Thai Journal of **Math**ematics
Volume 21 Number 4 (2023)
Pages 821–834

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Discrete and Computational Geometry, Graphs, and Games

On the Rainbow Mean Indexes of Caterpillars

Agnes D. Garciano, Reginaldo M. Marcelo*, Mari-Jo P. Ruiz and Mark Anthony C. Tolentino

Ateneo de Manila University, Quezon City, Philippines
e-mail : agarciano@ateneo.edu (A.D. Garciano); rmarcelo@ateneo.edu (R.M. Marcelo); mruiz@ateneo.edu (M.P. Ruiz); mtolentino@ateneo.edu (M.A.C. Tolentino)

Abstract Let G be a simple connected graph and c an edge coloring with colors that are positive integers. Given a vertex v of G, we define its **chromatic mean**, denoted by $\mathrm{cm}(v)$, as the average of the colors of the incident edges. If $\mathrm{cm}(v)$ is an integer for each $v \in V(G)$ and distinct vertices have distinct chromatic means, then c is called a **rainbow mean coloring**. The maximum chromatic mean of a vertex in the coloring c is called the **rainbow mean index** of c and is denoted by $\mathrm{rm}(c)$. On the other hand, the **rainbow mean index** of G , denoted by $\mathrm{rm}(G)$, is the minimum value of $\mathrm{rm}(c)$ among all rainbow mean colorings c of G. In this paper, we determine the rainbow mean indexes of families of caterpillars, including brooms, and double brooms.

MSC: 05C15; 05C78
Keywords: rainbow mean coloring; rainbow mean index; caterpillars; brooms; double stars; double brooms

Submission date: 31.12.2021 / Acceptance date: 23.07.2023

1. Introduction

Let $G = (V, E)$ be a simple connected graph and c an edge coloring with colors that are positive integers. For each vertex v of G, let E_v denote the set of all edges of G that are incident with v. The **chromatic mean** of v denoted by $\mathrm{cm}_c(v)$, or simply $\mathrm{cm}(v)$, is the average of the colors of the edges incident to v:

$$\mathrm{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg(v)}.$$

If $\mathrm{cm}(v)$ is an integer for each $v \in V(G)$ and distinct vertices have distinct chromatic means, then c is called a **rainbow mean coloring**. The maximum chromatic mean of a vertex in the coloring c is called the **rainbow mean index** of c and is denoted by $\mathrm{rm}(c)$. On the other hand, the **rainbow mean index** of G, denoted by $\mathrm{rm}(G)$, is the minimum value of $\mathrm{rm}(c)$ among all rainbow mean colorings c of G. Thus, $\mathrm{rm}(G) \geq n$ for a graph G of order $n \geq 3$.

*Corresponding author.

Published by The Mathematical Association of Thailand.

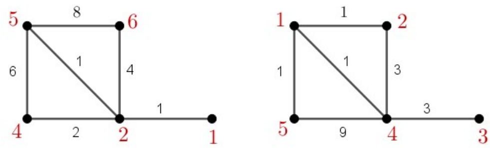


FIGURE 1. Rainbow mean colorings with mean indexes 6 and 5

Fig. 1 shows a graph G and two of its rainbow mean colorings with mean indexes 6 and 5. Hence it follows that $\mathrm{rm}(G) = 5$.

The rainbow mean coloring of a graph is an example of a vertex-distinguishing coloring which has received increased attention in the last few decades [1]. Chartrand et al. [2] introduced the rainbow mean coloring in 2019. They determined the rainbow mean indexes of paths, cycles, complete graphs, and stars. In [3], they provided a coloring for paths to support their result in [2]. Meanwhile, Hallas et al. [4] determined the rainbow mean indexes of several bipartite graphs including prisms, hypercubes, and complete bipartite graphs. In [5], the same authors determined the rainbow mean indexes of double stars, cubic caterpillars of even order, and the subdivision graph of stars.

The following results, among others, are proved in [2], [3] , and [6]. We assume that $n \geq 3$.

Theorem 1.1. [2, 6] *Every connected graph of order n has a rainbow mean coloring.*

Theorem 1.2. [2, 3] *If P_n is the path with order $n \neq 4$, then $rm(P_n) = n$. Moreover, if n is odd, then there is a rainbow mean coloring c with $rm(c) = n$ such that the end vertices of P_n have chromatic means 1 and n.*

Theorem 1.3. [2, 6] *If n is odd, then $rm(K_{1,n-1}) = n$. Moreover, there is a rainbow mean coloring c of $K_{1,n-1}$ for which the vertex of maximum degree has chromatic mean $\frac{n+1}{2}$.*

Theorem 1.4. [2, 6] *If G is connected of order $n \equiv 2 \pmod 4$ and all its vertices have odd degree, then $rm(G) \geq n + 1$.*

They also posed the following conjecture:

Conjecture: [2] *If G is connected with order $n \geq 3$, then $n \leq rm\,(G) \leq n + 2$.*

Illustrations of rainbow mean colorings of paths and stars referred to in Theorems 1.2 and 1.3 are shown in Fig. 2. Observe that in a rainbow mean coloring c of a graph, any pendant vertex v has $\mathrm{cm}(v) = c\,(e)$ where e is the unique edge incident with v.

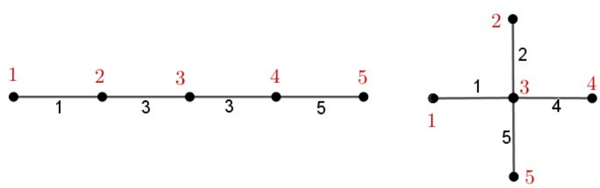


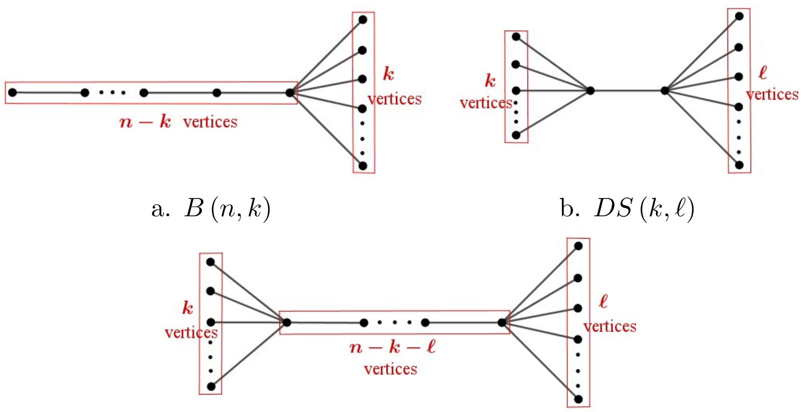
FIGURE 2. Rainbow mean colorings of P_5 and $K_{1,4}$.

2. Preliminaries

In this section, we derive some general results that will be used later. We also define the graphs that will be considered in the succeeding sections. From hereon, we assume that all graphs considered are simple and connected with order at least 3.

Let n and k be positive integers with $k \geq 2$ and $n-k \geq 3$. Let P_{n-k} be a path of order $n-k$ with terminal vertex u, and $K_{1,k}$ the star with central vertex v. Then the graph of order n obtained from P_{n-k} and $K_{1,k}$ by identifying the vertices u and v is called a **broom** and is denoted by $B(n,k)$ (Fig. 3a). We call the path P_{n-k} (as a subgraph of $B(n,k)$) the **handle** of the broom, with initial vertex the **tip** of the broom.

Now, let n, k, and ℓ be positive integers where $n \geq k+\ell+2$, $k \geq 2$, and $\ell \geq 2$. Let $P_{n-k-\ell}$ be a path with order $n-k-\ell$ and end vertices u and v. We form a graph G of order n by identifying u with the central vertex of $K_{1,k}$ and v with the central vertex of $K_{1,\ell}$. If $n-k-\ell = 2$, then G is called a **double star** and is denoted by $DS(k,\ell)$ (Fig. 3b). If $n-k-\ell > 2$, then G is called a **double broom** and is denoted by $DB(n,k,\ell)$ (Fig. 3c). We will also call the path $P_{n-k-\ell}$ (as a subgraph) the **handle** of the double broom.



a. $B(n,k)$ b. $DS(k,\ell)$

c. $DB(n,k,\ell)$

Figure 3. The broom, double star, and double broom.

For convenience, we call a rainbow mean coloring c of a graph G **optimal** if $\mathrm{rm}(c) = \mathrm{rm}(G)$. If G has order n and $\mathrm{rm}(G) = n$, then we say G is **Type 1** as in [4]; in this case, we will also call an optimal rainbow mean coloring a **Type 1 rainbow mean coloring**.

We present the following results which will be used later.

Lemma 2.1. *Let m be any positive integer and G a graph with rainbow mean coloring c. Let c' be an edge coloring of G defined by $c'(e) = c(e) + m$, for each edge $e \in E(G)$. Then c' is a rainbow mean coloring of G. Moreover, $rm(c') = rm(c) + m$.*

Proof. The result holds since

$$\mathrm{cm}_{c'}(v) = \frac{\sum_{e \in E_v} c'(e)}{\deg(v)} = \frac{\sum_{e \in E_v} (c(e)+m)}{\deg(v)} = \frac{\sum_{e \in E_v} c(e)}{\deg(v)} + m = \mathrm{cm}_c(v) + m.$$

∎

Lemma 2.2. *For $i = 1$ and 2, let G_i be a Type 1 graph with order n_i, and c_i an optimal rainbow mean coloring of G_i. Let u_i be in $V(G_i)$ such that $cm_1(u_1) = n_1$ and $cm_2(u_2) = 1$, where $cm_i(v)$ is the chromatic mean of $v \in V(G_i)$ relative to the coloring c_i. If G is the graph obtained from G_1 and G_2 by identifying vertices u_1 and u_2, then G is Type 1.*

Proof. Define c on $E(G)$ as follows:

$$c(e) = \begin{cases} c_1(e), & \text{if } e \in E(G_1), \\ c_2(e) + (n_1 - 1), & \text{if } e \in E(G_2). \end{cases}$$

Let u be the vertex obtained when u_1 and u_2 are identified in G, $d_i = \deg_{G_i}(u_i)$, for $i = 1$ or 2. For $v \in V(G)$, let $\mathrm{cm}(v)$ be the chromatic mean relative to the coloring c and $\mathrm{cm}_i(v)$ the chromatic mean of $v \in V(G_i)$ relative to the coloring c_i, $i = 1, 2$. Then,

$$\{\mathrm{cm}(v) : v \in V(G) - \{u\}\} = \{1, 2, \ldots, n_1 - 1\} \cup \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 - 1\}.$$

On the other hand,

$$\begin{aligned} \mathrm{cm}(u) &= \frac{\sum_{e \in E_u} c(e)}{d_1 + d_2} = \frac{\mathrm{cm}_1(u_1) \cdot d_1 + (\mathrm{cm}_2(u_2) + n_1 - 1) \cdot d_2}{d_1 + d_2} \\ &= \frac{n_1 \cdot d_1 + n_1 \cdot d_2}{d_1 + d_2}, \text{ since } \mathrm{cm}_1(u_1) = n_1 \text{ and } \mathrm{cm}_2(u_2) = 1 \\ &= n_1. \end{aligned}$$

Therefore, c is a rainbow mean coloring of G and G is Type 1. ■

The next lemma can be proved in a similar way as Lemma 2.1.

Lemma 2.3. *Let G be a Type 1 graph of order n and c an optimal rainbow mean coloring of G such that $c(e)$ is at most n for each $e \in E(G)$. Define $c'(e) = n + 1 - c(e)$ for each $e \in E(G)$. Then c' is also an optimal rainbow mean coloring of G and the vertex with chromatic mean 1 (respectively, n) under c, has chromatic mean n (respectively, 1) under c'.*

Recall that a **caterpillar** is a tree with the property that if all the pendant vertices are deleted, we obtain a path, called its **spine**. The rainbow mean indexes of cubic caterpillars of even order were obtained in [5]. In the following theorem, we present our result for a particular family of caterpillars in which vertices in the spine have even degree.

Theorem 2.4. *Let T be a caterpillar of order n with spine the path $(v_1, v_2, \ldots, v_r)$. Suppose further that r is odd, $\deg(v_i)$ is even for each i, and, in particular, $\deg(v_i) = 2$ if i is even. Then $rm(T) = n$.*

Proof. (Refer to Fig. 4 for an illustration.)

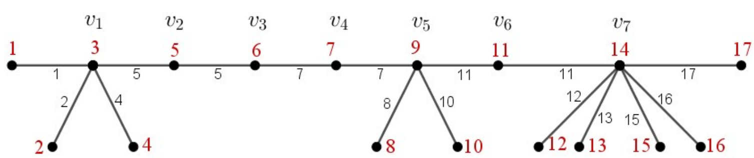


FIGURE 4. A caterpillar with Type 1 rainbow mean coloring

Suppose $\deg(v_i) = d_i$ for each i from 1 to r. Then, for each i, the subgraph $T[E_{v_i}]$ induced by E_{v_i} is a star K_{1,d_i}. Since $\deg(v_i) = d_i$ is even, it follows that K_{1,d_i} is

Type 1 by Theorem 1.3. Hence, we can find an optimal rainbow mean coloring c_1 of $H_1 = T[E_{v_1}] = K_{1,d_1}$ so that v_2 will have chromatic mean $d_1 + 1$. Similarly, we can find an optimal rainbow mean coloring c_2 of $T[E_{v_3}] = K_{1,d_3}$ so that v_2 will have chromatic mean 1. Applying Lemma 2.2 on H_1 and $T[E_{v_3}]$, we obtain a Type 1 rainbow mean coloring of the subgraph $H_2 = T\left[E_{v_1} \cup E_{v_3}\right]$ where $\mathrm{cm}(v_4) = \mathrm{rm}(H_2) = |V(H_2)|$. We then apply the same lemma on H_2 and $T[e_{v_5}] = K_{1,d_5}$, to obtain a Type 1 rainbow mean coloring of the subgraph $H_3 = T\left[E_{v_1} \cup E_{v_3} \cup E_{v_5}\right]$ where $\mathrm{cm}(v_6) = \mathrm{rm}(H_3) = |V(H_3)|$. We repeat the argument until we obtain a Type 1 rainbow mean coloring of

$$T = T\left[E_1 \cup E_3 \cup E_5 \cup \cdots \cup E_r\right],$$

showing that $\mathrm{rm}(T) = n$. ■

We note that double stars, brooms, and double brooms are examples of caterpillars. We will determine the rainbow chromatic indexes of brooms and double brooms in the next two sections. For double stars, Hallas et al. [5], constructed rainbow mean colorings and as a consequence proved the following result on their rainbow mean indexes.

Theorem 2.5. [5] *Let k and ℓ be integers with $k, \ell \geq 2$, and $G = DS(k, \ell)$ with order $n = k + \ell + 2$. Then*

$$rm(G) = \begin{cases} n+1, & \text{if } k \equiv \ell \pmod 4 \text{ where } k \text{ and } \ell \text{ are both even,} \\ n, & \text{otherwise.} \end{cases}$$

We provide an alternative proof which considers less cases. To simplify the notations, let $[N]$ denote the set $\{1, 2, \ldots, N\}$ and $\mathrm{sum}(S)$ the sum of the elements of a subset S of $[N]$ for any positive integer N.

Proof. By Theorem 1.4, it is enough to find a rainbow mean coloring c of G with $\mathrm{rm}(c) = n + 1$ if $k \equiv \ell \pmod 4$ and k and ℓ are both even; and $\mathrm{rm}(c) = n$, otherwise.

Let u and v be the vertices of G that have degrees $k + 1$ and $\ell + 1$, respectively. We make the following observation. Suppose $\{a, b\}$, S, and T are subsets of $[n + 1]$ that are mutually disjoint, $|S| = k$, $|T| = \ell$, and

$$a \cdot (k+1) - \mathrm{sum}(S) = b \cdot (\ell + 1) - \mathrm{sum}(T) > 0.$$

Then if c is an edge coloring of G such that

$$\{c(e) : e \in E_u - \{uv\}\} = S, \quad \{c(e) : e \in E_v - \{uv\}\} = T, \text{ and}$$
$$c(uv) = a \cdot (k+1) - \mathrm{sum}(S),$$

then c is a rainbow mean coloring of G with $\mathrm{rm}(u) = a$ and $\mathrm{rm}(v) = b$. Moreover, $\mathrm{rm}(c) = n + 1$ or n, depending on whether or not $n + 1$ is contained in $\{a, b\} \cup S \cup T$. Therefore, to complete the proof, we just need to identify a, b, S, and T that satisfy the necessary conditions.

Without loss of generality, let $\ell \geq k$ and $r = \ell - k$.

Case 1: If $r \equiv 0 \pmod 4$, and k and ℓ are both even. Let

$$a = n - 3 - \tfrac{r}{2}, \quad b = a + 1,$$
$$S = \{1, 3, 5, \ldots, n - 5 - r\} \cup \left\{n - 1 - \tfrac{r}{4}\right\}, \text{ and}$$
$$T = [n+1] - S - \{a, b, n\}.$$

First, $\{a, b\} \cap S = \emptyset$ since

$$n - 5 - r < n - 3 - \tfrac{r}{2} < n - 2 - \tfrac{r}{2} < n - 1 - \tfrac{r}{4}.$$

Hence, $\{a, b\}$, S, and T are mutually disjoint. Next,

$$|S| = \tfrac{n-5-r+1}{2} + 1 = \tfrac{n-2-r}{2} = k,$$

and so $|T| = n + 1 - k - 3 = \ell$. Now, since $r = n - 2k - 2$,

$$\begin{aligned} a \cdot (k+1) - \operatorname{sum}(S) &= \left(n - 3 - \tfrac{r}{2}\right)(k+1) - \left(\tfrac{n-4-r}{2}\right)^2 - \left(n - 1 - \tfrac{r}{4}\right) \\ &= \tfrac{1}{4}\left(2k - n + 2kn - 10\right) \\ &= \tfrac{1}{4}\left(n\left(2k-1\right) + 2k - 10\right). \end{aligned}$$

Since $n \geq 4$ and $k \geq 2$, $a \cdot (k+1) - \operatorname{sum}(S) > 0$. Now

$$\begin{aligned} b \cdot (\ell+1) - \operatorname{sum}(T) &= \left(n - 2 - \tfrac{r}{2}\right) \cdot (n - k - 1) - \left(\tfrac{1}{4}\left(-4k^2 - 2k + 2n^2 - 5n + 14\right)\right) \\ &= \tfrac{1}{4}\left(2k - n + 2kn - 10\right). \end{aligned}$$

Hence, we can construct a coloring c for G such that $\operatorname{rm}(c) = n + 1$. Therefore, $\operatorname{rm}(G) = n + 1$.

For the remaining cases, we identify a, b, S, and T, which can be shown to satisfy all the conditions to prove that $\operatorname{rm}(G) = n$. We omit the computations which are similar to those done in Case 1.

Case 2: If $r \equiv 0 \pmod 4$, k and ℓ are both odd, and $k \neq \ell$, let

$$\begin{gathered} a = \tfrac{3}{2}(k+1), \ b = a + \tfrac{r}{2}, \\ S = \left[\tfrac{3k+1}{2}\right] - \left[\tfrac{k+1}{2}\right], \text{ and } T = [n] - S - \{a, b\}. \end{gathered}$$

Case 3: If $r \equiv 0 \pmod 4$, k and ℓ are both odd, and $k = \ell$, let

$$\begin{gathered} a = \tfrac{3k-4+k \bmod 4}{2}, \ b = a + 2, \\ S = \{1, 3, 5, \ldots, 2k-3\} \cup \left\{2k - \tfrac{k \bmod 4 - 1}{2}\right\}, \text{ and} \\ T = [n] - S - \{a, b\}. \end{gathered}$$

Case 4: If $r \equiv 2 \pmod 4$ (and so k and ℓ are both odd or both even), let

$$\begin{gathered} a = n - 3 - \tfrac{r}{2}, \ b = a + 1, \\ S = \{1, 3, 5, \ldots, n-5-r\} \cup \left\{n - 1 - \tfrac{r+2}{4}\right\}, \text{ and} \\ T = [n] - S - \{a, b\}. \end{gathered}$$

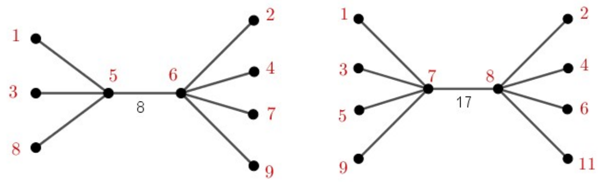
Case 5: If r is odd, let

$$\begin{gathered} a = n - \tfrac{r+7}{2}, \ b = a + 1, \\ S = \{1, 3, 5, \ldots, n-5-r\} \cup \{n-1\}, \text{ and} \\ T = [n] - S - \{a, b\}. \end{gathered}$$

■

Fig. 5 shows two examples of double stars with optimal rainbow mean colorings described in the proof of Theorem 2.5.

With the above result on double stars, we can apply Lemmas 2.2 and 2.3 repeatedly to obtain the rainbow mean index of another family of caterpillars.



a. $DS(3,4)$ b. $DS\,(4,4)$

FIGURE 5. Two double stars with optimal rainbow mean colorings.

Theorem 2.6. *Let T be a caterpillar of order n with spine the path $(v_1, v_2, \ldots, v_r)$ where $r \equiv 2 \pmod 3$ and $d_i = \deg(v_i)$ for $i = 1, 2, \ldots, r$. Suppose that $d_i \geq 2$ for each i, $d_i = 2$ if $3 \mid i$, and for each $0 \leq k \leq \lfloor \frac{r}{3} \rfloor$,*

$$d_{3k+1}d_{3k+2} \text{ is even, or } d_{3k+1} \not\equiv d_{3k+2} \pmod 4.$$

Then $rm\,(T) = n$.

To illustrate the proof, refer to the caterpillar in Fig. 6 obtained by applying Lemma 2.3 on $DS(2,3)$, $DS(5,5)$, and $DS(2,4)$.

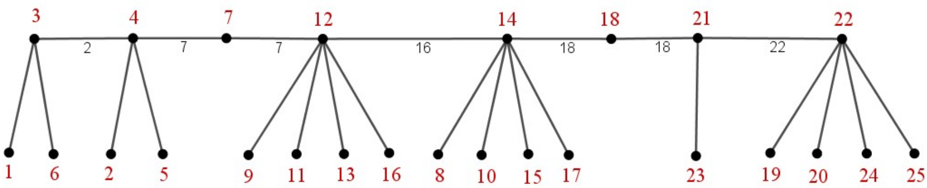


FIGURE 6. A Type 1 caterpillar with optimal rainbow mean coloring.

3. Brooms

In this section, we show that all brooms are Type 1 graphs.

Theorem 3.1. *Let k and n be positive integers such that $k \geq 2$ and $n \geq k+3$. Then $rm(B(n,k)) = n$ and there is a Type 1 rainbow mean coloring of $B(n,k)$ such that the tip has chromatic mean 1.*

Proof. We take cases. In each case we construct a rainbow mean coloring c of $B\,(n,k)$ with $\mathrm{rm}(c) = n$. Let $r = n-k$ and $(u_1, u_2, u_3, \cdots, u_r)$ be the handle of the broom with tip u_1. We shall call the k pendant edges that are incident to u_r **bristles** of the broom.

Case 1: Suppose n and k are odd. Then the broom $B\,(n,k)$ is a caterpillar that satisfies the assumptions in Theorem 2.4. Hence, we can find a Type 1 rainbow mean coloring for $B(n,k)$, which by construction assigns the chromatic mean 1 to the tip of the broom.

Case 2: Suppose n is odd and k is even. First, we consider the case $n = k+3$.

Define an edge coloring c_2 as follows. Recall that the handle has vertices u_1, u_2, and u_3 with u_1 the tip. Let $c_2\,(u_1u_2) = 1$ and $c_2\,(u_2u_3) = k+3$ as in Fig. 7. Color the k bristles using distinct elements from

$$T = [k+3] - \left\{1, \tfrac{k+4}{2}, \tfrac{k+6}{2}\right\}.$$

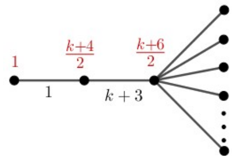


FIGURE 7. For the proof of Case 2 of Theorem 3.1.

Then it can be shown that

$$\left(\frac{k+6}{2}\right)(k+1) = k+3+\text{sum}(T),$$

or, equivalently, $\text{cm}(u_3) = \frac{k+6}{2}$. Therefore, c_2 is a rainbow mean coloring and $\text{rm}(B(k+3,k)) = k+3 = n$.

Now, suppose $n > k+3$. Then $n-k-2 > 1$ is odd. By Theorem 1.2, we can find a Type 1 rainbow mean coloring c_1 of the path P_{n-k-2} so that the chromatic mean of the terminal vertex is the order of the path. Let c_2 be the Type 1 rainbow mean coloring of $B(k+3,k)$ described above, and which assigns the chromatic mean 1 to the tip of the broom. Applying Lemma 2.2 on (P_{n-k-2}, c_1) and $(B(k+3,k), c_2)$, we get a Type 1 rainbow mean coloring of $B(n,k)$ such that the tip has chromatic mean 1.

Case 3: Suppose n is even and k is odd. First, suppose $n = k+5$ and $k \geq 5$. We define an edge coloring c_2 on $B(k+5,k)$ as follows. Let the edges on the handle of the broom be colored 1, 3, 5, and 7 as shown in Fig. 8(a). We then color the k bristles using distinct elements from $T = [k+5] - \left\{1,2,4,6,\frac{k+9}{2}\right\}$. Then the four vertices u_1, u_2, u_3, and u_4 of the handle have chromatic means 1, 2, 4, and 6, as shown in Fig. 8(a). We may now check that $\left(\frac{k+9}{2}\right)(k+1) = 7+\text{sum}(T)$, or, equivalently $\text{cm}(u_5) = \frac{k+9}{2}$. Hence, $\text{rm}(B(k+5,k)) = k+5 = n$.

Now, suppose $n > k+5$, and $k \geq 5$. Then $n-k-4 > 1$ is odd. By Theorem 1.2, we can find a Type 1 rainbow mean coloring c_1 of the path P_{n-k-4} so that the chromatic mean of the terminal vertex is the order of the path. Let c_2 be the Type 1 rainbow mean coloring constructed above for $B(k+5,k)$, and which assigns the chromatic mean 1 to the tip. By applying Lemma 2.2 on (P_{n-k-4}, c_1) and $(B(k+5,k), c_2)$, we get the desired result.

So far, we have proved Case 3 if $5 \leq k \leq n-5$. Since $3 \leq k \leq n-3$, we just need to consider the following subcases.

Subcase 3.1. Suppose $k = n-3$. Then $n \geq 6$. Define the coloring c (Fig. 8b) so that $c(u_1u_2) = 1$, $c(u_2u_3) = 2n-1$, and the bristles are colored using distinct elements from $T = [n] - \left\{1, n, \frac{n+4}{2}\right\}$. Then $\text{cm}(u_1) = 1$, $\text{cm}(u_2) = n$. It can be verified that $\left(\frac{n+4}{2}\right)(n-2) = 2n-1+\text{sum}(T)$, or, equivalently, $\text{cm}(u_3) = \frac{n+4}{2}$, so the result holds.

Subcase 3.2. Suppose $k = 3$. We just need to consider the values of $n \geq 8$. A Type 1 rainbow mean coloring c_2 for $B(8,3)$ with the tip having chromatic mean 1 is shown in Fig. 8(c). Now if $n \geq 10$, then $n-7 \geq 3$ is odd. By Theorem 1.2, we can find a Type 1 rainbow mean coloring c_1 of the path P_{n-7} so that the chromatic mean of the terminal vertex is the order of the path. If we apply Lemma 2.2 on (P_{n-7}, c_1) and $(B(8,3), c_2)$, we get the desired result.

Case 4: Suppose n and k are even. First, suppose $n = k+4$ and $k \geq 4$.

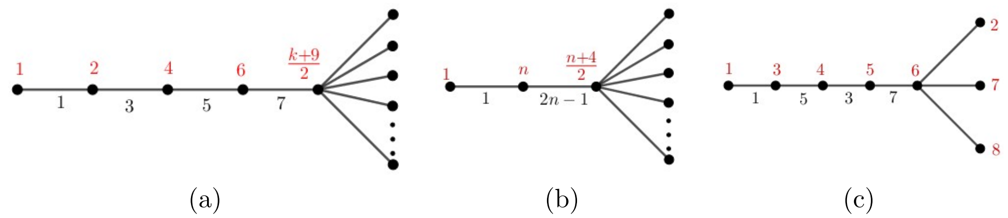


FIGURE 8. For the proof of Theorem 3.1 Case 3.

Let c_2 be an edge coloring for $B(k+4,k)$ defined as follows. Let $c_2(u_1u_2) = 1$, $c_2(u_2u_3) = k+3$, $c_2(u_3u_4) = k+1$ while the bristles are colored using distinct elements from $T = [k+4] - \{1, \frac{k+4}{2}, k+2, \frac{k+6}{2}\}$ (Refer to Fig. 9(a).) Then $\mathrm{cm}(u_1) = 1$, $\mathrm{cm}(u_2) = \frac{k+4}{2}$, and $\mathrm{cm}(u_3) = k+2$. It can be shown that $\left(\frac{k+6}{2}\right)(k+1) = k+1+\mathrm{sum}(T)$, or, equivalently, $\mathrm{cm}(u_4) = \frac{k+6}{2}$. Hence, $\mathrm{rm}(B(k+4,k)) = k+4 = n$.

As in the previous case, we can show that $\mathrm{rm}(B(n,k)) = n$ if $n > k+4$ and $k \geq 4$ by applying Lemma 2.2 on P_{n-k-3} and $B(k+4,k)$. Since $2 \leq k \leq n-4$, we just need to consider the case $k=2$. A Type 1 rainbow mean coloring c_2 for $B(6,2)$ is shown in Fig. 9(b). If $n \geq 8$, then $n-5 \geq 3$ is odd. As in the previous case, we can then construct a Type 1 rainbow mean coloring of $B(n,2)$ by applying Lemma 2.2 on (P_{n-5}, c_1) and $(B(6,2), c_2)$ where c_1 is an appropriate Type 1 rainbow mean coloring. ∎

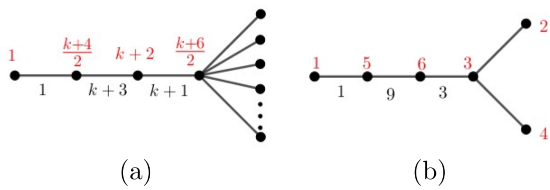


FIGURE 9. For the proof of Theorem 3.1 Case 4.

We observe that Lemma 2.3 applies in all the Type 1 rainbow mean colorings constructed in the proof of Theorem 3.1, except for the following cases: $B(n,2)$ and $B(n,n-3)$ when $n \geq 6$ is even. Therefore, except for these cases, we can find a Type 1 rainbow mean coloring such that the tip has chromatic mean equal to its order.

However, $B(8,2)$ has a Type 1 rainbow mean coloring c_1 such that the tip has chromatic mean equal to its order, and one such coloring is shown in Fig. 10. If $n \geq 10$ and n is even, then $n-7 \geq 3$ is odd, so by Theorem 1.2, P_{n-7} has a Type 1 rainbow mean coloring c_2 such that the initial vertex has chromatic mean 1. By applying Lemma 2.3 on $(B(8,2), c_1)$ and (P_{n-7}, c_2), we obtain a Type 1 rainbow mean coloring for $B(n,2)$ such that the tip of the broom has chromatic mean n. Therefore, we can make the following remark which will be important in computing the rainbow mean indexes of the double brooms in the next section.

Remark 3.2. Let k and n be positive integers such that $k \geq 2$ and $n \geq k+3$. Then there exists a Type 1 rainbow mean coloring of $B(n,k)$ such that the tip has chromatic mean equal to its order if $(n,k) \neq (6,2)$, n is odd, or $n \neq k+3$.

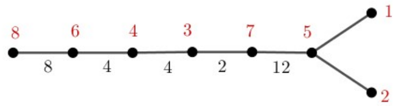


Figure 10. A Type 1 rainbow mean coloring of $B(8,2)$ where the tip has chromatic mean 8.

4. Double Brooms

Theorem 4.1. *Let k, ℓ, and n be positive integers with $k \geq 2$, $\ell \geq 2$, and $n \geq k+\ell+3$. Then $rm(DB\,(n,k,\ell)) = n$.*

Proof. We consider cases.

Case 1: Suppose at least one of k and ℓ, say ℓ, is odd. To obtain a Type 1 rainbow mean coloring for $DB(n,k,\ell)$, we apply Lemma 2.2 on one of the following pairs of graphs with appropriate Type 1 rainbow mean colorings c_1 and c_2:

(A) $(K_{1,k+1}, c_1)$ and $(B\,(n-k-1,\ell)\,, c_2)$ if k is odd
(B) $(B\,(n-\ell-1,k)\,, c_1)$ and $(K_{1,\ell+1}, c_2)$, and
(C) $(B\,(n-\ell-2,k)\,, c_1)$ and $(B\,(\ell+3,\ell)\,, c_2)$.

The existence of the coloring c_1 in (A), and of c_2 in (A), (B), and (C), is guaranteed by Theorems 1.3 and 3.1. Hence, the result holds when k is odd. So we may assume that k is even. Now, Remark 3.2 guarantees the existence of c_1 in (B) or (C), so long as $n-k-\ell \geq 4$. Since $n-k-\ell \geq 3$, we are then left with the case $n = k+\ell+3$, which we assume for the rest of Case 1. We divide the possibilities into subcases. Let u, v, and w be the vertices in the handle of the double broom, that have degrees $k+1$, 2, and $\ell+1$, respectively.

Subcase 1.1: If $\ell = k+1$, then $n = 2k+4$. Let $S = \{2,4,\ldots,2k+2\} - \{k+2\}$ and $T = [n] - (S \cup \{1,k+1,k+3\})$. Then, $\{1,k+1,k+3\}$, S, and T are mutually disjoint, $|S| = k$ and $|T| = k+1$. Define the edge coloring c so that the k pendant edges incident to u are colored using all the elements of S, the pendant edges incident to w are colored using T, while the edges incident to v are colored 1(See Fig. 11.). Then $\mathrm{cm}(v) = 1$. It can then be shown that $(k+1)^2 = 1 + \mathrm{sum}\,(S)$ and $(k+3)\,(k+2) = 1 + \mathrm{sum}\,(T)$, or, equivalently, $\mathrm{cm}(u) = k+1$ and $\mathrm{cm}(w) = k+3$, so the result follows.

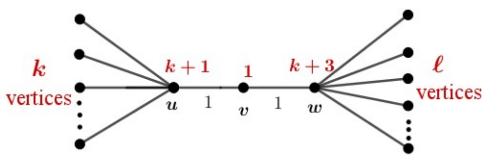


Figure 11. For the proof of Theorem 4.1 Subcase 1.1.

Subcase 1.2: If $\ell = k-1$, then $n = 2k+2$. Let $T = \{3,5,7,\ldots,2k-1\}$ and $S = [n] - (T \cup \{1,k,k+2\})$. Then $\{1,k,k+2\}$, S, and T are mutually disjoint, $|T| = k-1$ and $|S| = k$. As in Subcase 1.1, define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges incident to v are colored 1. Then $\mathrm{cm}(v) = 1$. It can then be shown that $(k+2)(k+1) = 1 + \mathrm{sum}(S)$ and $k^2 = 1 + \mathrm{sum}(T)$, or, equivalently, $\mathrm{cm}(u) = k+2$ and $\mathrm{cm}(w) = k$, so the result follows.

Subcase 1.3: If $\ell \geq k+3$, let $\ell = k + 2m + 1$ where $m \geq 1$. Then $n = 2k + 2m + 4$. Let

$$S = [k+2] - \left\{\tfrac{k}{2}+1, \tfrac{k}{2}+2\right\} \quad \text{and} \quad T = [n] - \left(S \cup \left\{\tfrac{k}{2}+1, 2m+\tfrac{3k}{2}+4, m+\tfrac{3k}{2}+4\right\}\right).$$

Then S, T, and $\left\{\frac{k}{2}+1, 2m+\frac{3k}{2}+4, m+\frac{3k}{2}+4\right\}$ are mutually disjoint, $|S| = k$ and $|T| = k + 2m + 1 = \ell$. Define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges uv and vw are colored 1 and $4m + 3k + 7$, as in Fig. 12. Then $\mathrm{cm}(v) = 2m + \frac{3k}{2} + 4$. To complete the proof, it can be shown that $\left(\frac{k}{2}+1\right)(k+1) = 1 + \mathrm{sum}(S)$ and $\left(m + \frac{3k}{2} + 4\right)(k + 2m + 2) = 4m + 3k + 7 + \mathrm{sum}(T)$, or, equivalently, $\mathrm{cm}(u) = \frac{k}{2} + 1$ and $\mathrm{cm}(w) = m + \frac{3k}{2} + 4$.

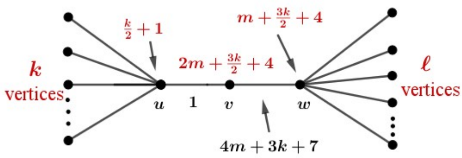


FIGURE 12. For the proof of Theorem 4.1 Subcase 1.3.

Subcase 1.4: If $k \geq \ell + 3$, let $k = \ell + 2m + 1$, where $m \geq 1$. Then $n = 2\ell + 2m + 4$. Let

$$T = \{2, 4, \ldots, 2\ell + 2m + 2\} - \{\ell+3, \ell+5, \ldots, \ell+2m+3\} \quad \text{and}$$
$$S = [n] - (T \cup \{\ell+m+2, \ell+2m+3, \ell+m+3\}).$$

Note that S, T, and $\{\ell+m+2, \ell+2m+3, \ell+m+3\}$ are mutually disjoint and $|T| = \ell$ while $|S| = k$. Define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges uv and vw are both colored $\ell + 2m + 3$ as in Fig. 13. Then $\mathrm{cm}(v) = \ell + 2m + 3$. It can be shown that $(\ell+m+2)(\ell+1) = \ell + 2m + 3 + \mathrm{sum}(T)$ and $(\ell+m+3)(\ell+2m+2) = \ell + 2m + 3 + \mathrm{sum}(S)$, or, equivalently, $\mathrm{cm}(u) = \ell + m + 3$ and $\mathrm{cm}(w) = \ell + m + 2$, thereby proving the result.

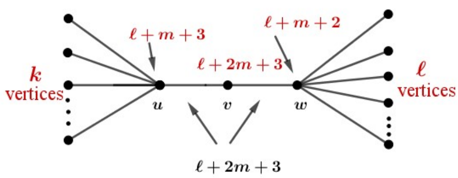


FIGURE 13. For the proof of Theorem 4.1 Subcase 1.4.

Case 2: Suppose k and ℓ are both even. As in Case 1, we apply Lemma 2.2, on the following pair of graphs with appropriate Type 1 rainbow mean colorings c_1 and c_2:

$$(B(n-\ell-2, k), c_1) \quad \text{and} \quad (B(\ell+3, \ell), c_2).$$

By Theorem 3.1, the Type 1 rainbow mean coloring c_2 exists. Now, by Remark 3.2, c_1 exists if $n-k-\ell-2 \geq 3$ and either $k \neq 2$ or $n-k-\ell-2 \neq 4$. We consider the remaining cases below.

Subcase 2.1: Suppose $n-k-\ell = 3$ and $k = \ell$. Then $n = 2k+3$. As in the previous case, let u, v, and w be the vertices in the handle of the double broom that have degrees $k+1$, 2, and $\ell+1$, respectively. Let

$$T = \{2, 4, \ldots, k\} \cup \{k+5, k+7, \ldots, 2k+3\} \text{ and } S = [n] - (T \cup \{k+1, k+2, k+3\}).$$

Then S, T, and $\{k+1, k+2, k+3\}$ are mutually disjoint, $|S| = k$, and $|T| = k = \ell$. Define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges uv and vw are colored $\frac{k}{2}+1$ and $\frac{3k}{2}+3$ as in Fig. 14. Then $\mathrm{cm}(v) = k+2$. It can be shown that $(k+3)(k+1) = \frac{3k}{2}+3+\mathrm{sum}(T)$ and $(k+1)^2 = \frac{k}{2}+1+\mathrm{sum}(S)$, or, equivalently, $\mathrm{cm}(u) = k+1$ and $\mathrm{cm}(w) = k+3$, thereby proving the result.

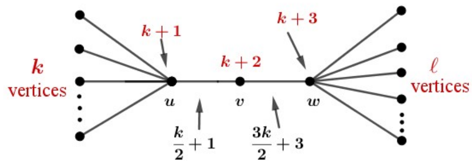


FIGURE 14. For the proof of Theorem 4.1 Subcase 2.1.

Subcase 2.2: Suppose $n-k-\ell = 3$ and $k \neq \ell$. Suppose $k < \ell$, and $\ell = k+2m$ where $m \geq 1$. Then $n = 2k+2m+3$. To simplify notations, we write

$$[N \,..\, M] = \{N, N+1, N+2, \ldots, M-1, M\},$$

for integers N and M if $N \leq M$; if $N > M$, we take $[N \,..\, M] = \emptyset$. Let

$$S = \left\{\tfrac{k}{2}\right\} \cup \left[\left(\tfrac{k}{2}+m+2\right) \,..\, (k+m)\right] \cup \left[(k+m+4) \,..\, \left(\tfrac{3k}{2}+m+3\right)\right], \text{ and}$$
$$T = [n] - (S \cup \{k+m+2, k+m+3, 2k+m+4\}).$$

Then S, T, and $\{k+m+2, k+m+3, 2k+m+4\}$ are mutually disjoint, $|S| = k$ and $|T| = n-3 = \ell$. Define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges uv and vw are colored $2k+2m+4$ and $2k+4$ as in Fig. 15. Then $\mathrm{cm}(v) = 2k+m+4$. It can be shown that $(k+m+3)(k+1) = 2k+2m+4+ \mathrm{sum}(S)$ and $(k+m+2)(k+2m+1) = 2k+4+\mathrm{sum}(T)$, or, equivalently, $\mathrm{cm}(u) = k+m+3$ and $\mathrm{cm}(w) = k+m+2$, thereby proving the result.

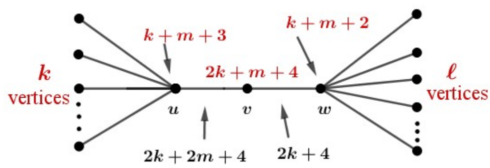


FIGURE 15. For the proof of Theorem 4.1 Subcase 2.2.

Subcase 2.3: Suppose $n-k-\ell=4$ and $k\neq\ell$. Suppose $k<\ell$ and $\ell=k+2m$, $m\geq 1$. Then $n=2k+2m+4$. In Subcase 2.2, we considered $DB\,(2k+2m+3,k,\ell)$ and found a Type 1 rainbow mean coloring c. Suppose the edge uv in Fig. 15 is replaced with a path (u,z,v) where z is a new vertex. Then the resulting graph is $DB\,(n,k,\ell)$. If we modify c so that both uz and zv are colored $2k+2m+4=n$, then we obtain a Type 1 rainbow mean coloring for $DB\,(n,k,\ell)$ (See Fig. 16.).

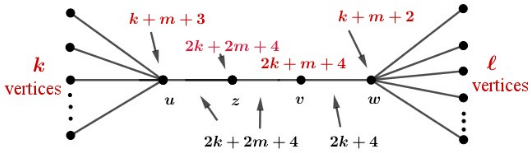


FIGURE 16. For the proof of Theorem 4.1 Subcase 2.3.

Subcase 2.4: Suppose $n-k-\ell=4$ and $k=\ell$. Then $n=2k+4$. Suppose the handle of the double broom has vertices u, z, v, and w as in the previous subcase. Let

$$S=\left[\left(\tfrac{k}{2}+2\right)\,..\,(k+1)\right]\cup\left[(k+5)\,..\,\left(\tfrac{3k}{2}+3\right)\right]\cup\left\{\tfrac{3k}{2}+5\right\}\quad\text{and}$$
$$T=[n]-(S\cup\{1,k+2,k+3,k+4\})\,.$$

Then S, T, and $\{1,k+2,k+3,k+4\}$ are mutually disjoint, $|S|=k$ and $|T|=k$. Define the edge coloring c so that the k pendant edges incident to u are colored using S, the pendant edges incident to w are colored using T, while the edges uz, zv, and vw are colored 1, 1, and $2k+5$, respectively, as in Fig. 17. Then $\mathrm{cm}(z)=1$ and $\mathrm{cm}(v)=k+3$. It can be shown that $(k+2)\,(k+1)=1+\mathrm{sum}\,(S)$ and $(k+4)\,(k+1)=2k+5+\mathrm{sum}\,(T)$, or, equivalently, $\mathrm{cm}(u)=k+2$ and $\mathrm{cm}(w)=k+4$, thereby proving the result.

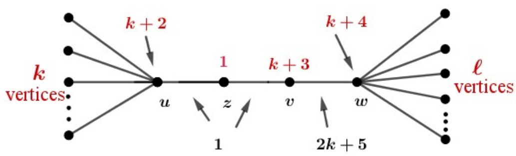


FIGURE 17. For the proof of Theorem 4.1 Subcase 2.4.

Subcase 2.5: Suppose $n-k-\ell=6$ and $k=\ell=2$. Hence, $n=10$, and we can find a Type 1 rainbow mean coloring of $DB\,(10,2,2)$ (See Fig. 18.)

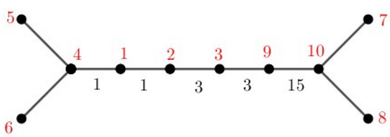


FIGURE 18. For the proof of Theorem 4.1 Subcase 2.5.

■

5. Conclusion

Since the concept of a rainbow mean coloring of a graph is a fairly new type of graph labeling, there are still many families of graphs whose rainbow chromatic means are unknown. While the focus of this paper is on caterpillars, brooms and double brooms, the lemmas used in the proofs provide creative tools for constructing rainbow mean colorings based on particular optimal colorings of subgraphs. These may also be used to determine the rainbow mean indexes of other graphs not covered in this or past papers. Note that the results given in this paper support the conjecture posed in [2]. Further investigations on other families of trees and determining their type will be a good step towards a more general result on trees.

References

[1] H. Escuadro, F. Okamoto, P. Zhang, S. Arumugam, Vertex-distinguishing colorings of graphs: A survey of recent developments, AKCE Int. J. Graphs Comb. 4 (3) (2020) 277–299.

[2] G. Chartrand, J. Hallas, E. Salehi, P. Zhang, Rainbow mean colorings of graphs, Discrete Math. Lett. 2 (2019) 18–25.

[3] G. Chartrand, J. Hallas, E. Salehi, P. Zhang, A note on rainbow mean indexes of paths, Discrete Math. Lett. 8 (2022) 57–59.

[4] J. Hallas, E. Salehi, P. Zhang, Rainbow mean colorings of bipartite graphs, Bull. Inst. Combin. Appl. 88 (2020) 78–97.

[5] J. Hallas, E. Salehi, P. Zhang, On Rainbow Mean Colorings of Trees, in: Hoffman, F. (eds) Combinatorics, Graph Theory and Computing, SEICCGTC 2020, Springer Proceedings in Mathematics & Statistics, Vol. 388, Springer, Cham, 2022.

[6] A. Ali, G. Chartrand, P. Zhang, Irregularity in Graphs, Springer, Cham, 2021.