Thai Journal of **Math**ematics
Volume 21 Number 4 (2023)
Pages 855–861

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Discrete and Computational Geometry, Graphs, and Games

Non-crossing Monotone Paths and Cycles through Specified Points of Labeled Point Sets

Toshinori Sakai[1] **and Jorge Urrutia**[2]

[1] *Department of Mathematical Sciences, School of Science, Tokai University, Japan*
e-mail : sakai@tokai-u.jp

[2] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, México*
e-mail : urrutia@matem.unam.mx

Abstract Let P be a set of n points in convex position in the plane, each of which is assigned a different number, called a label. A path whose vertices are points of P is called monotone if the labels of its vertices increase while traversing it starting from one of its end vertices. We show that for any element $q \in P$, there is a non-crossing monotone path containing q connecting at least $\lceil \sqrt{2(n-1)} \rceil$ elements. This bound is almost tight. A simple polygon with vertices in P is called monotone if it consists of a monotone path and the line segment connecting its endpoints. The polygon is monotonically increasing (respectively decreasing) if when we traverse it in the clockwise direction starting at one of its vertices, the labels of its vertices increase (respectively decrease). We call such a simple polygon an increasing (respectively a decreasing) cycle. We also prove that any set P of $(l-1)(m-1)+2$ labeled points in general position in the plane, and for any point $q \in P$ on the boundary of the convex hull of P, there exists an increasing cycle containing q with at least $l+1$ elements, or a decreasing cycle containing q with at least $m+1$ elements.

MSC: 52C10; 11B99
Keywords: labeled point sets; circular permutations; monotone paths; monotone cycles; monotone subsequences

Submission date: 14.02.2022 / Acceptance date: 29.11.2022

1. Introduction

Let P be a set of points in the plane. The elements of P are said to be *in general position* if no three of them are collinear; and *in convex position* if they are all on the boundary of the convex hull of P. P is called a *labeled point set* if a number, called a *label*, is assigned to each of its elements. The label of $p \in P$ will be denoted by $\ell(p)$. In this paper, we assume that the labels assigned to each element are *all different*. We

Published by The Mathematical Association of Thailand.

denote by $\mathcal{P}_n$ the family of labeled point sets P with n elements in general position, and by $\mathcal{C}_n$ the family of labeled point sets P with n elements in convex position.

For $P \in \mathcal{P}_n$, a non-crossing polygonal line connecting $k\,(\leq n)$ elements $p_1, \ldots, p_k$ of P in this order is called a *monotone path of* P if the sequence $\ell(p_1), \ell(p_2), \ldots, \ell(p_k)$ is monotonically increasing or decreasing (Figure 1(a)). A *simple* polygon whose boundary consists of a monotone path of P and a segment connecting its endpoints is called a *monotone cycle of* P (Figure 1(b)). In this paper, we assume that any simple polygon is oriented *clockwise*, i.e., when one travels along the cycle, the interior region bounded by the cycle is on the right-hand side. A monotone cycle is said to be monotonically increasing (resp. decreasing) if the sequence of the labels is monotonically increasing (resp. decreasing) when we read the labels in clockwise order from an appropriate element.

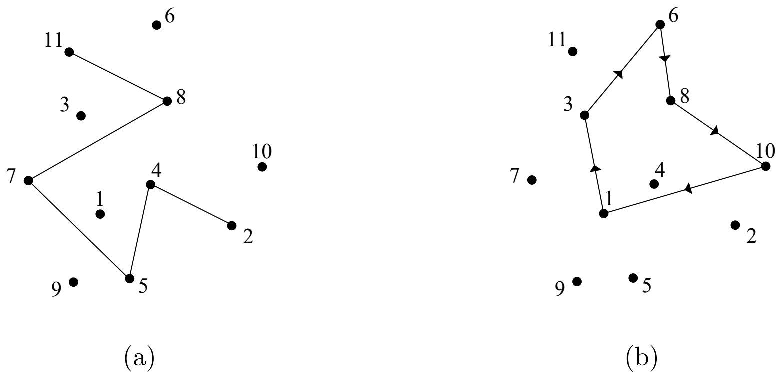

FIGURE 1. (a) A monotone path and (b) a monotone (monotonically increasing) cycle.

In 1935, Erdős and Szekeres [1] showed the following theorem:

Theorem 1.1. *Any sequence of* $(l-1)(m-1)+1$ *distinct real numbers contains either a monotonically increasing subsequence with at least* l *terms, or a monotonically decreasing subsequence with at least* m *terms (these bounds are tight).*

From Theorem 1.1, it follows that any sequence of n distinct real numbers contains a monotonically increasing or decreasing subsequence with at least $\lceil \sqrt{n}\, \rceil$ terms.

1.1. Monotone Paths

The problem of finding non-crossing monotone long paths in labeled point sets is a natural variation of Theorem 1.1, and this problem was first studied by Czyzowicz et al.[2]. They proved that any $P \in \mathcal{C}_n$ contains a vertex set of a monotone path with at least $\lceil \sqrt{2n}\, \rceil$ elements. This bound was improved by Sakai and Urrutia [3] to $\lceil \sqrt{3n - 3/4} - 1/2\, \rceil$ by giving a simple proof for a result by Chung [4] concerning the number of the terms of a longest *unimodal* (see the second paragraph of Section 2) or "anti"-unimodal subsequence in a sequence of numbers, and then applying it to labeled point sets. In this paper, we show the following result:

Theorem 1.2. *For any* $P \in \mathcal{C}_n$ *and for any element* $q \in P$*, there is a monotone path of* P *with at least* $\lceil \sqrt{2(n-1)}\, \rceil$ *vertices containing* q*.*

We also construct a point set of $\mathcal{C}_n$ that shows that this bound is almost tight.

1.2. Monotone Cycles

Each point set P of $\mathcal{C}_n$, or the cycle connecting the vertices of the convex hull of P, can be identified with a circular permutation. A circular permutation is said to be monotone (resp. monotonically increasing, monotonically decreasing) if it corresponds to a monotone (resp. monotonically increasing, monotonically decreasing) cycle. Using Theorem 1.1, we see that for any $P \in \mathcal{C}_n$, there is a monotone cycle with at least $\lceil \sqrt{n-1} \rceil + 1$ vertices, and this bound is tight [3]. This result implies that any circular permutation of n distinct real numbers contains a monotone circular subpermutation with at least $\lceil \sqrt{n-1} \rceil + 1$ terms. Czabarka and Wang [5] gave a result for circular permutations in the following style that is similar to Theorem 1.1, and they also characterize all circular permutations that show that these bounds are tight:

Theorem 1.3. *Let C be a circular permutation of $(l-1)(m-1)+2$ distinct real numbers. Then C contains either a monotonically increasing circular subpermutation with at least $l+1$ terms, or a monotonically decreasing circular subpermutation with at least $m+1$ terms.*

In this paper, we show the following Theorem 1.4 concerning monotone cycles for sets of points in *general* position, from which Corollary 1.5, an extension of Theorem 1.3 follows.

Theorem 1.4. *For any $P \in \mathcal{P}_{(l-1)(m-1)+2}$ and for any $q \in P$ on the boundary of the convex hull of P, there exists either a monotonically increasing cycle of P with at least $l+1$ vertices containing q, or a monotonically decreasing cycle of P with at least $m+1$ vertices containing q. These bounds are tight.*

Corollary 1.5. *Let C be a circular permutation of $(l-1)(m-1)+2$ distinct real numbers. Then for any term k of C, there exists either a monotonically increasing circular subpermutation with at least $l+1$ terms including k, or a monotonically decreasing circular subpermutation with at least $m+1$ terms including k. These bounds are tight.*

The tightness of Theorem 1.4 and Corollary 1.5 follows from the tightness of Theorem 1.3.

2. Proof of Theorem 1.2

Let $P \in \mathcal{C}_n$ and denote by $p_1, p_2, \ldots, p_n$ the elements of P in clockwise order along the boundary of the convex hull of P from an element. We say that P and the sequence of the labels $\ell(p_1)$, $\ell(p_2)$, $\ldots$, $\ell(p_k)$ are *corresponding*. Depending on the choice of the element for p_1, there are n sequences corresponding to P in general.

Let $S = \{a_k\}_{1 \le k \le n}$ be a sequence of n distinct real numbers. S is called a *unimodal* sequence if $a_1 < a_2 < \cdots < a_i$ and $a_i > a_{i+1} > \cdots > a_n$ for some i (we allow the possibility that $i = 1$ or $i = n$, i.e., $a_1 > a_2 > \cdots > a_n$ or $a_1 < a_2 < \cdots < a_n$). Sequences obtained by circular shifts of a unimodal sequence are called *bitonic* sequences: to be more precise, S is bitonic if

(i) $a_1 > a_2 > \cdots > a_i < a_{i+1} < \cdots < a_j > a_{j+1} > \cdots > a_n (> a_1)$; or

(ii) $a_1 < a_2 < \cdots < a_i > a_{i+1} > \cdots > a_j < a_{j+1} < \cdots < a_n (< a_1)$

for some i and j, where we allow the possibility that $i = 1$ or $j = n$ for each case. If $P \in \mathcal{C}_n$ is corresponding to some bitonic sequence, then P is also corresponding to a unimodal sequence.

Observation 2.1. Let $P \in \mathcal{C}_n$, $S = \{a_i\}_{1\le i\le n}$ a sequence corresponding to P, and $S' : a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ a bitonic subsequence of S. Then the path connecting the elements in increasing (or decreasing) order of their labels is a monotone path of P. Conversely, by reading the labels of the vertices of a monotone path of P along the boundary of the convex hull of P, we obtain a bitonic subsequence of S.

Thus to prove Theorem 1.2, it suffices to show:

Theorem 2.1. *Let $P \in \mathcal{C}_n$ and $q \in P$. Then a sequence corresponding to P contains a bitonic subsequence with at least $\lceil \sqrt{2(n-1)} \rceil$ terms including $\ell(q)$.*

Proof. We may assume that the labels assigned to the points are $1, 2, \ldots, n$. Starting from q, write the elements of P as $p_0(= q), p_1, \ldots, p_{n-1}$ in clockwise order along the boundary of the convex hull of P. Write $\ell(p_i) = a_i$ for $0 \le i \le n-1$, and define a sequence $S = \{b_i\}_{1\le i\le 2(n-1)}$ as follows:

$$b_{2i-1} = a_i \quad \text{and} \quad b_{2i} = \begin{cases} 1 - a_i & (\text{if } a_i < a_0), \\ 2n+1-a_i & (\text{if } a_i > a_0) \end{cases} \quad \text{for } 1 \le i \le n-1.$$

By Theorem 1.1, there is a monotone subsequence S' of S with L terms, where $L \ge \lceil \sqrt{2(n-1)} \rceil$. First consider the case where S' is increasing. In this case,

$$1 - a_{i_1} < \cdots < 1 - a_{i_{k-1}} < 0 < a_{i_k} < \cdots < a_{i_m} < n < 2n+1-a_{i_{m+1}} < \cdots < 2n+1-a_{i_L}$$

for some k and m (we allow the possibility that $k = 1$ or $m = L$). Thus

(i) $a_0 > a_{i_1}$ and $a_{i_1}, \ldots, a_{i_{k-1}}$ is decreasing if $k \ge 2$;

(ii) $a_{i_k}, \ldots, a_{i_m}$ is increasing;

(iii) $a_{i_{m+1}}, \ldots, a_{i_L}$ is decreasing and $a_{i_L} > a_0$ if $m \le L-1$.

Note that the last term a_{i_m} in (ii) and the first term $a_{i_{m+1}}$ in (iii) can be an identical term in the sequence $\{a_i\}_{0\le i\le n-1}$ (this occurs when consecutive terms $b_{2i_m-1}(= a_{i_m})$ and $b_{2i_m}(= 2n+1-a_{i_m})$ of S are contained in S'). Now by removing $a_{i_{m+1}}$ from $\{a_{i_j}\}_{1\le j\le L}$ and adding $a_0 = \ell(p_0)$ as the first term, we obtain a bitonic subsequence with $L \ge \lceil \sqrt{2(n-1)} \rceil$ terms.

We can argue similarly in the case where S' is decreasing. ∎

The bound stated in Theorem 2.1 (or Theorem 1.2) is almost tight. To show this, we construct a sequence S corresponding to $P \in \mathcal{C}_n$ such that any bitonic subsequence that contains the maximum term of S has at most $\sqrt{2n-1}+1$ terms for $n = 2k^2+2k+1$, where $k \ge 0$ is an integer. For a sequence $A : \ a_1, a_2, \ldots, a_{k(k+1)/2}$, we define its permutation $\mathcal{P}(A)$ as follows: first we relabel the terms of A as

$$\begin{array}{rrrrrr} \alpha_{11}, & \alpha_{21}, & \alpha_{31}, & \alpha_{41}, & \ldots, & \alpha_{k1}, \\ & \alpha_{22}, & \alpha_{32}, & \alpha_{42}, & \ldots, & \alpha_{k2}, \\ & & \alpha_{33}, & \alpha_{43}, & \ldots, & \alpha_{k3}, \\ & & & \ldots & \ldots & \ldots \\ & & & & & \alpha_{kk} \end{array}$$

in the same order as A. Denote by $\mathcal{P}(A)$ the sequence obtained by rearranging these terms as follows:

$$\alpha_{11},\ \alpha_{22},\ \alpha_{21},\ \alpha_{33},\ \alpha_{32},\ \alpha_{31},\ \ldots,\ \alpha_{kk},\ \ldots,\ \alpha_{k2},\ \alpha_{k1}.$$

For instance, $\mathcal{P}(A)$ for the sequence A of consecutive integers $1, 2, \ldots, 10$ is

$$\mathcal{P}(A):\ 1,\ 5,\ 2,\ 8,\ 6,\ 3,\ 10,\ 9,\ 7,\ 4 \quad \text{(Figure 2)}.$$

We denote by $\mathcal{P}(A)^{-1}$ the sequence obtained by arranging the terms of $\mathcal{P}(A)$ in reverse order.

Now let A_1 and A_2 be the sequences of $k(k+1)/2$ consecutive odd integers:

$$A_1:\ 1,\ 3,\ \ldots,\ k(k+1)-1,$$
$$A_2:\ k(k+1)+1,\ k(k+1)+3,\ \ldots,\ 2k(k+1)-1;$$

and let A_3 and A_4 be the sequences of $k(k+1)/2$ consecutive even integers:

$$A_3:\ 2,\ 4,\ \ldots,\ k(k+1),$$
$$A_4:\ k(k+1)+2,\ k(k+1)+4,\ \ldots,\ 2k(k+1).$$

Let S be the sequence obtained by arranging the terms of the sequence $\mathcal{P}(A_2)$, followed by the terms of $\mathcal{P}(A_1)$, the integer $n = 2k(k+1)+1$, and the terms of $\mathcal{P}(A_3)^{-1}$ and $\mathcal{P}(A_4)^{-1}$, see Figure 3. Then we can verify that any bitonic subsequence containing n has at most $2k+2 = \sqrt{2n-1}+1$ terms.

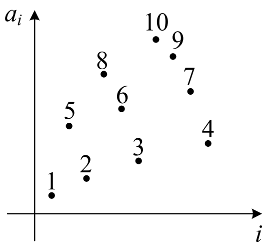

Figure 2. $\mathcal{P}(A)$ for $A:\ 1, 2, 3, \ldots, 10$.

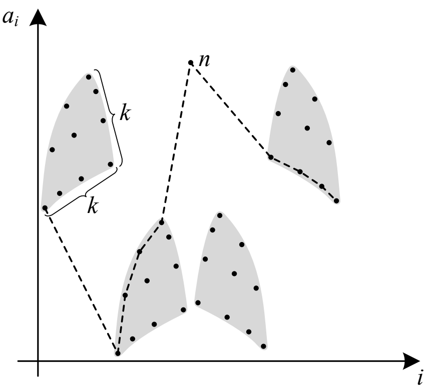

Figure 3. The broken line represents a bitonic subsequence with $2k+2$ terms.

3. Proof of Theorem 1.4

Let $n = (l-1)(m-1)+2$, $p_0 = q$, and denote by $p_1, p_2, \ldots, p_{n-1}$ the elements of $P - \{p_0\}$ in clockwise order around p_0, where p_{n-1}, p_0 and p_1 are consecutive vertices that appear in this order along the boundary of the convex hull of P, see Figure 4. We may assume again that the labels assigned to the points are $1, 2, \ldots, n$.

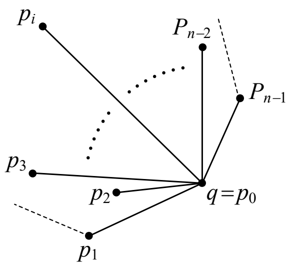

FIGURE 4. $p_0 = q$ and $p_1, p_2, \ldots, p_{n-1}$ in clockwise order.

Write $\ell(p_i) = a_i$ for $0 \leq i \leq n-1$, and define a sequence $S = \{b_i\}_{1\leq i\leq n-1}$ by

$$b_i = \begin{cases} a_i & (\text{if } a_i > a_0), \\ a_i + n & (\text{if } a_i < a_0) \end{cases} \quad \text{for } 1 \leq i \leq n-1 \tag{3.1}$$

(so, S is a permutation of $n-1$ integers $a_0+1, a_0+2, \ldots (a_0-1)+n$). Since S contains $n-1 = (l-1)(m-1)+1$ terms, it follows from Theorem 1.1 that S contains an increasing subsequence with at least l terms, or a decreasing subsequence with at least m terms. First consider the case where there is an increasing subsequence with $L \geq l$ terms. In this case, the subsequence $S' = \{b_{i_j}\}_{1\leq j\leq L}$ is in the following form:

$$a_{i_1},\ a_{i_2},\ \ldots,\ a_{i_{h-1}},\ a_{i_h}+n,\ a_{i_{h+1}}+n,\ \ldots,\ a_{i_L}+n$$

(we allow the possibility that $h = 1$ or $h = L+1$). In this case, we have

$$a_0 < a_{i_1} < a_{i_2} < \cdots < a_{i_{h-1}} \quad \text{and} \quad a_{i_h} < a_{i_{h+1}} < \cdots < a_{i_L} < a_0$$

by (3.1), and hence

$$a_{i_h} < a_{i_{h+1}} < \cdots < a_{i_L} < a_0 < a_{i_1} < a_{i_2} < \cdots < a_{i_{h-1}}.$$

Therefore the cycle connecting the points $p_{i_h}, p_{i_{h+1}}, \ldots, p_{i_L}, p_0, p_{i_1}, p_{i_2}, \ldots, p_{i_{h-1}}$ and p_{i_h} in this order is a monotonically increasing cycle of P with $L+1 \geq l+1$ vertices containing $p_0 = q$.

Similarly, in the case where S contains a decreasing subsequence with at least m terms, we obtain a monotonically decreasing cycle of P with at least $m+1$ vertices containing q. ∎

Acknowledgements

The authors would like to thank the anonymous reviewers for their careful reading of our manuscript and their comments and suggestions.

References

[1] P. Erdős, G. Szekeres, A combinatorial problem in geometry, Compos. Math. 2 (1935) 463–470.

[2] J. Czyzowicz, E. Kranakis, D. Krizanc, J. Urrutia, Maximal length common non-intersecting paths, Proc. 8th Can. Conf. Comput. Geom., August 1996, Ottawa, 180–189.

[3] T. Sakai, J. Urrutia, Monotonic polygons and paths of weighted point sets, Comput. Geom. Graphs Appl. (LNCS 7033, 2011), 164–175.

[4] F.R.K. Chung: On unimodal subsequences, J. Comb. Theory A 29 (1980) 267–279.

[5] É. Czabarka, Z. Wang, Erdős-Szekeres theorem for cyclic permutations, Involve 12 (2019) 351–360.