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On the Set Chromatic Number of the Middle Graph of Extended Stars and Related Tree Families

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Abstract Suppose G is a simple, undirected, finite, nontrivial, and connected graph and that $c : V(G) \to \mathbb{N}$ is a vertex coloring, not necessarily proper, of G. As introduced by Chartrand et al., c is called a set coloring of G if $NC(u) \neq NC(v)$ for every pair of adjacent vertices u and v; here, $NC(x)$ denotes the set of colors of all the neighbors of the vertex x. Moreover, the set chromatic number of G, denoted by $\chi_s(G)$, is the minimum number of colors that can be used to construct a set coloring of G. On the other hand, the middle graph $M(G)$ of a graph G is defined as the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are adjacent if and only if u and v are adjacent edges in G; or $u \in V(G)$, $v \in E(G)$, and u is incident to v in G. In this paper, we study set colorings in relation to the middle graph of some tree families. We establish lower bounds for the set chromatic number of these graphs and we algorithmically construct set colorings for them. For most cases, we find that the set chromatic number for these graphs is given by min-max formulas.

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1. Introduction

All the graphs to be considered in this work are simple, undirected, finite, nontrivial, and connected. As coined in [1], a vertex coloring or edge coloring of a graph is *neighbor-distinguishing* if it induces a vertex labelling such that any two adjacent vertices receive distinct labels. Naturally, the classical proper vertex coloring is neighbor-dsitinguishing.

Among several neighbor-disinguishing colorings that have been introduced (such as those in [1–6]), the topic of this work is set coloring, which was introduced in [7]. Given a graph G and $c : V(G) \to \mathbb{N}$ a vertex coloring of G, we have the following definitions: For any subset S of $V(G)$, we let $c(S) = \{c(s) : s \in S\}$. The *neighborhood color set* of a

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vertex v is defined as the set $NC(v) = c(N_G(v))$, where $N_G(v) = \{w : vw \in E(G)\}$. The coloring c is called a *set coloring* of G if $NC(v) \neq NC(w)$ for any two adjacent vertices v and w. If, in addition, $|c(V(G))| = k$ (i.e. c uses k colors), then we call c a *set k-coloring* of G. The *set chromatic number* of G is denoted by $\chi_s(G)$ and is defined as the smallest integer k for which G has a set k-coloring.

Set colorings have been previously studied in relation to different graph operations involving different graph families. For instance, there have been studies in relation to the join [8, 9], corona and vertex/edge deletions [7], and comb product [9]. Meanwhile, similar studies have been carried out involving other neighbor-distinguishing colorings [10–13].

Previously, in [14], the authors studied set colorings in relation to a graph operation called middle graph. Introduced by Hamada and Yoshimura [15], the *middle graph* $M(G)$ of a graph G is defined to be the intersection graph of $V' \cup E(G)$, where V' is the set of all singletons each containing a vertex of G. Alternatively, we may think of $M(G)$ as the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are adjacent if and only if u and v are adjacent edges in G; or $u \in V(G)$, $v \in E(G)$, and u is incident to v in G. Note that two adjacent vertices of G are not adjacent in $M(G)$. The results in [14] include bounds for $\chi_s(M(G))$ as well as the set chromatic number of the middle graph of stars, double stars, paths, cycles, and tadpoles.

This work aims to continue [14] by considering the middle graph of other families of trees and to contribute to the literature on the active research area of graph colorings and graph operations. In particular, the results in this paper reveal that the set chromatic number of the middle graph of some tree families is given by min-max formulas.

2. Some Known Results on Set Colorings

We will use the following notations: For any positive integer m, we denote by $\mathbb{N}_m$ the set $\{1, 2, ..., m\}$. For a vertex v in a graph G, we denote by $S_G(v)$ the set of all pendant neighbors, in G, of v (i.e., all end-vertices of G adjacent to v).

The following family of graphs has been studied in [7]. Let n, t be integers such that $n \geq 2$ and $0 \leq t \leq n$. The graph $G_{n,t}$ is the graph whose vertex set may be denoted by $\{v_1, v_2, ..., v_n\} \cup \{u_1, u_2, ..., u_t\}$ and whose edge set is $\{v_iv_j : i \neq j\} \cup \{v_ku_k : k = 1, 2, ..., t\}$. Note that the vertices $v_1, v_2, ..., v_n$ form a complete subgraph of $G_{n,t}$ with order n. The set chromatic number of $G_{n,t}$ is given by the following proposition.

Proposition 2.1 ([7], Proposition 2.6)**.** *For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.*

Let us now present relevant results from [14]. The first result provides a lower bound for $\chi_s(M(G))$, where G is a graph with at least one pendant vertex. We include the proof, for completeness.

Lemma 2.2 ([14])**.** *Let G be any graph and assume that G has a vertex v that has at least one pendant neighbor. Then*

$$\chi_s(M(G)) \geq |S_G(v)| + 1.$$

Proof. We set $Q :=$ the set of all nonpendant neighbors, in G, of v, $T := \{vq \in E(G) : q \in Q\}$, $S := S_G(v)$, and $R := \{vs \in E(G) : s \in S\}$. Note that T and R can also be viewed as subsets of $V(M(G))$.

Let c be a set k-coloring of $M(G)$. First, suppose that $Q \neq \emptyset$ and let t be a fixed vertex from T. Then the clique H of $M(G)$ formed by v, t, and all the vertices in R is isomorphic to $K_{|S|+2}$. Moreover, permuting colors if necessary, we may assume that $c(V(H)) = \mathbb{N}_\ell$

for some $\ell \le k$. Let X be the maximal subset of $V(H)$ such that for all $x \in X$, there exists $y \in X \setminus \{x\}$ for which $c(x) = c(y)$. If $X = \emptyset$, then $|S| + 1 \le |V(H)| = \ell \le k$, and so we may assume that $X \ne \emptyset$. Since the remaining vertices in $V(H) \setminus X$ receive unique colors, we must have $|V(H)| - |X| + 1 \le \ell$ and so $|X| \ge |S| - \ell + 3$. The possible neighborhood color sets of vertices in X are given as follows:

(1) If $v \in X$, then $NC(v) = \mathbb{N}_\ell \cup c(T \setminus \{t\})$.
(2) If $t \in X$, then $NC(t) = \mathbb{N}_\ell \cup c(T \setminus \{t\}) \cup c\big(N_{M(G)}(t) \setminus (V(H) \cup T)\big)$.
(3) If $r \in R \cap X$ and $s \in S \cap N_{M(G)}(r)$, then:
  (a) If $c(s) \notin \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell \cup \{c(s)\} \cup c(T \setminus \{t\})$.
  (b) If $c(s) \in \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell \cup c(T \setminus \{t\})$.

Note that (1) and (3b) provide the same neighborhood color set; aside from these two, all the other possible neighborhood color sets above may all be distinct from each other. Since there are $k - \ell$ colors not in $\mathbb{N}_\ell$, (3a) provides for $k - \ell$ distinct neighborhood color sets. Hence, the maximum number of possible neighborhood color sets available for vertices in X is $k - \ell + 2$. So we must have $k - \ell + 2 \ge |X|$. Then $k - \ell + 2 \ge |S| - \ell + 3$, which implies that $k \ge |S| + 1$.

Now, suppose $Q = \emptyset$. Since G is connected, it must be isomorphic to a star $K_{1,m}$. Since $M(K_{1,m}) \cong G_{m+1,m}$, our desired conclusion follows from Proposition 2.1. ■

Using Lemma 2.2, the following theorem was proved in [14]. The optimal set coloring is constructed using Algorithm 1 in [14].

Theorem 2.3 ([14]). *Let T be a tree of height 2 rooted at a vertex v_0 with $\deg(v_0) \ge 4$. If there is an internal vertex w with $\deg(w) = \Delta(G) \ge \deg(v_0)+1$, then $\chi_s(M(T)) = \deg(w)$.*

3. Extended Stars

In this section, we determine the set chromatic number of the middle graph of extended star graphs, which we define below. Such graphs have been used, for example, in [16].

Definition 3.1. Let m, n be positive integers. We define the *extended star* graph $T_{m,n}$ as follows: If $m = 1$, then $T_{1,n}$ is the star graph $K_{1,n+1}$. If $m \ge 2$, then $T_{m,n}$ is the graph obtained from the star $K_{1,m}$ by connecting n pendant vertices to each pendant vertex of $K_{1,m}$.

By Observation 3.2 in [14], we have $\chi_s(M(T_{1,n})) = n + 2$ for any positive integer n. So we may assume that $m \ge 2$.

Recall that $n! = 1 \times 2 \times \cdots \times n$ for any positive integer n, while $0! = 1$. Moreover, given non-negative integers n and r, the number of combinations, denoted by $C(n, r)$, of n objects taken r at a time is given by $C(n, r) = \frac{n!}{r!(n-r)!}$. We also recall the following identities: $C(n, 0) = 1$ and $C(n - 1, r - 1) + C(n - 1, r) = C(n, r)$.

We now introduce the following: For positive integers a, b, c, we denote by $q(a, b, c)$ the smallest positive integer k for which

$$\sum_{\alpha=0}^{\min\{k-c,b\}} C(k - c, \alpha) \ge a - c.$$

Note that, necessarily, $q(a, b, c) \ge c$; moreover, $q(a, b, c) = c$ if and only if $a - c \le 1$. We then have the following property:

Observation 3.1. If $a \leq a'$ and $c \leq c'$, then $q(a,b,c) \leq q(a',b,c')$.

Proof. If $a-c \leq 1$, then $q(a,b,c) = c \leq c' = q(a',b',c')$, which is the desired inequality.

From now, we assume that $a-c \geq 2$. If $a \leq a'$, then it is clear that $q(a,b,c) \leq q(a',b,c)$.

Now, suppose c is a positive integer and consider $q' := q(a,b,c+1)$. Set $q = q(a,b,c)$, $r = \min\{q-c, b\}$, and $r' = \min\{q'-(c+1), b\}$.

Claim 1: If $a-c \geq 3$, then $C(q'-(c+1), r') \leq a-c-2$.
Proof of Claim 1. If $r' = 0$, then $q' = c+1$, which can only happen if $a-(c+1) \leq 1$ or $a-c \leq 2$ but this contradicts the assumption. Thus, we must have $r' \geq 1$. Suppose, on the contrary, that $C(q'-(c+1), r') \geq a-c-1$. Then $C((q'-1)-(c+1), r'-1) + C((q'-1)-(c+1), r') = C(q'-(c+1), r') \geq a-c-1 = a-(c+1)$, which contradicts the minimality of q'.

Claim 2: $q \leq q'$
Proof of Claim 2. First, suppose $a-c=2$; then $a-(c+1)=1$. It follows that $q = q(a,b,c) = c+1 = q(a,b,c+1) = q'$, as desired.

Now, suppose $a-c \geq 3$. We will show that q' satisfies $\displaystyle\sum_{\alpha=0}^{\min\{q'-c,b\}} C(q'-c,\alpha) \geq a-c$.

First, we have

$$
\begin{aligned}
&\sum_{\alpha=0}^{\min\{q'-c,b\}} C(q'-c,\alpha) \\
&\qquad \geq \sum_{\alpha=0}^{r'} C(q'-c,\alpha) \\
&\qquad = C(q'-c,0) + \sum_{\alpha=1}^{r'} [C(q'-(c+1),\alpha-1) + C(q'-(c+1),\alpha)] \\
&\qquad = C(q'-(c+1),0) + \sum_{\alpha=1}^{r'} [C(q'-(c+1),\alpha-1) + C(q'-(c+1),\alpha)] \\
&\qquad = 2\left[\sum_{\alpha=0}^{r'} C(q'-(c+1),\alpha)\right] - C(q'-(c+1),r') \\
&\qquad \geq 2[a-(c+1)] - C(q'-(c+1),r') \\
&\qquad \geq a-c \quad \text{(by Claim 1)}.
\end{aligned}
$$

Hence, by the minimality of q, we have $q \leq q'$.

By Claim 2 and induction, we have $q(a,b,c) \leq q(a,b,c')$ whenever $c \leq c'$. ■

For the rest of this section, we will denote the vertices of $T_{m,n}$ as follows:

(1) The root vertex is v_0.
(2) The children of v_0 are $v_1, v_2, ..., v_m$.
(3) For each $i \in \mathbb{N}_m$, the children of v_i are $v_{i,1}, v_{i,2}, ..., v_{i,n}$.

We now establish a lower bound for $\chi_s(M(T_{m,n}))$.

Lemma 3.2. *Let $m \geq 2$ and $n \geq 1$ be integers. Then*

$$\chi_s(M(T_{m,n})) \geq \min\big\{\max\{n+2, q(m+2,n+1,1)\}, \max\{n+1, q(m+3,n+1,3)\}\big\}.$$

Proof. Let $G = M(T_{m,n})$. Let c be a set k-coloring of G. We will consider the following two subgraphs of G:

H_1 = subgraph of G induced by the vertices v_0 and v_0v_i, for $i \in \mathbb{N}_m$,

H_2 = subgraph of G induced by v_0v_1, v_1, and $v_1v_{1,j}$ for $j \in \mathbb{N}_n$.

Then $H_1 \cong K_{m+1}$ and $H_2 \cong K_{n+2}$. For $i \in \{1, 2\}$, we define the following:

$C_i = c(V(H_i))$, $\quad \ell_i = |C_i|$,

$X_i = \{x \in V(H_i) : c(x) = c(y) \text{ for some } y \in V(H_i) \setminus \{x\}\}$.

Moreover, if $X_i \neq \emptyset$, then $|V(H_i)| - |X_i| + 1 \leq \ell_i$ or $|X_i| \geq |V(H_i)| - \ell_i + 1$.

We have the following possible neighborhood color sets for vertices in X_1.

(1a) If $v_0 \in X_1$, then $NC(v_0) = C_1$.

(1b) If $v_0v_i \in X_1$, $i \in \mathbb{N}_m$, then $NC(v_0v_i) = C_1 \cup \{c(v_i)\} \cup \{c(v_iv_{i,j}) : j \in \mathbb{N}_n\}$.

Similarly, we have the following possible neighborhood color sets for vertices in X_2.

(2a) If $v_0v_1 \in X_2$, then $NC(v_0v_1) = C_2 \cup \{c(v_0)\} \cup \{c(v_0v_i) : i \neq 1\}$.

(2b) If $v_1 \in X_2$, then $NC(v_1) = C_2$.

(2c) If $v_1v_{1,j} \in X_2$, $j \in \mathbb{N}_n$, then $NC(v_1v_{1,j}) = C_2 \cup \{c(v_{1,j})\}$.

Lemma 2.2 implies that $k \geq n + 1$. If $X_1 = \emptyset$ (i.e. $\ell_1 = m + 1$), then $k \geq m + 1 \geq q(m+3, n+1, 3)$. Then $k \geq \max\{n + 1, q(m + 3, n + 1, 3)\}$. Now, we assume that $X_1 \neq \emptyset$ (i.e. $1 \leq \ell_1 \leq m$).

Without any other restriction, (1a)-(1b) imply that vertices in X_1 will have neighborhood color sets of the form $C_1 \cup \mathcal{S}$, where $\mathcal{S} \subseteq \mathbb{N}_k \setminus C_1$ and $0 \leq |\mathcal{S}| \leq n + 1$. Thus, the maximum number of possible neighborhood color sets for vertices in X_1 is $\sum_{\alpha=0}^{\min\{k-\ell_1,n+1\}} C(k - \ell_1, \alpha)$. Since vertices in X_1 should have distinct neighborhood color sets, we must have

$$\sum_{\alpha=0}^{\min\{k-\ell_1,n+1\}} C(k - \ell_1, \alpha) \geq |X_1|. \tag{3.1}$$

Since $|X_1| \geq m - \ell_1 + 2$, it follows that

$$\sum_{\alpha=0}^{\min\{k-\ell_1,n+1\}} C(k - \ell_1, \alpha) \geq m - \ell_1 + 2 \quad \Longrightarrow \quad k \geq q(m + 2, n + 1, \ell_1). \tag{3.2}$$

We now proceed by cases based on the value of ℓ_1. Note that $c(v_0v_1) \in C_1 \cap C_2$.

Case 1. Suppose $\ell_1 \leq 2$.

If $X_2 = \emptyset$, then $k \geq \ell_2 \geq n + 2$. We now show that $k \geq n + 2$ even if $X_2 \neq \emptyset$. Since $|V(H_1)| \geq 3$, we may assume that $v_0v_1 \in X_1$, which implies that $c(v_0v_1) \in \{c(v_0)\} \cup \{c(v_0v_i) : i \neq 1\}$. Moreover, we have $|\{c(v_0)\} \cup \{c(v_0v_i) : i \neq 1\}| \leq 2$. Thus, if $v_0v_1 \in X_2$, then $|NC(v_0v_1)| \leq |C_2| + 1$. Thus, given (2a)-(2c), the possible neighborhood color sets of vertices in X_2 are all of the form $C_2 \cup \{\alpha\}$, where α may or may not be in C_2. Since $k - \ell_2$ colors are not used in X_2, there are at most $1 + k - \ell_2$ possible neighborhood color sets for the vertices in X_2. Therefore, $1 + k - \ell_2 \geq |X_2| \geq n + 2 - \ell_2 + 1$, which implies that $k \geq n + 2$.

On the other hand, (3.2) and Observation 3.1 imply that $k \geq q(m + 2, n + 1, 1)$. The conclusion follows.

Case 2. Suppose $\ell_1 \geq 3$ and $|X_1| \geq m$.

In this case, (3.1) becomes

$$\sum_{\alpha=0}^{\min\{k-\ell_1,n+1\}} C(k-\ell_1,\alpha) \geq m,$$

which implies that $k \geq q(m+\ell_1, n+1, \ell_1)$. Since $\ell_1 \geq 3$, we have $k \geq q(m+3, n+1, 3)$. Since Lemma 2.2 implies that $k \geq n+1$ as well, the conclusion follows.

Case 3. Suppose $\ell_1 \geq 3$ and $|X_1| \leq m-1$.

We may assume that $v_0v_1 \notin X_1$. First, let us assume that $v_0v_1 \in X_2$, which implies that $c(v_0v_1) \in \{c(v_1)\} \cup \{c(v_1v_{1,j} : j \in \mathbb{N}_n\}$. Thus, the neighborhood color set of v_0v_1 has the same form as (1b) even if $v_0v_1 \notin X_1$. So the right-hand side of (3.1) becomes $|X_1|+1$ and we obtain

$$\sum_{\alpha=0}^{\min\{k-\ell_1,n+1\}} C(k-\ell_1,\alpha) \geq m-\ell_1+3 \quad \Longrightarrow \quad k \geq q(m+3,n+1,3).$$

With Lemma 2.2, we have $k \geq \max\{n+1, q(m+3,n+1,3)\}$.

On the other hand, suppose $v_0v_1 \notin X_2$. If $X_2 = \emptyset$, then $k \geq n+2$. Now, suppose $X_2 \neq \emptyset$. Then only (2b) and (2c) are applicable and the maximum number of possible neighborhood color sets for vertices in X_2 is $1+k-\ell_2$, which implies that $k \geq n+2$ as well. At the same time, $k \geq q(m+2,n+1,3) \geq q(m+2,n+1,1)$ by (3.2) and Observation 3.1. Thus, $k \geq \max\{n+2, q(m+2,n+1,1)\}$.

Therefore, in any of the Cases 1-3, we have

$$k \geq \min\bigl\{\max\{n+2, q(m+2,n+1,1)\}, \max\{n+1, q(m+3,n+1,3)\}\bigr\},$$

as desired. ∎

When $n \geq m \geq 4$, the extended star $T_{m,n}$ is a tree satisfying the assumptions of Theorem 2.3. Moreover, when $n \geq m \geq 4$, we have $n+2 \geq m+2 \geq q(m+2,n+1,1)$ and $n+1 \geq m+1 \geq q(m+3,n+1,3)$. Thus, we have the following.

Corollary 3.3. *If $n \geq m \geq 4$, then*

$$\begin{aligned}\chi_s(M(T_{m.n})) &= n+1 \\ &= \min\bigl\{\max\{n+2, q(m+2,n+1,1)\}, \max\{n+1, q(m+3,n+1,3)\}\bigr\}.\end{aligned}$$

With Lemma 3.2, we can actually extend Corollary 3.3 to the general case where $m \geq 2$ and $n \geq 1$. To show this, we prove the following two lemmas, where we construct set colorings of $M(T_{m,n})$.

Lemma 3.4. *Let $m \geq 2$ and $n \geq 1$ be integers. Set $p = \max\{n+2, q(m+2,n+1,1)\}$. Then $M(T_{m,n})$ is set p-colorable.*

Proof. Let $S_1, S_2, ..., S_{2^{p-1}-1}$ be the nonempty subsets of $\mathbb{N}_p \setminus \{1\}$ arranged in such a way that $|S_i| \leq |S_{i+1}|$ for all i. Let us construct a coloring $c : V(M(T_{m,n})) \to \mathbb{N}_p$ as follows:

(A1) Set $c(v_0) = c(v_0v_i) = 1$ for $i \in \mathbb{N}_m$.
(A2) Set $c(v_i) = \min S_i$ for $i \in \mathbb{N}_m$.
(A3) Set $\alpha_i = |S_i \setminus \{c(v_i)\}|$ for $i \in \mathbb{N}_m$.
  (a) If $\alpha_i = 0$, set $c(v_iv_{i,j}) = 1$ for $j \in \mathbb{N}_n$.

TABLE 1. The neighbors and neighborhood color set of each vertex in $M(T_{m,n})$ under the coloring c (Lemma 3.4)

Vertex	Neighbors	NC
v_0	v_0v_i $\forall i \in \mathbb{N}_m$	$\{1\}$
v_0v_i, $i \in \mathbb{N}_m$	v_0, v_i, v_0v_j $\forall j \neq i$, $v_iv_{i,j}$ $\forall j \in \mathbb{N}_n$	$\{1\} \cup S_i$
v_i, $i \in \mathbb{N}_m$	v_0v_i, $v_iv_{i,j}$ $\forall j \in \mathbb{N}_n$	$\{1\} \cup [S_i \setminus \{c(v_i)\}]$
$v_iv_{i,j}$, $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$	v_0v_i, v_i, $v_{i,j}$, $v_iv_{i,q}$ $\forall q \neq j$	If $c(v_iv_{i,j}) \neq 1$: $\{1\} \cup [S_i \setminus c(v_iv_{i,j})]$ If $c(v_iv_{i,j}) = 1$: $\{1, c(v_{i,j})\} \cup S_i$
$v_{i,j}$, $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$	$v_iv_{i,j}$	If $c(v_iv_{i,j}) \neq 1$: $\{s_j\}$ If $c(v_iv_{i,j}) = 1$: $\{1\}$

(b) If $\alpha_i > 0$, suppose $S_i \setminus \{c(v_i)\} = \{s_1, s_2, ..., s_{\alpha_i}\}$. Set $c(v_iv_{i,j}) = s_j$ for $1 \leq j \leq \alpha_i$; set $c(v_iv_{i,j}) = 1$ for $\alpha_i + 1 \leq j \leq n$.

(A4) For each $i \in \mathbb{N}_m$:
  (a) For all $j \in \mathbb{N}_n$ for which $c(v_iv_{i,j}) \neq 1$, set $c(v_{i,j}) = 1$.
  (b) Let $W_i = \{v_{i,j} : c(v_iv_{i,j}) = 1\}$. Set $c(W_i)$ so that the vertices in W_i receive distinct colors from $\mathbb{N}_p \setminus (\{1\} \cup S_i)$.

First, we prove that c can always be constructed. For (A2), we need $2^{p-1} - 1 \geq m$. Since $p \geq q(m+2, n+1, 1)$, we have

$$\sum_{\alpha=0}^{p-1} C(p-1, \alpha) \geq m+1 \quad \Longrightarrow \quad 2^{p-1} - 1 \geq m.$$

For (A3b), we verify that $\alpha_i \leq n$ so that all colors in $S_i \setminus \{c(v_i)\}$ are used. Indeed, if $p = n+2$, then $\alpha_i = |S_i \setminus \{c(v_i)\}| \leq p-2 = n$. On the other hand, if $p = q(m+2, n+1, 1)$, then

$$\sum_{\alpha=0}^{\min\{p-1,n+1\}} C(p-1, \alpha) \geq m+1 \quad \Longrightarrow \quad \sum_{\alpha=1}^{\min\{p-1,n+1\}} C(p-1, \alpha) \geq m.$$

But $\sum_{\alpha=1}^{\min\{p-1,n+1\}} C(p-1, \alpha)$ is the number of nonempty α-subsets of $\mathbb{N}_p \setminus \{1\}$, where $\alpha \leq n+1$. Since this number is at least m, we must have $|S_m| \leq n+1$, which implies that $\alpha_i \leq n$ for all $i \in \mathbb{N}_m$. For (A4b), we need $|W_i| \leq |\mathbb{N}_p \setminus (\{1\} \cup S_i)|$. Indeed, $|W_i| = n - \alpha_i \leq p - (\alpha_i + 2) = |\mathbb{N}_p \setminus (\{1\} \cup S_i)|$. Thus, it is always possible to construct the coloring c.

Now, we show that c uses exactly p colors. First, suppose $p = n+2$. Since $|S_1| = 1$, we have $\alpha_1 = 0$. By (A3a) and (A4b), we have $W_1 = \{v_{1,j} : j \in \mathbb{N}_n\}$; that is, $|W_1| = n = p-2$. By (A4b), $c(W_1) = \mathbb{N}_p \setminus (\{1\} \cup S_1)$. Meanwhile, the colors in $\{1\} \cup S_1$ are used in (A1) and (A2). Thus, all colors from $\mathbb{N}_p$ are used by c in this case. On the other hand, suppose $p = q(m+2, n+1, 1)$. Clearly,

$$\sum_{\alpha=0}^{\min\{(m+1)-1,n+1\}} C((m+1)-1, \alpha) \geq m+1$$

for any value of n. By the minimality of p, we must have $p \leq m+1$. Thus, in (A2), all the 1-subsets of $\mathbb{N}_p \setminus \{1\}$ are used by c. Meanwhile, the color 1 is used in (A1). Therefore, all colors from $\mathbb{N}_p$ are also used in this case.

Finally, we present in Table 1 the neighbors and neighborhood color set of each vertex in $M(T_{m,n})$. Through Table 1, it can be easily verified that c is a set coloring. ∎

As an example, consider $T_{4,2}$. Since $\max\{4, q(6,3,1)\} = 4$, Lemma 3.4 implies that $M(T_{4,2})$ is set 4-colorable. Figure 1 shows a set 4-coloring of $M(T_{4,2})$ constructed using (A1)-(A4) from Lemma 3.4.

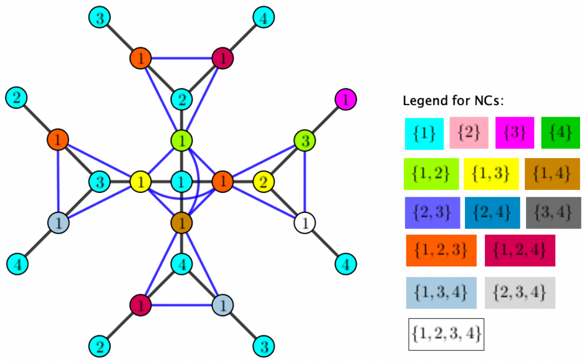

FIGURE 1. A set coloring $c : V(M(T_{4,2})) \to \mathbb{N}_4$ constructed using (A1)-(A4) from Lemma 3.4

Lemma 3.5. *Let $m \geq 2$ and $n \geq 1$ be integers. Set $p = \max\{n+1, q(m+3, n+1, 3)\}$. Then $M(T_{m,n})$ is set p-colorable.*

Proof. Note that since $m \geq 2$, we have $p \geq q(m+3, n+1, 3) \geq 4$. Let $S_1, S_2, ..., S_{2^{p-3}}$ be the subsets of $\mathbb{N}_p \setminus \mathbb{N}_3$ arranged in such a way that $|S_i| \leq |S_{i+1}|$ for all i. Note that $S_1 = \emptyset$. We construct a coloring $c : V(M(T_{m,n})) \to \mathbb{N}_p$ as follows:

(B1) Set $c(v_0) = 3, c(v_0 v_m) = 2$, and $c(v_0 v_i) = 1$ for all $i \in \mathbb{N}_{m-1}$.

(B2) Set $c(v_1) = 1$ and $c(v_i) = \min S_i$ for $i \in \mathbb{N}_m$.

(B3) Set $c(v_1 v_{1,j}) = 1$ for $j \in \mathbb{N}_n$.

For each $i \in \mathbb{N}_m \setminus \{1\}$, set $\alpha_i = |S_i \setminus \{c(v_i)\}|$ and

(a) if $\alpha_i = 0$, set $c(v_i v_{i,j}) = 1$ for $j \in \mathbb{N}_n$ if $i \neq m$; moreover, set $c(v_m v_{m,j}) = 2$ for $j \in \mathbb{N}_n$;

(b) if $\alpha_i > 0$, suppose $S_i \setminus \{c(v_i)\} = \{s_1, s_2, ..., s_{\alpha_i}\}$. If $2 \leq i \leq m-1$, set $c(v_i v_{i,j}) = s_j$ for $1 \leq j \leq \alpha_i$ and $c(v_i v_{i,j}) = 1$ for $\alpha_i + 1 \leq j \leq n$. If $i = m$, set $c(v_m v_{m,j}) = s_j$ for $1 \leq j \leq \alpha_m$ and $c(v_m v_{m,j}) = 2$ for $\alpha_m + 1 \leq j \leq n$.

(B4) For each $i \in \mathbb{N}_m$:

(a) If $i = 1$, set $c(\{v_{1,j} : j \in \mathbb{N}_n\}) \subseteq \mathbb{N}_p \setminus \{1\}$ so that the vertices in $\{v_{1,j} : j \in \mathbb{N}_n\}$ receive different colors.

(b) If $2 \leq i \leq m-1$, let $W_i = \{v_{1,j} : c(v_i v_{i,j}) \notin S_i\}$ and set $c(W_i) \subseteq \mathbb{N}_p \setminus S_i$ so that the vertices in W_i receive different colors.
Then set $c(\{v_{i,j} : j \in \mathbb{N}_n\} \setminus W_i) = \{1\}$.

(c) If $i = m$, let $W_m = \{v_{m,j} : c(v_m v_{m,j}) \notin S_m\}$ and set $c(W_m) \subseteq \mathbb{N}_p \setminus S_m$ so that the vertices in W_m receive different colors.
Then set $c(\{v_{m,j} : j \in \mathbb{N}_n\} \setminus W_m) = \{2\}$.

Table 2. The neighbors and neighborhood color set of each vertex in $M(T_{m,n})$ under the coloring c (Lemma 3.5)

Vertex		Neighbors	NC
v_0		$v_0v_i\ \forall i \in \mathbb{N}_m$	$\mathbb{N}_2$
v_0v_i	$i=1$	$v_0,\ v_0v_j\ \forall j \neq 1,\ v_1,$ $v_1v_{1,j}\ \forall j \in \mathbb{N}_n$	$\mathbb{N}_3$
	$2 \leq i \leq m-1$	$v_0,\ v_0v_j\ \forall j \neq i,\ v_i,$ $v_iv_{i,j}\ \forall j \in \mathbb{N}_n$	$\mathbb{N}_3 \cup S_i$
	$i=m$	$v_0,\ v_0v_j\ \forall j \neq m,\ v_m,$ $v_mv_{m,j}\ \forall j \in \mathbb{N}_n$	$\mathbb{N}_3 \cup S_m$ if $\alpha_m \leq n-1$
			$\{1,3\} \cup S_m$ if $\alpha_m = n$
v_i	$i=1$	$v_0v_1,\ v_1v_{1,j}\ \forall j \in \mathbb{N}_n$	$\{1\}$
	$2 \leq i \leq m-1$	$v_0v_i,\ v_iv_{i,j}\ \forall j \in \mathbb{N}_n$	$[\{1\} \cup S_i] \setminus \{c(v_i)\}$
	$i=m$	$v_0v_m,\ v_mv_{m,j}\ \forall j \in \mathbb{N}_n$	$[\{2\} \cup S_m] \setminus \{c(v_m)\}$
$v_iv_{i,j}$	$i=1,\ j \in \mathbb{N}_n$	$v_0v_1,\ v_{1,j},$ $v_1v_{1,q}\ \forall q \neq j$	$\{1, c(v_{1,j})\}$ where $c(v_{1,j}) \neq 1$
	$2 \leq i \leq m-1,$ $j \in \mathbb{N}_n$	$v_0v_i,\ v_{i,j},$ $v_iv_{i,q}\ \forall q \neq j$	If $c(v_iv_{i,j}) \notin S_i$: (i.e. $\alpha_i + 1 \leq j \leq n$ or $\alpha_i = 0$) $\{1, c(v_{i,j})\} \cup S_i$ where $c(v_{i,j}) \in \mathbb{N}_p \setminus S_i$
			If $c(v_iv_{i,j}) \in S_i$: (i.e. $1 \leq j \leq \alpha_i$) $\{1\} \cup [S_i \setminus \{s_j\}]$
	$i=m,\ j \in \mathbb{N}_n$	$v_0v_m,\ v_{m,j},$ $v_mv_{m,q}\ \forall q \neq j$	If $c(v_mv_{m,j}) \notin S_m$: (i.e. $\alpha_m + 1 \leq j \leq n$ or $\alpha_m = 0$) $\{2, c(v_{m,j})\} \cup S_m$ where $c(v_{m,j}) \in \mathbb{N}_p \setminus S_m$
			If $c(v_mv_{m,j}) \in S_m$: (i.e. $1 \leq j \leq \alpha_m$) $\{2\} \cup [S_m \setminus \{s_j\}]$
$v_{i,j}$	$i=1,\ j \in \mathbb{N}_n$	$v_1v_{i,j}$	$\{1\}$
	$2 \leq i \leq m-1,$ $j \in \mathbb{N}_n$	$v_iv_{i,j}$	$\{s_j\}$ if $1 \leq j \leq \alpha_i$
			$\{1\}$ if $\alpha_i + 1 \leq j \leq n$
	$i=m,\ j \in \mathbb{N}_n$	$v_mv_{m,j}$	$\{s_j\}$ if $1 \leq j \leq \alpha_m$
			$\{2\}$ if $\alpha_m + 1 \leq j \leq n$

As in the previous lemma, we prove that c can always be constructed. In fact, similar arguments allow us to conclude that: for(B2), we have $2^{p-3} \geq m$, which follows from $p \geq q(m+3, n+1, 3)$; for (B3b), we have $\alpha_i \leq n$ so that all colors in $S_i \setminus \{c(v_i)\}$ are used; and for (B4b-c), we have $|W_i| \leq |\mathbb{N}_p \setminus S_i|$ for $i \in \mathbb{N}_m \setminus \{1\}$.

We also show that c uses exactly p colors. If $p = n+1$, then by (B4a), we have $c(\{v_{1,j} : j \in \mathbb{N}_n\}) = \mathbb{N}_p \setminus \{1\}$. Since c also uses the color 1, all colors from $\mathbb{N}_p$ are used by c. Now, suppose $p = q(m+3, n+1, 3)$. Note that

$$\sum_{\alpha=0}^{\min\{(m+2)-1, n+1\}} C((m+2)-3, \alpha) \geq m$$

for any value of $n \geq 2$. By the minimality of p, we must have $p \leq m+2$ or $p-3 \leq m-1$. Thus, all 1-subsets of $\mathbb{N}_p \setminus \mathbb{N}_3$ are used in (B2). Since c also uses the colors 1, 2, 3 in (B1), all colors from $\mathbb{N}_p$ are also used by c in this case.

Finally, we present in Table 2 the neighbors and neighborhood color set of each vertex in $M(T_{m,n})$. Through Table 2, it can be easily verified that c is a set coloring. ∎

Consider $T_{4,2}$ again. Since $\max\{3, q(7, 3, 3)\} = 5$, Lemma 3.5 implies that $M(T_{4,2})$ is set 5-colorable. Figure 2 shows a set 5-coloring of $M(T_{4,2})$ constructed using (B1)-(B4) from Lemma 3.4. Note, however, that the coloring in Figure 2 is not optimal since we have previously found a set 4-coloring for $M(T_{4,2})$.

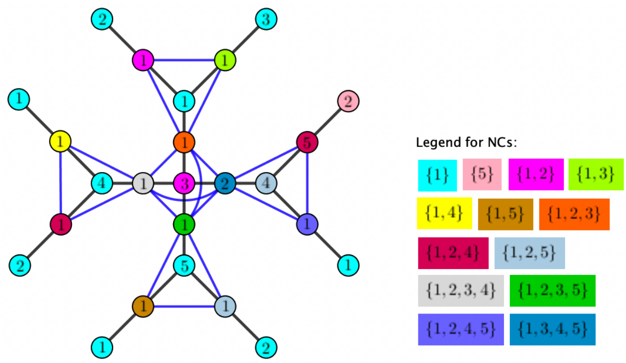

FIGURE 2. A set coloring $c : V(M(T_{4,2})) \to \mathbb{N}_5$ constructed using (B1)-(B4) from Lemma 3.5

Lemma 3.4 and Lemma 3.5 imply that, for integers $m \geq 2$ and $n \geq 1$, we have $\chi_s(M(T_{m,n})) \leq \min\{\max\{n+2, q(m+2, n+1, 1)\}, \max\{n+1, q(m+3, n+1, 3)\}\}$. Moreover, with Lemma 3.2 and our discussion on the case where $m = 1$, we have completely determined the set chromatic number of the middle graph of extended stars.

Theorem 3.6. *Let m and n be positive integers. Then $\chi_s(M(T_{1,n})) = n + 2$ while*

$$\chi_s(M(T_{m,n})) = \min\{\max\{n + 2, q(m + 2, n + 1, 1)\}, \max\{n + 1, q(m + 3, n + 1, 3)\}\}$$

for $m \geq 2$.

4. Banana Trees

A natural extension of the family of extended stars is the family of banana trees. In the literature, an (n, k)-*banana tree* is a graph obtained by connecting one leaf of each of n copies of $K_{1,k}$ with a single root vertex that is distinct from all the stars. For the sake of consistency with the previous section, we will adopt a different notation, presented below, for banana trees.

Definition 4.1. Let m, n be positive integers We define the *banana tree* $B_{m,n}$ to be the graph with vertex set $V = \{v_0\} \cup \{v_i, u_i : i \in \mathbb{N}_m\} \cup \bigcup_{i\in\mathbb{N}_m} S_i$, where $|S_i| = n$ for all $i \in \mathbb{N}_m$, and edge set $E = \{v_0v_i, v_iu_i : i \in \mathbb{N}_m\} \cup \bigcup_{i\in\mathbb{N}_m} \{u_is : s \in S_i\}$.

In this section, we completely determine the set chromatic number of $M(B_{m,n})$ for positive integers m and n. To this end, we begin by establishing the following lower bound.

Lemma 4.2. *Let $m \geq 2$ and $n \geq 1$ be integers. Then*

$$\chi_s(M(B_{m,n})) \geq \min\left\{\max\left\{\left\lceil \frac{\sqrt{8m-7}+3}{2}\right\rceil, n+1\right\}, \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2}\right\rceil, n+2\right\}\right\}.$$

Proof. Let $G = M(B_{m,n})$ and let c be a set k-coloring of G. We will consider the following two subgraphs of G:

H_1 = subgraph of G induced by the vertices v_0 and v_0v_i, for $i \in \mathbb{N}_m$,

H_2 = subgraph of G induced by v_1u_1, u_1, and u_1s for all $s \in S_1$.

Then $H_1 \cong K_{m+1}$ and $H_2 \cong K_{n+2}$. For $i \in \{1, 2\}$, we define the following:

$C_i = c(V(H_i)), \quad \ell_i = |C_i|,$

$X_i = \{x \in V(H_i) : c(x) = c(y)$ for some $y \in V(H_i) \setminus \{x\}\}$. Moreover, if $X_i \neq \emptyset$, then $|X_i| \geq |V(H_i)| - \ell_i + 1$.

Lemma 2.2 implies that $k \geq n+1$. If $X_1 = \emptyset$ (i.e. $\ell_1 = m+1$), then $k \geq m + 1 \geq \frac{\sqrt{8m-7}+3}{2}$. Then $k \geq \max\left\{\frac{\sqrt{8m-7}+3}{2}, n+1\right\}$. Now, we assume that $X_1 \neq \emptyset$ (i.e. $1 \leq \ell_1 \leq m$).

We have the following possible neighborhood color sets for vertices in X_1.

(1a) If $v_0 \in X_1$, then $NC(v_0) = C_1$.

(1b) If $v_0v_i \in X_1$, $i \in \mathbb{N}_m$, then $NC(v_0v_i) = C_1 \cup \{c(v_i), c(v_iu_i)\}$.

Similarly, we have the following possible neighborhood color sets for vertices in X_2.

(2a) If $v_1u_1 \in X_2$, then $NC(v_1u_1) = C_2 \cup \{c(v_1), c(v_0v_1)\}$.

(2b) If $u_1 \in X_2$, then $NC(u_1) = C_2$.

(2c) If $u_1s \in X_2$, $s \in S_1$, then $NC(u_1s) = C_2 \cup \{c(s)\}$.

Given (1a)-(1b) and without other restrictions, the maximum number of possible neighborhood color sets for vertices in X_1 is $C(k-\ell_1, 0) + C(k-\ell_1, 1) + C(k-\ell_2, 2)$. Since $|X_1| \geq m+2-\ell_1$ and vertices in X_1 must have distinct neighborhood color sets, we must have

$$C(k-\ell_1, 0) + C(k-\ell_1, 1) + C(k-\ell_2, 2) \geq m + 2 - \ell_1,$$

which implies that

$$k \geq \frac{\sqrt{8(m-\ell_1)+9} + 2\ell_1 - 1}{2}. \tag{4.1}$$

We now proceed by cases depending on the value of ℓ_1.

Case 1. Suppose $\ell_1 = 1$. Then $X_1 = V(H_1)$ and we may assume that $C_1 = \{1\}$.

Case 1.1. Suppose c has the property that for all $i \in \mathbb{N}_m$, $|\{c(v_0v_i), c(v_i), c(v_iu_i)\}| = 3$. Thus, for all $i \in \mathbb{N}_m$, we have $1 \notin \{c(v_i), c(v_iu_i)\}$ and $|\{c(v_i), c(v_iu_i)\}| = 2$. Moreover, since c is a set coloring, we must have $\{c(v_i), c(v_iu_i)\} \neq \{c(v_j), c(v_ju_j)\}$ for all distinct $i, j \in \mathbb{N}_m$. It follows that $C(k-1, 2) \geq m$, which implies that $k \geq \frac{\sqrt{8m+1}+3}{2} \geq \frac{\sqrt{8m-7}+3}{2}$. Therefore, $k \geq \max\left\{\frac{\sqrt{8m-7}+3}{2}, n+1\right\}$.

Case 1.2. Suppose, without loss of generality, that $|\{c(v_0v_1), c(v_1), c(v_1u_1)\}| \leq 2$. If $X_2 = \emptyset$, then $k \geq n+2$. Now, suppose $X_2 \neq \emptyset$. It follows that if $v_1u_1 \in X_2$, then $NC(v_1u_1)$ has at most $\ell_2 + 1$ elements, as the neighborhood color sets in (2c) have. Thus, the maximum number of possible neighborhood color sets for vertices in X_2 is $1 + k - \ell_2$. Since vertices in X_2 must have distinct neighborhood color sets, we must have $1 + k - \ell_2 \geq |X_2| \geq n + 3 - \ell_2$, which implies that $k \geq n+2$. Therefore, with (4.1) and $\ell_1 = 1$, we have $k \geq \max\left\{\frac{\sqrt{8m+1}+1}{2}, n+2\right\}$.

Case 2. Suppose $\ell_1 \geq 2$. Note that, in this case, the right-hand side of (4.1) achieves its minimum when $\ell_1 = 2$. Thus, (4.1) and Lemma 2.2 imply $k \geq \max\left\{\frac{\sqrt{8m-7}+3}{2}, n+1\right\}$.

Therefore, we have $k \geq \min\left\{\max\left\{\frac{\sqrt{8m-7}+3}{2}, n+1\right\}, \max\left\{\frac{\sqrt{8m+1}+1}{2}, n+2\right\}\right\}$. Since k must be an integer, the desired conclusion follows. ∎

Let us now briefly discuss the banana trees $B_{1,n}$ and $B_{2,n}$. First, we define a *double-star graph* to be a tree containing exactly two non-pendant vertices; moreover, we denote by $S_{\alpha,\beta}$ the double-star graph with degree sequence $(\alpha+1, \beta+1, 1, ..., 1)$. Clearly, the banana tree $B_{1,n}$ is isomorphic to $S_{n,1}$ for $n \geq 1$. Thus, by Proposition 4.1 in [14], $\chi_s(M(B_{1,1})) = 3$ while $\chi_s(M(B_{1,n})) = n+1$ for $n \geq 2$.

Now, let us consider $B_{2,n}$. For positive integers s, t_1, t_2 with $t_1 \geq t_2$, we define the *double broom* DB_{s,t_1,t_2} to be the graph obtained by identifying one vertex of the path P_s to the central vertex of the star K_{1,t_1} and identifying the other endvertex of P_s to the central vertex of K_{1,t_2}. Clearly, $B_{2,n}$ is isomorphic to $DB_{5,n,n}$ for $n \geq 1$. Thus, by Theorem 2.2 in [17], we have $\chi_s(M(B_{2,1})) = 3$ while $\chi_s(M(B_{2,n})) = n+1$ for $n \geq 2$.

We observe that for $m \leq 2$ and $n \geq 1$, except when $(m,n) = (1,1)$, the lower bound in Lemma 4.2 actually coincides with the set chromatic number of $M(B_{m,n})$.

We now algorithmically construct set colorings for $M(B_{m,n})$ for $m \geq 3$ and $n \geq 1$.

Lemma 4.3. *Let $m \geq 3$ and $n \geq 1$ be integers. Set $p = \max\left\{\left\lceil\frac{\sqrt{8m-7}+3}{2}\right\rceil, n+1\right\}$. Then $M(B_{m,n})$ is set p-colorable.*

Proof. Note that $p \geq 4$ since $m \geq 3$. Let $T_1, T_2, ..., T_{p-2}$ be the 1-subsets of $\{3, 4, ..., p\}$ and $T_{p-1}, T_p, ..., T_{C(p-2,1)+C(p-2,2)}$ be the 2-subsets of $\{3, 4, ..., p\}$. We construct a coloring $c : V(M(B_{m,n})) \to \mathbb{N}_p$ as follows:

- (A1) Set $c(v_0) = 1$, $c(v_0v_i) = 1$ for all $i \in \mathbb{N}_{m-1}$, and $c(v_0v_m) = 2$.
- (A2) For each $i \in \mathbb{N}_{\min\{p-2,m-1\}}$:
  - (a) Set $c(v_i) = 2$ and $\{c(v_iu_i)\} = T_i$.
  - (b) Let $r_i \in \mathbb{N}_p \setminus \{1, 2, c(v_iu_i)\}$ and set $c(u_i) = r_i$.
  - (c) Set $c(u_is) = c(v_iu_i)$ for all $s \in S_i$.
  - (d) Set $c(S_i) \subseteq \mathbb{N}_p \setminus \{r_i\}$ so that the vertices in S_i receive different colors.
- (A3) Suppose $p \leq m$. For each $i \in \{p-1, ..., m-1\}$:
  - (a) Set $c(\{v_i, v_iu_i\}) = T_i$.
  - (b) Set $c(u_i) = c(v_iu_i)$ and $c(u_is) = c(v_iu_i)$ for all $s \in S_i$.
  - (c) Set $c(S_i) \subseteq \mathbb{N}_p \setminus \{c(v_iu_i)\}$ so that vertices in S_i receive different colors.
- (A4) For $i = m$:
  - (a) Set $c(v_m) = 1$ and $c(v_mu_m) = 3$.
  - (b) Set $c(u_m) = 3$ and $c(u_ms) = 3$ for all $s \in S_m$.
  - (c) Set $c(S_m) \subseteq \mathbb{N}_p \setminus \{3\}$ so that vertices in S_m receive different colors.

We show that c can always be constructed. For (A2a) and (A3a), we need $C(p-2,1) + C(p-2,2) \geq m-1$, which follows from $p \geq \frac{\sqrt{8m-7}+3}{2}$. For (A2b), we need $p \geq 4$, which follows from $m \geq 3$. For (A2d), (A3c), and (A4c), we need $p-1 \geq n$ or $p \geq n+1$, which is as assumed.

We now show that c uses exactly p colors. First, suppose $p = n+1$. Thus, $p-1 = n = |c(S_1)|$, which implies $c(S_1) = \mathbb{N}_p \setminus \{r_1\}$ by (A2d). Since $c(u_1) = r_1$, all colors from $\mathbb{N}_p$ are used. Now, let us suppose that $p = \left\lceil\frac{\sqrt{8m-7}+3}{2}\right\rceil$. Since $\frac{\sqrt{8m-7}+3}{2} \leq m+1$ for $m \geq 2$, we have $p \le m+1$; that is, $p-2 \le m-1$. Thus, in (A2), all 1-subsets of $\{3,4,\dots,p\}$ are used. Moreover, since the colors 1 and 2 are used in (A1), then c also uses all the colors from $\mathbb{N}_p$ in this case.

Finally, we present in Table 3 the neighbors and neighborhood color set of each vertex in $M(B_{m,n})$. Through Table 3, it can be easily verified that c is a set coloring. ∎

TABLE 3. The neighbors and neighborhood color set of each vertex in $M(B_{m,n})$ under the coloring c (Lemma 4.3)

Vertex	Neighbors	NC	
v_0	$v_0v_i, i \in \mathbb{N}_m$	$\{1,2\}$	
$v_0v_i, i \in \mathbb{N}_m$	v_0, v_0v_j $(i \neq j)$, v_i, v_iu_i	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	$\{1,2\} \cup T_i$ $\{1,2\} \cup T_i$ $\{1,3\}$
$v_i, i \in \mathbb{N}_m$	v_0v_i, v_iu_i	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	$\{1\} \cup T_i$ $\{1\} \cup [T_i \setminus \{c(v_i)\}]$ $\{2,3\}$
$v_iu_i, i \in \mathbb{N}_m$	v_0v_i, v_i, u_i, u_is for $s \in S_i$	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	$\{1,2,r_i\} \cup T_i$ $\{1\} \cup T_i$ $\{1,2,3\}$
$u_i, i \in \mathbb{N}_m$	v_iu_i, u_is for $s \in S_i$	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	T_i $\{c(v_iu_i)\} = T_i \setminus \{c(v_i)\}$ $\{3\}$
$u_is, i \in \mathbb{N}_m$ & $s \in S_i$	v_iu_i, u_i, s, u_it for $t \in S_i \setminus \{s\}$	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	$T_i \cup \{r_i, c(s)\}$ $\{c(v_iu_i), c(s)\}$ $\{3, c(s)\}$
$s \in S_i, i \in \mathbb{N}_m$	u_is	$i \in \mathbb{N}_{\min\{p-2,m-1\}}$: $i \in \{p-1,\dots,m-1\}$: $i = m$:	$\{c(v_iu_i)\}$ $\{c(v_iu_i)\}$ $\{3\}$

As an example, consider $B_{4,3}$. Since $\max\left\{\left\lceil \frac{\sqrt{25}+3}{2} \right\rceil, 4\right\} = 4$, then $M(B_{4,3})$ is set 4-colorable by Lemma 4.3. In Figure 3, we present the set coloring $c : V(M(B_{4,3})) \to \mathbb{N}_4$ constructed using the algorithm (A1)-(A4) from Lemma 4.3.

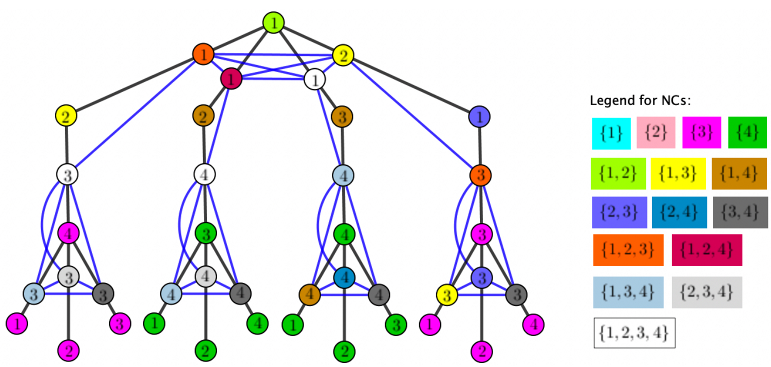

FIGURE 3. A set coloring $c : V(M(B_{4,3})) \to \mathbb{N}_4$ constructed using the algorithm (A1)-(A4) from Lemma 4.3

Lemma 4.4. *Let $m \geq 3$ and $n \geq 1$ be integers. Set $p = \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil, n+2\right\}$. Then $M(B_{m,n})$ is set p-colorable.*

Proof. Note that $p \geq 4$ since $m \geq 3$. Let $T_1, T_2, ..., T_{p-1}$ be the 1-subsets of $\{2,3,...,p\}$ and $T_p, T_{p+1}, ..., T_{C(p-1,1)+C(p-1,2)}$ be the 2-subsets of $\{2,3,...,p\}$. We construct a coloring $c : V(M(B_{m,n})) \to \mathbb{N}_p$ as follows:

- (B1) Set $c(v_0) = 1$ and $c(v_0v_i) = 1$ for all $i \in \mathbb{N}_m$.
- (B2) For each $i \in \mathbb{N}_{\min\{p-1,m\}}$:
  - (a) Set $\{c(v_i)\} = T_i$ and $c(v_iu_i) = 1$.
  - (b) Let $r_i \in \mathbb{N}_p \setminus [\{1\} \cup T_i]$ and set $c(u_i) = r_i$.
  - (c) Set $c(u_is) = 1$ for all $s \in S_i$.
  - (d) Set $c(S_i) \subseteq \mathbb{N}_p \setminus [T_i \cup \{r_i\}]$ so that the vertices in S_i receive different colors.
- (B3) Suppose $p \leq m$. For each $i \in \{p, ..., m\}$:
  - (a) Set $c(\{v_i, v_iu_i\}) = T_i$.
  - (b) Set $c(u_i) = c(v_i)$ and $c(u_is) = c(v_i)$ for all $s \in S_i$.
  - (c) Set $c(T_i) \subseteq \mathbb{N}_p \setminus S_i$ so that vertices in S_i receive different colors.

We show that c can always be constructed. For (B2a) and (B3a), we need $C(p-1,1)+C(p-1,2) \geq m$, which follows from $p \geq \frac{\sqrt{8m+1}+1}{2}$. For (B2b), we need $p \geq 3$, which follows from $m \geq 3$. For (B2d) and (B3c), we need $p-2 \geq n$ or $p \geq n+2$, which is as assumed.

We now show that c uses exactly p colors. We proceed by cases. First, suppose $p = n+2$. Thus, $p-2 = n = |c(S_1)|$, which implies $c(S_1) = \mathbb{N}_p \setminus [T_1 \cup \{r_1\}]$ by (B2d). Since $c(u_1) = r_1$ and $\{c(v_1)\} = T_1$, all colors from $\mathbb{N}_p$ are used. Now, let us suppose that $p = \left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil$. Since $\frac{\sqrt{8m+1}+1}{2} \leq m+1$ for $m \geq 1$, we have $p \leq m+1$; that is, $p-1 \leq m$. Thus, in (B2), all 1-subsets of $\{2,3,...,p\}$ are used. Moreover, since the color 1 is used in (A1), then c also uses all the colors from $\mathbb{N}_p$ in this case.

Finally, we present in Table 4 the neighbors and neighborhood color set of each vertex in $M(B_{m,n})$. Through Table 4, it can be easily verified that c is a set coloring. ∎

Table 4. The neighbors and neighborhood color set of each vertex in $M(B_{m,n})$ under the coloring c (Lemma 4.4)

Vertex	Neighbors	NC	
v_0	$v_0v_i, i \in \mathbb{N}_m$	$\{1\}$	
$v_0v_i, i \in \mathbb{N}_m$	v_0, v_0v_j $(i \neq j)$, v_i, v_iu_i	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1\} \cup T_i$ $\{1\} \cup T_i$
$v_i, i \in \mathbb{N}_m$	v_0v_i, v_iu_i	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1\}$ $\{1\} \cup [T_i \setminus \{c(v_i)\}]$
$v_iu_i, i \in \mathbb{N}_m$	v_0v_i, v_i, u_i, u_is for $s \in S_i$	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1, r_i\} \cup T_i$ $\{1\} \cup [T_i \setminus \{c(v_iu_i)\}]$
$u_i, i \in \mathbb{N}_m$	v_iu_i, u_is for $s \in S_i$	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1\}$ T_i
$u_is, i \in \mathbb{N}_m$ & $s \in S_i$	v_iu_i, u_i, s, u_it for $t \in S_i \setminus \{s\}$	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1, r_i, c(s)\}$ $T_i \cup \{c(s)\}$
$s \in S_i, i \in \mathbb{N}_m$	u_is	$i \in \mathbb{N}_{\min\{p-1,m\}}$: $i \in \{p, ..., m\}$:	$\{1\}$ $\{c(v_i)\}$

We consider $B_{4,3}$ again. Since $\max\left\{\left\lceil\frac{\sqrt{33}+1}{2}\right\rceil, 5\right\} = 5$, then $M(B_{4,3})$ is set 5-colorable by Lemma 4.4. In Figure 3, we present the set coloring $c : V(M(B_{4,3})) \to \mathbb{N}_5$ constructed using the algorithm (B1)-(B3) from Lemma 4.4. Note, however, that the coloring in Figure 4 is not optimal since we have previously found a set 4-coloring for $M(B_{4,3})$.

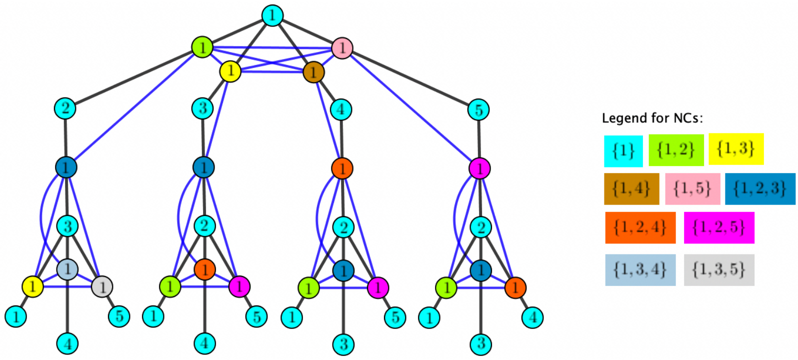

FIGURE 4. A set coloring $c : V(M(B_{4,3})) \to \mathbb{N}_5$ constructed using the algorithm (B1)-(B3) from Lemma 4.4

Lemma 4.3 and Lemma 4.4 imply that, for integers $m \geq 3$ and $n \geq 1$, we have $\chi_s(M(B_{m,n})) \geq \min\left\{\max\left\{\left\lceil\frac{\sqrt{8m-7}+3}{2}\right\rceil, n+1\right\}, \max\left\{\left\lceil\frac{\sqrt{8m+1}+1}{2}\right\rceil, n+2\right\}\right\}$. Moreover, with Lemma 4.2 and our discussion for the cases where $m \leq 2$, we have the following result.

Theorem 4.5. *Let m and n be positive integers. Then $\chi_s(M(B_{1,1})) = 3$ while for $(m,n) \neq (1,1)$, we have*

$$\chi_s(M(B_{m,n})) = \min\left\{\max\left\{\left\lceil\frac{\sqrt{8m-7}+3}{2}\right\rceil, n+1\right\}, \max\left\{\left\lceil\frac{\sqrt{8m+1}+1}{2}\right\rceil, n+2\right\}\right\}.$$

Extended Banana Trees

We conclude this section by considering an extension of banana trees. Let p, m, n be positive integers. Suppose $H_1, H_2, ..., H_m$ are m copies of the path P_{p+2} and $J_1, J_2, ..., J_m$ are m copies of the star $K_{1,n}$. Moreover, suppose $H_i = v_{i,1}v_{i,2}\cdots v_{i,p+2}$ for $i \in \mathbb{N}_m$. We define the graph $BT_{p,m,n}$ to be the graph obtained by identifying all the vertices $v_{i,1}$, $i \in \mathbb{N}_m$, to be a single vertex v_0 and then, for each $i \in \mathbb{N}_m$, identifying $v_{i,p+2}$ to the central vertex of J_i. The graph $BT_{3,4,4}$ is shown in Figure 5.

For positive integers s and t, we define the *broom* $Brm_{s,t}$ to be the graph obtained by identifiying an endvertex of the path P_s and the central vertex of the star $K_{1,t}$. It is clear that $BT_{p,1,n}$ is isomorphic to $Brm_{p+2,n}$. In fact, when $n = 1$, $BT_{p,1,1}$ is isomorphic to the path P_{p+3}. Thus, by Proposition 3.4 in [14] and Theorem 2.1 in [17], we have $\chi_s(M(BT_{p,1,1})) = 3$ while $\chi_s(M(BT_{p,1,n})) = n + 1$ for $n \geq 2$.

When $m = 2$, we observe that $BT_{p,2,n}$ is isomorphic to the double broom $DB_{2p+3,n,n}$. Moreover, $BT_{p,2,1}$ is isomorphic to the path P_{2p+5}. Thus, by Proposition 3.4 in [14] and Theorem 2.2 in [17], we have $\chi_s(M(BT_{p,2,1})) = 3$ while $\chi_s(M(BT_{p,2,n})) = n+1$ for $n \geq 2$.

Now, $BT_{1,m,n}$ is isomorphic to the banana tree $B_{m,n}$. Thus, combined with the preceding discussion, we may assume that $p \geq 2$ and $m \geq 3$. Using a similar argument as in Lemma 4.2, we obtain the following lower bound.

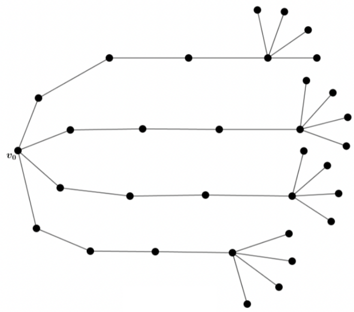

FIGURE 5. The graph $BT_{3,4,4}$

Lemma 4.6. *Let $p \geq 2$ and $m \geq 3$ be integers. Then*

$$\chi_s(M(BT_{p,m,n})) \geq \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil, n+1\right\}.$$

As an example, consider $BT_{2,3,1}$. We have $\chi_s(M(BT_{2,3,1})) \geq \max\left\{\left\lceil \frac{\sqrt{25}+1}{2} \right\rceil, 2\right\} = 3$, by Lemma 4.6. Moreover, Figure 6 shows a set 3-coloring of $M(BT_{2,3,1})$. Therefore, $\chi_s(M(BT_{2,3,1})) = 3$.

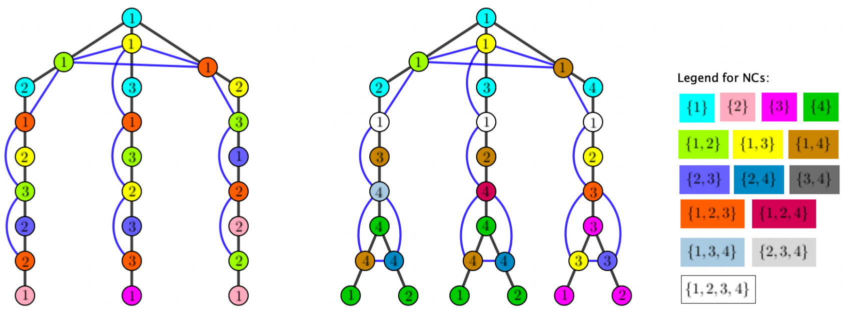

FIGURE 6. A set 3-coloring of $M(BT_{2,3,1})$ (left) and a set 4-coloring of $M(BT_{2,3,2})$ (right)

On the other hand, consider $BT_{2,3,2}$, for which $\chi_s(M(BT_{2,3,2})) \geq \max\left\{\left\lceil \frac{\sqrt{25}+1}{2} \right\rceil, 3\right\} = 3$, by Lemma 4.6. However, it can be easily shown that $M(BT_{2,3,2})$ does not have a set 3-coloring but has a set 4-coloring as shown in Figure 6. Therefore, $\chi_s(M(BT_{2,3,2})) = 4$.

In fact, the inequality in Lemma 4.6 is actually an equality except only for the case $(p, m, n) = (2, 3, 2)$. This will be shown through the following two lemmas, where we construct optimal set colorings of $M(BT_{p,m,n})$. The first lemma is for the case where $p = 2$.

Lemma 4.7. *Let $m \geq 3$ and $n \geq 1$ be integers such that $(m, n) \neq (3, 2)$. Set $k = \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil, n+1\right\}$. Then $M(BT_{2,m,n})$ is set k-colorable.*

Proof. The case $(m, n) = (3, 1)$ has been discussed earlier. So we only consider the following cases: $(m = 3; n \geq 3)$ and $(m \geq 4; n \geq 1)$. In either case, we have $k \geq 4$. Suppose

$$V(BT_{2,m,n}) = \{v_0\} \cup \{v_i, u_i, w_i : i \in \mathbb{N}_m\} \cup \bigcup_{i \in \mathbb{N}_m} S_i,$$

where $|S_i| = n$ for all $i \in \mathbb{N}_m$, and with

$$E(BT_{2,m,n}) = \{v_0 v_i, v_i u_i, u_i w_i : i \in \mathbb{N}_m\} \cup \bigcup_{i \in \mathbb{N}_m} \{w_i s : s \in S_i\}.$$

Recall that $V(M(BT_{2,m,n})) = V(BT_{2,m,n}) \cup E(BT_{2,m,n})$. Let $T_1, T_2, ..., T_{k-1}$ be the 1-subsets of $\{2, 3, ..., k\}$ and let $T_k, T_{k+1}, ..., T_{C(k-1,1)+C(k-1,2)}$ be the 2-subsets of $\{2, 3, ..., k\}$. We now construct a coloring $c : V(M(BT_{2,m,n})) \to \mathbb{N}_k$ as follows:

(C1) Set $c(v_0) = 1$ and $c(v_0 v_i) = 1$ for all $i \in \mathbb{N}_m$.

(C2) For each $i \in \mathbb{N}_{\min\{k-1,m\}}$:
- (a) Set $\{c(v_i)\} = T_i$ and $c(v_i u_i) = 1$.
- (b) Let $r_i \in \mathbb{N}_k \setminus [\{1\} \cup T_i]$. Set $c(u_i) = r_i$.
- (c) Let $z_i \in \mathbb{N}_k \setminus [\{1, r_i\} \cup T_i]$. Set $c(u_i w_i)$, $c(w_i)$, and $c(w_i s)$, for all $s \in S_i$, to be all equal to z_i.
- (d) Set $c(S_i) \subseteq \mathbb{N}_k \setminus \{z_i\}$ so that the vertices in S_i receive different colors.

(C3) Suppose $k \leq m$. Then for each $i \in \{k, ..., m\}$:
- (a) Set $c(\{v_i, v_i u_i\}) = T_i$.
- (b) Set $c(u_i) = 1$.
- (c) Set $c(u_i w_i), c(w_i)$, and $c(w_i s)$, for all $s \in S_i$, to be all equal to $c(v_i)$.
- (d) Set $c(S_i) \subseteq \mathbb{N}_k \setminus \{c(v_i)\}$ so that the vertices in S_i receive different colors.

Similar to earlier proofs, it can be easily verified that c is a set k-coloring. ∎

As an example, consider $BT_{2,4,3}$. Since $\max\left\{\left\lceil \frac{\sqrt{33}+1}{2} \right\rceil, 4\right\} = 4$, then $M(BT_{2,4,3})$ is set 4-colorable by Lemma 4.7. In Figure 7, we present the set coloring $c : V(M(BT_{2,4,3})) \to \mathbb{N}_4$ constructed using the algorithm (C1)-(C3) from Lemma 4.7.

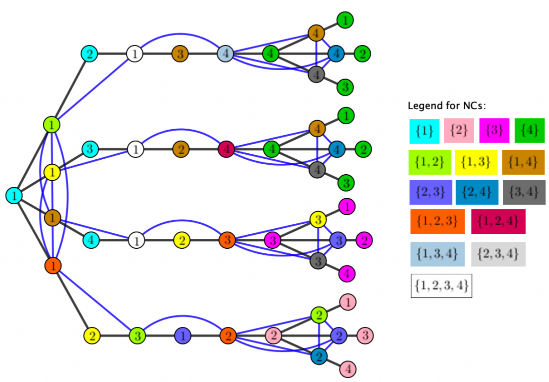

FIGURE 7. A set coloring $c : V(M(BT_{2,4,3})) \to \mathbb{N}_4$ constructed using the algorithm (C1)-(C3) from Lemma 4.7

We now construct optimal set colorings of $M(BT_{p,m,n})$ for the general case $p \geq 3$.

Lemma 4.8. *Let $p \geq 3$, $m \geq 3$, and $n \geq 1$ be integers. Set $k = \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil, n+1\right\}$. Then $M(BT_{p,m,n})$ is set k-colorable.*

Proof. We will use the vertex labels shown in Figure 8 to refer to the vertices of $M(BT_{p,m,n})$. In Figure 8, each S_i, $i \in \mathbb{N}_m$, is a set of n pendant vertices. Note that $M(BT_{p,m,n})$ also has vertices of the form $v_0 x_{i,1}$, for $i \in \mathbb{N}_m$, and vertices of the form $w_i s$, for $i \in \mathbb{N}_m$ and $s \in S_i$.

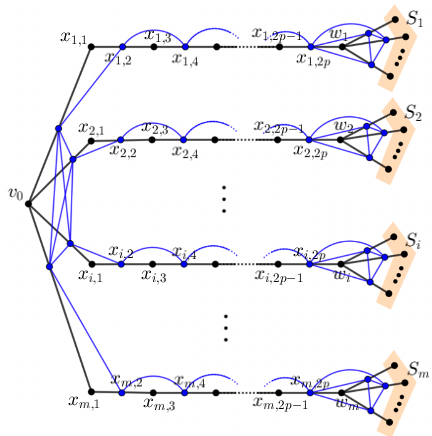

FIGURE 8. Vertex labels of $M(BT_{p,m,n})$

Let $T_1, T_2, ..., T_{C(k-1,1)+C(k-1,2)}$ be the 1-subsets and 2-subsets of $\{2, 3, ..., k\}$. We now construct a coloring $c : V(M(BT_{p,m,n})) \to \mathbb{N}_k$ as follows:

(D1) Set $c(v_0) = 1$ and $c(v_0 x_{i,1}) = 1$ for all $i \in \mathbb{N}_m$.

(D2) For each $i \in \mathbb{N}_m$:

(a) If $|T_i| = 1$, let $\alpha_i \in T_i$, $\beta_i = 1$, and $\gamma_i \in \mathbb{N}_k \setminus \{\alpha_i, \beta_i\}$.
On the other hand, if $|T_i| = 2$, let $\alpha_i \in T_i$, $\beta_i \in T_i \setminus \{\alpha_i\}$, and $\gamma_i = 1$.

(b) For each $j \in \mathbb{N}_{2p}$, set

$$c(x_{i,j}) = \begin{cases} \alpha_i, & j \equiv 1 \pmod 3, \\ \beta_i, & j \equiv 2 \pmod 3, \\ \gamma_i, & j \equiv 0 \pmod 3. \end{cases}$$

(c) Set $c(w_i) = c(x_{i,2p})$ and $c(w_i s) = c(x_{i,2p})$ for all $s \in S_i$.

(d) Set $c(S_i) \subseteq \mathbb{N}_k \setminus \{c(x_{i,2p})\}$ so that the vertices in S_i receive different colors.

Similar to earlier proofs, it can be easily verified that c is a set k-coloring. ∎

With Lemma 4.6, Lemma 4.7, and Lemma 4.8, as well as the discussion on the cases where $m \leq 2$, we have the following result.

Theorem 4.9. *Let $p \geq 2, m \geq 1, n \geq 1$ be integers. Then*

$$\chi_s(M(BT_{p,m,n})) = \begin{cases} 3, & \text{if } m = n = 1, \\ 4, & \text{if } (p,m,n) = (2,3,2), \\ \max\left\{\left\lceil \frac{\sqrt{8m+1}+1}{2} \right\rceil, n+1\right\}, & \text{otherwise.} \end{cases}$$

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