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The Distribution of a Forward Stochastic Disease-Model

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Abstract : In this paper, we derive the distribution of a disease-model which is not possible to have backward transitons. The distribution is the sums of gamma distributions. In special cases, the results reduce to some AIDS medels and uniform forward model.

Keywords : forward stochastic disease-model, homogeneous Markov process, first passage probability distribution, random time, generator matrix.
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1 Introduction and main result

This work, we consider the distribution of the disease-model as shown in Figure 1. In this model, I_1 corresponds to the exposure state and a patient dies at death-state (I_{n+1}) where $n \geq 2$. We assume that it is not possible to have backward transition from I_i to I_j for $j < i$ and call this model *forward disease-model*.

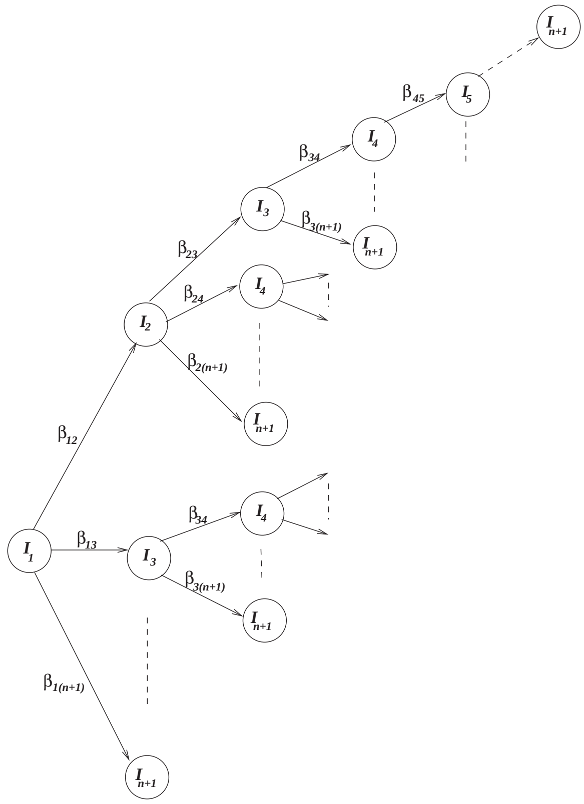

Figure 1

For every $t \geq 0$, we let X_t be a random variable whose value is the state at time t. So the state space of $\{X_t \mid t \geq 0\}$ is $\{I_1, I_2, \ldots, I_{n+1}\}$ where I_{n+1} is the absorbing state and $I_1, I_2, \ldots, I_n$ are transition states. In this work, we assume that $\{X_t \mid t \geq 0\}$ is a homogeneous continuous Markov process. From Chapter 5 of Sidney (2002), we know that the transition matrix $P(t) = \left[p_{ij}(t)\right]_{(n+1)\times(n+1)}$ of $\{X_t \mid t \geq 0\}$ is satisfied the followings:

$$\left.\begin{aligned} &p_{ij}(t) = v_{ij}t + o(t) && \text{where } t \to 0 \text{ and } i \neq j, \\ &p_{ii}(t) = 1 - v_{ii}t + o(t) && \text{where } t \to 0, \\ &\quad v_{ij} \geq 0, \quad v_{ii} = \sum_{j \neq i} v_{ij}, && \end{aligned}\right\} \tag{1.1}$$

where v_{ij} is the transition rate at which X_t jumps from i to j. From Figure 1, we know that

$$v_{ij} = \begin{cases} 0, & \text{if } i > j, \\ \gamma_i, & \text{if } i = j, \\ \beta_{ij}, & \text{if } i < j, \end{cases} \tag{1.2}$$

where $\gamma_i = \sum_{l=i+1}^{n+1} \beta_{il}$ for $i = 1, 2, \ldots, n$ and $\gamma_{n+1} = 0$. In this paper, we obtain the distribution of the random time which I_1 is absorbed into I_{n+1}, that is, the random time that a patient will die since he has been infective. Here is our main result.

Theorem 1.1. *Let W be the random time that a patient will die since he has been infective. Assume that γ_i's are distinct. Then*

(i) the probability density function f_1 of W can be written as

$$f_1(t) = \sum_{k=1}^{n} \left(\sum_{l=k}^{n} p_{1k} p'_{kl} \beta_{l(n+1)} \right) e^{-\gamma_k t},$$

and

(ii) the average time from infective state I_1 to the death-state I_{n+1} is

$$\sum_{k=1}^{n} \sum_{l=k}^{n} \frac{p_{1k} p'_{kl} \beta_{l(n+1)}}{\gamma_k^2}$$

where

$$p_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} \frac{\beta_{j_q j_{q+1}}}{\gamma_j - \gamma_{j_q}}, & \textit{if } i < j, \\ 1, & \textit{if } i = j, \\ 0, & \textit{if } i > j, \end{cases}$$

and

$$p'_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} p_{j_q j_{q+1}}, & \textit{if } i < j, \\ 1, & \textit{if } i = j, \\ 0, & \textit{if } i > j. \end{cases}$$

Notice that if $\beta_{i(i+2)} = \beta_{i(i+3)} = \cdots = \beta_{i(n+1)} = 0$ for $i = 1, 2, \ldots n-1$, this model reduces to the AIDS (Acquired Immunodeficiency Syndrome) model considered by Longini et al. (1989a, 1989b, 1991 and 1992). In case of uniform forward model, i.e., the transition rate β_{ij}'s are equal, it is easy to see that γ_i's are distinct. So Theorem 1.1 can be applied to this case. Hence it is reasonable to assume that γ_i's are distinct.

2 Proof of the main result

For each i, let W_i be the random time that I_i is absorbed into I_{n+1}. Then W_i is referred to as the first passage time I_i and $f_i(t)$, the probability density function, the first passage probability density of I_i. Let

$$f(t) = \left[f_1(t), f_2(t), \ldots, f_n(t)\right]^T,$$

where X^T denotes the transpose of a matrix X. Let

$$A = \begin{bmatrix} \gamma_1 & -\beta_{12} & -\beta_{13} & \cdots & -\beta_{1n} \\ 0 & \gamma_2 & -\beta_{23} & \cdots & -\beta_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & -\beta_{(n-1)n} \\ 0 & 0 & \cdots & 0 & \gamma_n \end{bmatrix}$$

We observe that the generator matrix of (X_t) is

$$\begin{bmatrix} -A & \mu \\ 0_n & 0 \end{bmatrix}$$

where $\mu = \left[\beta_{1(n+1)}, \beta_{2(n+1)}, \ldots, \beta_{n(n+1)}\right]^T$ and $0_n = (\underbrace{0, 0, \ldots, 0}_{n})$. Since $\det A = \gamma_1\gamma_2\cdots\gamma_n > 0$, by Tan and Byers (1993), we obtain that

$$f(t) = \exp(-At)A1_n \tag{2.1}$$

where $1_n = (\underbrace{1, 1, \ldots, 1}_{n})^T$ and

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j.$$

From Curtis (1984) Chapter 7, there are the Jordan canonical form J of A which is of the form

$$J = \begin{bmatrix} \gamma_1 & 0 & 0 & \dots & 0 \\ 0 & \gamma_2 & 0 & \dots & 0 \\ 0 & 0 & \gamma_3 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & \gamma_n \end{bmatrix}$$

and the invertible matrix P such that

$$A = PJP^{-1}.$$

From Appendices 1–2, we know that $P = \left(p_{ij}\right)$ and $P^{-1} = \left(p'_{ij}\right)$ where p_{ij} and p'_{ij} are defined as follows:

$$p_{ij} = \begin{cases} \sum_{k=1}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} \dfrac{\beta_{j_q j_{q+1}}}{\gamma_j - \gamma_{j_q}}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \end{cases}$$

and

$$p'_{ij} = \begin{cases} \sum_{k=1}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} p_{j_q j_{q+1}}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}$$

Hence

$$\begin{aligned} e^{-At} &= e^{-\left(PJP^{-1}\right)t} \\ &= Pe^{-Jt}P^{-1} \\ &= P \begin{bmatrix} e^{-\gamma_1 t} & 0 & \cdots & 0 \\ 0 & e^{-\gamma_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{-\gamma_n t} \end{bmatrix} P^{-1} \\ &= \left(\alpha_{ij}\right) \end{aligned} \tag{2.2}$$

where $\alpha_{ij} = \sum_{k=1}^{n} p_{ik}\, p'_{kj}\, e^{-\gamma_k t}$. From (2.1), (2.2) and the fact that

$$A1_n = \left[\gamma_1 - \sum_{l=1}^{n}\beta_{1l}, \gamma_2 - \sum_{l=1}^{n}\beta_{2l}, \ldots, \gamma_n - \sum_{l=1}^{n}\beta_{nl},\right]^T = \left[\beta_{1(n+1)}, \beta_{2(n+1)}, \ldots, \beta_{n(n+1)}\right]^T,$$

we have

$$\begin{aligned}
f_1(t) &= (1, 0, \ldots, 0)e^{-At}A1_n \\
&= \left[\sum_{k=1}^{n} p_{1k}\, p'_{k1}\, e^{-\gamma_k t},\ \sum_{k=1}^{n} p_{1k}\, p'_{k2}\, e^{-\gamma_k t},\ \ldots,\ \sum_{k=1}^{n} p_{1k}\, p'_{kn}\, e^{-\gamma_k t}\right] \\
&\quad \times \left[\beta_{1(n+1)}, \beta_{2(n+1)}, \ldots, \beta_{n(n+1)}\right]^T \\
&= \sum_{l=1}^{n}\left(\sum_{k=1}^{n} p_{1k}\, p'_{kl}\, e^{-\gamma_k t}\right)\beta_{l(n+1)} \\
&= \sum_{l=1}^{n}\sum_{k=1}^{l} p_{1k}\, p'_{kl}\, e^{-\gamma_k t}\beta_{l(n+1)} \\
&= \sum_{k=1}^{n}\left(\sum_{l=k}^{n} p_{1k}\, p'_{kl}\beta_{l(n+1)}\right) e^{-\gamma_k t}.
\end{aligned}$$

and the average time $E(W) = \int_0^\infty t f_1(t)\,dt = \sum_{k=1}^{n}\sum_{l=k}^{n}\frac{p_{1k}\, p'_{kl}\beta_{l(n+1)}}{\gamma_k^2}$.

3 Appendices

Appendix 1. *Let* $P = \left(p_{ij}\right)$ *and* $Q = \left(q_{ij}\right)$ *be* $n \times n$ *matrices such that*

$$p_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \end{cases}$$

and

$$q_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{j-i}(-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j}\ \prod_{q=0}^{k-1} p_{j_q j_{q+1}}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases} \tag{3.1}$$

Then $P = Q^{-1}$.

Proof. Let $PQ = \left(a_{ij}\right)$.
Case 1 $i = j$.
We see that

$$a_{ii} = \sum_{r=1}^{n} p_{ir} q_{ri} = p_{ii} q_{ii} = 1.$$

Case 2 $i > j$.
We see that

$$a_{ij} = \sum_{r=1}^{n} p_{ir} q_{rj} = \underbrace{0 + 0 + \cdots + 0}_{n} = 0.$$

Case 3 $i < j$.
First, notice that

$$a_{ij} = \sum_{r=1}^{n} p_{ir} q_{rj} = \sum_{r=i}^{j} p_{ir} q_{rj}. \tag{3.2}$$

If $j = i + 1$, then

$$a_{ij} = p_{i(i+1)} + q_{i(i+1)} = p_{i(i+1)} - p_{i(i+1)} = 0.$$

Now, suppose that $j \geq i + 2$. From (3.1) and (3.2),

$$\begin{aligned}
&a_{ij} \\
&= p_{ij} + q_{ij} + \sum_{r=i+1}^{j-1} p_{ir} q_{rj} \\
&= p_{ij} + \sum_{k=1}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} p_{j_q j_{q+1}} + \sum_{r=i+1}^{j-1} \sum_{k=1}^{j-r} (-1)^k \sum_{r=j_0<j_1<\cdots<j_k=j} p_{ir} \prod_{q=0}^{k-1} p_{j_q j_{q+1}} \\
&= \sum_{k=2}^{j-i} (-1)^k \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} p_{j_q j_{q+1}} + \sum_{k=1}^{j-i-1} (-1)^k \sum_{r=i+1}^{j-k} \sum_{r=j_0<j_1<\cdots<j_k=j} p_{ir} \prod_{q=0}^{k-1} p_{j_q j_{q+1}} \\
&= \sum_{k=1}^{j-i-1} (-1)^{k+1} \sum_{i=j_0<j_1<\cdots<j_{k+1}=j} \prod_{q=0}^{k} p_{j_q j_{q+1}} + \sum_{k=1}^{j-i-1} (-1)^k \sum_{i=j_0<j_1<\cdots<j_{k+1}=j} \prod_{q=0}^{k} p_{j_q j_{q+1}} \\
&= 0.
\end{aligned}$$

From Cases 1–3, we have $PQ = I_n$. Hence $P = Q^{-1}$. $\square$

Appendix 2. *Let* $A = \left(a_{ij}\right)$ *be an* $n \times n$ *matrix such that* *(i)* $a_{ij} = 0$ *for all* $i > j$,

(ii) $a_{11}, a_{22}, \ldots, a_{nn}$ *are all distinct.*

Then the Jordan canonical form J *of* A *is*

$$J = \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix}$$

and $A = PJP^{-1}$ *where* $P = \left(p_{ij}\right)$ *is defined by*

$$p_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{j-i} \sum_{i=j_0<j_1<\cdots<j_k=j} \prod_{q=0}^{k-1} \frac{a_{j_q j_{q+1}}}{a_{jj} - a_{j_q j_q}}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j \end{cases}$$

and $P^{-1} = \left(p'_{ij}\right)$ *where* p'_{ij} *is defined by* (3.1).

Proof. It is obvious from Appendix 1 that the matrix $\left(p'_{ij}\right)$ is P^{-1}. Note that $PJP^{-1} = \left(b_{ij}\right)$ where

$$b_{ij} = \sum_{r=1}^{n} p_{ir} a_{rr} p'_{rj}.$$

It is trivial that

$$b_{ii} = \sum_{r=1}^{n} p_{ir} a_{rr} p'_{ri} = p_{ii} a_{ii} p'_{ii} = a_{ii}.$$

If $i > j$, then $p_{ir} p'_{rj} = 0$ for every $r = 1, 2, \ldots, n$. Hence $b_{ij} = 0$. Suppose that $i < j$. If $j = i + 1$, then

$$\begin{aligned} b_{i(i+1)} &= \sum_{r=1}^{n} p_{ir} a_{rr} p'_{r(i+1)} \\ &= a_{ii} p'_{i(i+1)} + p_{i(i+1)} a_{(i+1)(i+1)} \\ &= -a_{ii} p_{i(i+1)} + p_{i(i+1)} a_{(i+1)(i+1)} \\ &= p_{i(i+1)} \left(a_{(i+1)(i+1)} - a_{ii}\right) \\ &= a_{i(i+1)}. \end{aligned}$$

To prove the rest, we first show that for any i, j with $j > i$

$$(a_{jj} - a_{ii})\, p_{ij} = \sum_{l=i+1}^{j} a_{il} p_{lj}.$$

Let $j = i + r$ where $r \geq 1$. We see that

$$\begin{aligned}
&p_{i(i+r)} \\
&= \sum_{k=1}^{r} \sum_{i=j_0<j_1<\cdots<j_k=i+r} \prod_{q=0}^{k-1} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}} \\
&= \sum_{k=1}^{r} \sum_{l=i+1}^{i+r-k+1} \frac{a_{il}}{a_{(i+r)(i+r)} - a_{ii}} \sum_{l=j_0<j_1<\cdots<j_{k-1}=i+r} \prod_{q=0}^{k-2} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(a_{(i+r)(i+r)} - a_{ii}\right) p_{i(i+r)} \\
&= \sum_{k=1}^{r} \sum_{l=i+1}^{i+r-k+1} a_{il} \sum_{l=j_0<j_1<\cdots<j_{k-1}=i+r} \prod_{q=0}^{k-2} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}} \\
&= \sum_{l=i+1}^{i+r} a_{il} \sum_{k=1}^{i+r-l+1} \sum_{l=j_0<j_1<\cdots<j_{k-1}=i+r} \prod_{q=0}^{k-2} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}} \\
&= \sum_{l=i+1}^{i+r} a_{il} \sum_{k=0}^{i+r-l} \sum_{l=j_0<j_1<\cdots<j_k=i+r} \prod_{q=0}^{k-1} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}} \\
&= \sum_{l=i+1}^{i+r} a_{il} \sum_{k=1}^{i+r-l} \sum_{l=j_0<j_1<\cdots<j_k=i+r} \prod_{q=0}^{k-1} \frac{a_{j_q j_{q+1}}}{a_{(i+r)(i+r)} - a_{j_q j_q}} \\
&= \sum_{l=i+1}^{i+r} a_{il} p_{l(i+r)}.
\end{aligned}$$

Now, let $j \geq i+2$. Then

$$
\begin{aligned}
b_{ij}
&= \sum_{r=1}^{n} p_{ir} a_{rr} p'_{rj} \\
&= a_{ii} p'_{ij} + \sum_{r=i+1}^{j-1} p_{ir} a_{rr} p'_{rj} + p_{ij} a_{jj} \\
&= \left(a_{ii} \sum_{k=1}^{j-i} (-1)^k \sum_{i=\alpha_0<\alpha_1<\cdots<\alpha_k=j} \prod_{q=0}^{k-1} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{r=i+1}^{j-1} p_{ir} a_{rr} \sum_{k=1}^{j-r} (-1)^k \sum_{r=\gamma_0<\gamma_1<\cdots<\gamma_k=j} \prod_{q=0}^{k-1} p_{\gamma_q \gamma_{q+1}} \right) + p_{ij} a_{jj} \\
&= (a_{jj} - a_{ii})\, p_{ij} + \left(a_{ii} \sum_{k=2}^{j-i} (-1)^k \sum_{i=\alpha_0<\alpha_1<\cdots<\alpha_k=j} \prod_{q=0}^{k-1} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=1}^{j-i-1} \sum_{r=i+1}^{j-k} (-1)^k a_{rr} p_{ir} \sum_{r=\gamma_0<\gamma_1<\cdots<\gamma_k=j} \prod_{q=0}^{k-1} p_{\gamma_q \gamma_{q+1}} \right) \\
&= \sum_{l=i+1}^{j} a_{il} p_{lj} + \left(a_{ii} \sum_{k=2}^{j-i} (-1)^k \sum_{i=\alpha_0<\alpha_1<\cdots<\alpha_k=j} \prod_{q=0}^{k-1} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{rr} p_{ir} \sum_{r=\gamma_0<\gamma_1<\cdots<\gamma_{k-1}=j} \prod_{q=0}^{k-2} p_{\gamma_q \gamma_{q+1}} \right) \\
&= \left(a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} \right) + \left(a_{ii} \sum_{k=2}^{j-i} (-1)^k \sum_{i=\alpha_0<\alpha_1<\cdots<\alpha_k=j} \prod_{q=0}^{k-1} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{rr} p_{ir} \sum_{r=\gamma_0<\gamma_1<\cdots<\gamma_{k-1}=j} \prod_{q=0}^{k-2} p_{\gamma_q \gamma_{q+1}} \right) \\
&= a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^k a_{ii} p_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{rr} p_{ir} \sum_{r=\gamma_0<\gamma_1<\cdots<\gamma_{k-1}=j} \prod_{q=0}^{k-2} p_{\gamma_q \gamma_{q+1}} \right) \\
&= a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} (a_{rr} - a_{ii})\, p_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} \left(a_{ir} + \sum_{L=i+1}^{r-1} a_{iL} p_{Lr} \right) \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&= a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} \sum_{L=i+1}^{r-1} a_{iL} p_{Lr} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&= a_{ij} + \sum_{l=i+1}^{j-1} a_{il} p_{lj} - \sum_{r=i+1}^{j-1} a_{ir} p_{rj} \\
&\quad + \left(\sum_{k=3}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&\quad + \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} \sum_{L=i+1}^{r-1} a_{iL} p_{Lr} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) \\
&= a_{ij}
\end{aligned}
$$

where we claim that

$$
\begin{aligned}
&\left(\sum_{k=3}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) + \\
&\qquad \left(\sum_{k=2}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} \sum_{L=i+1}^{r-1} a_{iL} p_{Lr} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}} \right) = 0.
\end{aligned}
$$

To prove the above claim, first, we consider

$$
\sum_{k=3}^{j-i} \sum_{r=i+1}^{j-k+1} (-1)^{k-1} a_{ir} \sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j} \prod_{q=0}^{k-2} p_{\alpha_q \alpha_{q+1}}.
$$

We can see that

$$\begin{aligned}
&\sum_{k=3}^{j-i}\sum_{r=i+1}^{j-k+1}(-1)^{k-1}a_{ir}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j}\prod_{q=0}^{k-2}p_{\alpha_q\alpha_{q+1}}\\
&\quad=\sum_{k=3}^{j-i}\sum_{r=i+1}^{j-k+1}(-1)^{k-1}\sum_{s=r+1}^{j-k+2}a_{ir}p_{rs}\sum_{s=\gamma_0<\gamma_1<\cdots<\gamma_{k-2}=j}\prod_{q=0}^{k-3}p_{\gamma_q\gamma_{q+1}}\\
&\quad=\sum_{k=3}^{j-i}\sum_{s=i+2}^{j-k+2}\sum_{r=i+1}^{s-1}(-1)^{k-1}a_{ir}p_{rs}\sum_{s=\gamma_0<\gamma_1<\cdots<\gamma_{k-2}=j}\prod_{q=0}^{k-3}p_{\gamma_q\gamma_{q+1}}.
\end{aligned}$$

Next, we consider

$$\sum_{k=2}^{j-i}\sum_{r=i+1}^{j-k+1}(-1)^{k-1}\sum_{L=i+1}^{r-1}a_{iL}p_{Lr}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j}\prod_{q=0}^{k-2}p_{\alpha_q\alpha_{q+1}}.$$

We obtain that

$$\begin{aligned}
&\sum_{k=2}^{j-i}\sum_{r=i+1}^{j-k+1}(-1)^{k-1}\sum_{L=i+1}^{r-1}a_{iL}p_{Lr}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j}\prod_{q=0}^{k-2}p_{\alpha_q\alpha_{q+1}}\\
&\quad=\sum_{k=2}^{j-i}\sum_{r=i+2}^{j-k+1}\sum_{L=i+1}^{r-1}(-1)^{k-1}a_{iL}p_{Lr}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}=j}\prod_{q=0}^{k-2}p_{\alpha_q\alpha_{q+1}}\\
&\quad=\sum_{k=3}^{j-i+1}\sum_{r=i+2}^{j-k+2}\sum_{L=i+1}^{r-1}(-1)^{k-2}a_{iL}p_{Lr}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-2}=j}\prod_{q=0}^{k-3}p_{\alpha_q\alpha_{q+1}}\\
&\quad=\sum_{k=3}^{j-i}\sum_{r=i+2}^{j-k+2}\sum_{L=i+1}^{r-1}(-1)^{k-2}a_{iL}p_{Lr}\sum_{r=\alpha_0<\alpha_1<\cdots<\alpha_{k-2}=j}\prod_{q=0}^{k-3}p_{\alpha_q\alpha_{q+1}}.
\end{aligned}$$

Hence the claim is proved. □

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