus $C_R(x) \subseteq C_R(y)$. Similarly, $C_R(y) \subseteq C_R(x)$, so $C_R(x) = C_R(y)$. The converse is obvious. ∎

Next, we define a relation $\sim$ on $R\backslash Z(R)$. For any $x, y \in R\backslash Z(R)$, $x \sim y$ if and only if $xy = yx$. It turns out that $\sim$ is an equivalence relation.

Lemma 3.15. *Let R be a CC-ring. Then $\sim$ is an equivalence relation on $R\backslash Z(R)$.*

Proof. Let $x, y, z \in R\backslash Z(R)$. Since $x \in C_R(x)$, $x \sim x$. Then $\sim$ is reflexive. Suppose that $x \sim y$. Then $xy = yx$. Thus $y \sim x$. Hence $\sim$ is symmetric. Suppose that $x \sim y$ and $y \sim z$. Then $xy = yx$ and $yz = zy$. By Lemma 3.14, we get $C_R(x) = C_R(y)$ and $C_R(y) = C_R(z)$, that is, $C_R(x) = C_R(z)$. Thus $xz = zx$, and so $x \sim z$. Therefore, $\sim$ is transitive. As a result, $\sim$ is an equivalence relation. ■

This equivalence relation $\sim$ on $R\backslash Z(R)$ induces a partition of $R\backslash Z(R)$, where the equivalence classes are given by $[x] = \{y \in R\backslash Z(R) \mid x \sim y\}$. Notice that $[x] = C_R(x)\backslash Z(R)$. In particular, if R is a finite CC-ring, then we can partition $R\backslash Z(R)$ into $C_R(x_1)\backslash Z(R)$, $C_R(x_2)\backslash Z(R), ..., C_R(x_k)\backslash Z(R)$ for some $k \in \mathbb{N}$ and $x_1, x_2, x_3, ..., x_k \in R\backslash Z(R)$.

In 1932, Whitney [17] proved the classical inequalities $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for every graph G. Surprisingly, the vertex-connectivity and the edge-connectivity are equal in the case of non-commuting graph of finite CC-rings.

Theorem 3.16. *Let R be a finite CC-ring. Then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$.*

Proof. Let R be a finite CC-ring. Then $V(\Gamma_R) = R\backslash Z(R)$ can be partitioned into k equivalence classes with respect to $\sim$ for some $k \in \mathbb{N}$. Suppose that x and y belong to the same class. Then $x \sim y$, that is, $xy = yx$. Thus x and y are not adjacent. On the other hand, suppose that x and y belong to the different classes. Then $x \nsim y$, that is, $xy \neq yx$. Thus x and y are adjacent. It implies that every two vertices x and y, x adjacent to y if and only if x and y belong to different classes. Therefore, Γ_R is a complete $k-$partite graph. By Lemma 2.11 , we get $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. ■

By Theorem 3.16, we can determine edge-connectivity and vertex-connectivity of Γ_R where R is a ring of order p^n, p is a prime number and $n \in \{2, 3\}$.

Lemma 3.17. *Let p be a prime number and R be a finite non-commutative ring with identity.*

(1) If $|R| = p^2$, then $|Z(R)| = 1$ and $|C_R(x)| = p$ for all $x \in R \setminus Z(R)$.
(2) If $|R| = p^3$, then $|Z(R)| = p$ and $|C_R(x)| = p^2$ for all $x \in R \setminus Z(R)$.

Proof. Let p be a prime number and R be a finite non-commutative ring with identity.

Suppose that $|R| = p^2$. Let $x \in R\backslash Z(R)$. Because $C_R(x)$ is an additive subgroup of R, $|C_R(x)| \in \{1, p, p^2\}$. By Lemma 2.12, $|C_R(x)| < p^2$. Since $0, x \in C_R(x)$, $|C_R(x)| \geq 2$. Therefore, $|C_R(x)| = p$. Similarly, $Z(R)$ is an additive subgroup of $C_R(x)$, so $|Z(R)| \in \{1, p\}$. By Lemma 2.12, we have $|Z(R)| < p$, so $|Z(R)| = 1$.

Assume that $|R| = p^3$. By Lemma 2.7, we get $|Z(R)| = p$. Let $x \in R\backslash Z(R)$. Since $C_R(x)$ is an additive subgroup of R, $|C_R(x)| \in \{1, p, p^2, p^3\}$. By Lemma 2.12, we get $p < |C_R(x)| < p^3$, so $|C_R(x)| = p^2$. ■

Lemma 3.18. *Let p be a prime number and R be a non-commutative ring of order p^2. Then R is a CC-ring.*

Proof. Let x be a non-central element of R. By Lemma 3.17, we get $|C_R(x)| = p$. Also by Lemma 2.10, we get $C_R(x)$ is commutative. ■

Theorem 3.19. *Let p be a prime number and R be a non-commutative ring of order p^2. Then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^2 - p$.*

Proof. By Lemma 3.18, we get R is a CC-ring. Also, by Theorem 3.16, we have $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. Because $\deg(x) = |R| - |C_R(x)|$ for every $x \in V(\Gamma_R)$, we get $\delta(\Gamma_R) = |R| - \max\limits_{x \in R \setminus Z(R)} |C_R(x)|$. By Lemma 3.17, $|C_R(x)| = p$ for any $x \in R\setminus Z(R)$. Therefore, $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^2 - p$. ∎

Lemma 3.20. *Let p be a prime number and R be a non-commutative ring of order p^3 with identity. Then R is a CC-ring.*

Proof. Let R be a non-commutative ring of order p^3 with identity. Let $x \in R\setminus Z(R)$. By Lemma 3.17, we get $|C_R(x)| = p^2$. Observe that $1 \in C_R(x)$, so $C_R(x)$ is a ring with identity. By Lemma 2.6, $C_R(x)$ is commutative. Therefore, R is a CC-ring. ∎

Theorem 3.21. *Let p be a prime number and R be a finite non-commutative ring of order p^3 with identity. Then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^3 - p^2$.*

Proof. By Lemma 3.20, R is a CC-ring. Also by Theorem 3.16, we get $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. By Lemma 3.17, $|C_R(x)| = p^2$ for any $x \in R\setminus Z(R)$. Then $\delta(\Gamma_R) = |R| - \max\limits_{x \in R \setminus Z(R)} |C_R(x)| = p^3 - p^2$. Therefore, $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^3 - p^2$. ∎

Next, we consider a ring R of order p^n where p is a prime number and $n \in \{4, 5\}$. The edge-connectivity and the vertex-connectivity of Γ_R both depend on $|Z(R)|$. The next lemma indicates all possibilities of $|Z(R)|$.

Lemma 3.22. *Let p be a prime number and R be a finite non-commutative ring with identity.*

(1) If $|R| = p^4$, then $|Z(R)| \in \{p, p^2\}$.
(2) If $|R| = p^5$, then $|Z(R)| \in \{p, p^2, p^3\}$.

Proof. Let p be a prime number and R be a finite non-commutative ring with identity.

Suppose that $|R| = p^4$. Since $Z(R)$ is an additive subgroup of R, $|Z(R)| \in \left\{1, p, p^2, p^3, p^4\right\}$. Because $0, 1 \in Z(R)$, $|Z(R)| \geq 2$. Moreover, $|Z(R)| < p^4$ by Lemma 2.12. Thus, $|Z(R)| \in \left\{p, p^2, p^3\right\}$. By Lemma 2.8, $|Z(R)| \neq p^3$. Therefore, $|Z(R)| \in \left\{p, p^2\right\}$.

Assume that $|R| = p^5$. Since $Z(R)$ is an additive subgroup of R and $|R| = p^5$, $|Z(R)| \in \left\{1, p, p, p^2, p^3, p^4, p^5\right\}$. Note that $|Z(R)| < p^5$ and $|Z(R)| \neq p^4$ by Lemma 2.12 and Lemma 2.8, respectively. Thus, $|Z(R)| \in \left\{1, p, p^2, p^3\right\}$. Since $0, 1 \in Z(R)$, $|Z(R)| > 2$. Therefore, $|Z(R)| \in \left\{p, p^2, p^3\right\}$. ∎

If R is a ring of order p^4 where p is a prime number, then two possibilities for $|Z(R)|$ arise from Lemma 3.22. They yield different possibilities for $|C_R(x)|$ where x is a non-central element of R.

Lemma 3.23. *Let p be a prime number and R be a finite non-commutative ring of order p^4 with identity such that $|Z(R)| = p$. Then $|C_R(x)| \in \left\{p^2, p^3\right\}$ for any $x \in R\setminus Z(R)$.*

Proof. Let $x \in R \setminus Z(R)$. Since $C_R(x)$ is an additive subgroup of R, p^4 is a multiple of $|C_R(x)|$. Then $|C_R(x)| \in \left\{1, p, p^2, p^3, p^4\right\}$. By Lemma 2.12, we have $p < |C_R(x)| < p^4$, so $|C_R(x)| \in \left\{p^2, p^3\right\}$. ∎

Lemma 3.24. *Let p be a prime number and R be a finite non-commutative ring of order p^4 with identity such that $|Z(R)| = p^2$. Then $|C_R(x)| = p^3$ for any $x \in R\backslash Z(R)$.*

Proof. Let x be a non-central element of R. Since $C_R(x)$ is an additive subgroup of R, $|C_R(x)| \in \{1, p, p^2, p^3, p^4\}$. By Lemma 2.12, we get $p^2 < |C_R(x)| < p^4$, so $|C_R(x)| = p^3$. ∎

Theorem 3.25. *Let p be a prime number and R be a finite non-commutative ring of order p^4 with identity. Then the connectivity of Γ_R is one of the following case:*

(1) If $|Z(R)| = p$, then either

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^4 - p^2 \quad \text{or} \quad \kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^4 - p^3.$$

(2) If $|Z(R)| = p^2$, then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^4 - p^3$.

Proof. Let R be a finite non-commutative ring of order p^4 with identity. By Lemma 2.9, R is a CC-ring. Also by Theorem 3.16, we get $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. Since $\deg(x) = |R| - |C_R(x)|$ for all $x \in V(\Gamma_R)$, we have

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = \min_{x\in V(\Gamma_R)} (|R| - |C_R(x)|) = |R| - \max_{x\in V(\Gamma_R)} |C_R(x)|.$$

Next, we consider the following two cases:

Case 1: Suppose that $|Z(R)| = p$. By Lemma 3.23, we have $|C_R(x)| \in \{p^2, p^3\}$. If $|C_R(x)| = p^2$ for all $x \in R\backslash Z(R)$, then

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x\in V(\Gamma_R)} |C_R(x)| = p^4 - p^2.$$

On the other hand, if there exists $x \in R\backslash Z(R)$ such that $|C_R(x)| = p^3$, then

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x\in V(\Gamma_R)} |C_R(x)| = p^4 - p^3.$$

Case 2: Suppose that $|Z(R)| = p^2$. By Lemma 3.24, we have $|C_R(x)| = p^3$ for all $x \in R\backslash Z(R)$. Then

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x\in V(\Gamma_R)} |C_R(x)| = p^4 - p^3.$$ ∎

Finally, if R is a ring of order p^5 where p is a prime number, there are three possibilities for $|Z(R)|$ by Lemma 3.22. It will affect the edge-connectivity and the vertex-connectivity of Γ_R as well.

Lemma 3.26. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^2$. Then $|C_R(x)| \in \{p^3, p^4\}$ for any $x \in R\backslash Z(R)$.*

Proof. Let x be a non-central element of R. Since $C_R(x)$ is an additive subgroup of R, p^5 is a multiple of $|C_R(x)|$. Then $|C_R(x)| \in \{1, p, p^2, p^3, p^4, p^5\}$. By Lemma 2.12, we get $p^2 < |C_R(x)| < p^5$, so $|C_R(x)| \in \{p^3, p^4\}$. ∎

Lemma 3.27. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^2$. Then R is a CC-ring.*

Proof. Let R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^2$. Suppose that x is a non-central element of R. By Lemma, 3.26, we get $|C_R(x)| \in \{p^3, p^4\}$. Since $1 \in C_R(x)$, $C_R(x)$ is a ring with identity.

Case 1: Let $|C_R(x)| = p^3$. We will show that $C_R(x)$ is a commutative ring. Assume, to the contrary, that $C_R(x)$ is a non-commutative ring. By Lemma 2.7, $|Z(C_R(x))| = p$.

This is a contradiction since $Z(R) \subseteq Z(C_R(x))$. Consequently, $C_R(x)$ is a commutative ring.

Case 2: Let $|C_R(x)| = p^4$. Suppose that $C_R(x)$ is a non-commutative ring. By Lemma 3.22, we get $|Z(C_R(x))| \in \{p, p^2\}$. Since $Z(R) \subseteq Z(C_R(x))$, $|Z(C_R(x))| \geq |Z(R)| = p^2$. Then $|Z(C_R(x))| = p^2$, so $Z(R) = Z(C_R(x))$. Since $x \in Z(C_R(x))$ and $x \notin Z(R)$, we get $Z(R) \subsetneq Z(C_R(x))$, which is a contradiction. Then $C_R(x)$ is a commutative ring.

As a result, R is a CC-ring. ∎

Theorem 3.28. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^2$. Then*

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^5 - p^3 \quad \text{or} \quad \kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^5 - p^4.$$

Proof. Let R be a finite non-commutative ring with identity of order p^5 such that $|Z(R)| = p^2$. By Lemma 3.27, R is a CC-ring. Also, by Theorem 3.16, we get $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. Since $\deg(x) = |R| - |C_R(x)|$ for all $x \in V(\Gamma_R)$, we get

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = \min_{x \in V(\Gamma_R)} (|R| - |C_R(x)|) = |R| - \max_{x \in V(\Gamma_R)} |C_R(x)|.$$

Suppose that $x \in R \backslash Z(R)$. By Lemma 3.26, we have $|C_R(x)| \in \{p^3, p^4\}$. If $|C_R(x)| = p^3$ for all $x \in R \backslash Z(R)$, then

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x \in V(\Gamma_R)} |C_R(x)| = p^5 - p^3.$$

If there exists $x \in R \backslash Z(R)$ such that $|C_R(x)| = p^4$, then

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x \in V(\Gamma_R)} |C_R(x)| = p^5 - p^4.$$

∎

Lemma 3.29. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^3$. Then $|C_R(x)| = p^4$ for any $x \in R \backslash Z(R)$.*

Proof. Let x be a non-central element of R. Since $C_R(x)$ is an additive subgroup of R, we have $|C_R(x)| \in \{1, p, p^2, p^3, p^4, p^5\}$. By Lemma 2.12, we get $p^3 < |C_R(x)| < p^5$, so $|C_R(x)| = p^4$. ∎

Lemma 3.30. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^3$. Then R is a CC-ring.*

Proof. Let x be a non-central element of R. By Lemma 3.29, we get $|C_R(x)| = p^4$. We will show that $C_R(x)$ is commutative. Assume, to the contrary, that $C_R(x)$ is a non-commutative ring. Since $1 \in C_R(x)$, $C_R(x)$ is a non-commutative ring of order p^4 with identity. By Lemma 3.22, $|Z(C_R(x))| \in \{p, p^2\}$. Since $Z(R)$ is a subring of $Z(C_R(x))$ and $x \in Z(C_R(x)) \backslash Z(R)$, we get $Z(R) \subsetneq Z(C_R(x))$. Then $|Z(C_R(x))| > |Z(R)| = p^3$, which is a contradiction. Then $C_R(x)$ is a commutative ring. Consequently, R is a CC-ring. ∎

Corollary 3.31. *Let p be a prime number and R be a finite non-commutative ring of order p^5 with identity such that $|Z(R)| = p^3$. Then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = p^5 - p^4$.*

Proof. Let R be a finite non-commutative ring with identity of order p^5 such that $|Z(R)| = p^3$. By Lemma 3.30, R is a CC-ring. By Theorem 3.16, we have $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. Also by Lemma 3.29, $|C_R(x)| = p^4$ for all $x \in R \backslash Z(R)$. Therefore,

$$\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R) = |R| - \max_{x \in V(\Gamma_R)} |C_R(x)| = p^5 - p^4.$$

∎

If R is a ring of order p^5 and $|Z(R)| = p$ where p is a prime number, then R may not be a CC-ring as illustrated in the following example.

Example 3.32. Let $R = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix} \middle| \; a, b, c, d, e \in \mathbb{Z}_2 \right\}$. Then R is a non-commutative ring of order 32 and $Z(R) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$. Then it is easy to see that $C_R\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix} \middle| \; a, c, d, e \in \mathbb{Z}_2 \right\}$ which is non-commutative since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

4. Concluding Remark

In this paper, we studied the edge-connectivity and the vertex-connectivity of non-commuting graphs of a finite non-commutative ring R. We proved that $\lambda(\Gamma_R) = \delta(\Gamma_R)$ and obtained a lower bound and an upper bound for the edge-connectivity and the vertex-connectivity of Γ_R. In particular, we showed that if R is a finite CC-ring, then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$. Then the more general problem is to determine the following conjecture.

Conjecture: Let R be a finite non-commutative ring. Then $\kappa(\Gamma_R) = \lambda(\Gamma_R) = \delta(\Gamma_R)$.

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