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Linear-Time Rectilinear Drawings of Subdivisions of Triconnected Cubic Planar Graphs with Orthogonally Convex Faces

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Abstract A graph is called planar if it admits a planar drawing on the plane, i.e., no two edges create a crossing except possibly at their common endpoint. In a rectilinear drawing Γ of a planar graph, each vertex is drawn as a point and each edge is drawn as either horizontal or vertical line segment. A face in Γ is called *orthogonally convex* if every horizontal or vertical line segment connecting two points within the face does not intersect any other face. We examine the decision problem that takes a planar graph as an input and seeks for a rectilinear drawing where the faces are drawn as orthogonally convex polygons. A linear-time algorithm for this problem is known for biconnected planar graphs, but the algorithm relies on complex data structures and linear-time planarity testing, which are challenging to implement. In this paper, we give a necessary and sufficient condition for a subdivision of a triconnected cubic planar graph to admit such a drawing, and design a linear-time algorithm to check the condition and compute a desired drawing, if it exists. As a byproduct of our results we show that if a subdivision of a triconnected cubic planar graph G admits a rectilinear drawing, then it must also admit a rectilinear drawing with orthogonally convex faces.

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1. Introduction

Graph drawing is a well known research area that lies at the intersection of graph theory, topology and geometry. Automatic graph drawings have attracted researchers because of their vital and important applications in VLSI circuit layout, networks, computer architecture, circuit schematics etc. [1–3]. Among various drawing styles, 'orthogonal drawings' of planar graphs have occupied a big area due to their practical applications, specially in circuit schematics, entity relationship diagrams, data flow diagrams etc. [4–9]. An *orthogonal drawing* of a planar graph G is a drawing of G in which each vertex is mapped to a *point*, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common ends. A *bend* is a point where an edge changes its direction in a drawing. Every planar graph of the maximum degree four has an orthogonal drawing, but may need bends. Finding an orthogonal drawing of a planar graph of the maximum degree four with the minimum number of bends is an NP-hard problem [10]. However, polynomial algorithms are known for finding bend-minimum orthogonal drawings of plane graphs (with fixed embedding) of maximum degree four and some restricted classes of planar graphs of maximum degree three [6–9, 11].

An orthogonal drawing D of a plane graph is a *no-bend orthogonal drawing or a rectilinear drawing* if D has no bend. Clearly all rectilinear drawings are orthogonal drawings but not vice versa because of requirements of some bends in some orthogonal drawings. Figure 1(b) illustrates a rectilinear drawing of the planar graph in Figure 1(a). Not every plane graph has a rectilinear drawing. The plane graph in Figure 1(c) has no rectilinear drawing since the triangle abc can not be drawn by horizontal and vertical line segments without bend, but the plane graph in Figure 1(c) has an orthogonal drawing with some bends as illustrated in Figure 1(d).

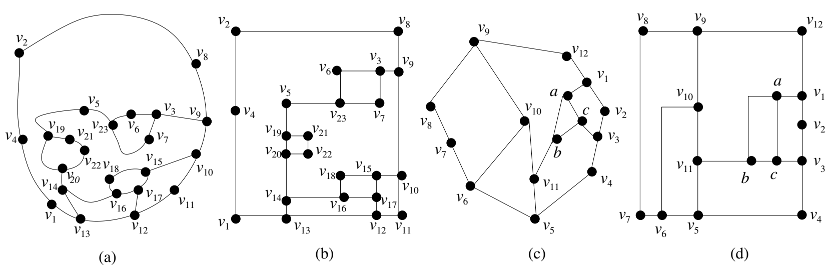

Figure 1. (a) A plane graph Γ for which a rectilinear drawing exists, and (b) a corresponding rectilinear drawing of Γ. (c) A plane graph for which there is no rectilinear drawing, and (d) a corresponding orthogonal drawing with two bends.

Rahman et al. [11] gave a linear-time algorithm to determine whether a plane graph Γ with $\Delta \leq 3$ has a rectilinear drawing or not and to find a rectilinear drawing of Γ, if it exists. Since a planar graph G may have an exponential number of planar embeddings, the problem of finding rectilinear drawings of planar graphs is a non-trivial problem and is solvable in polynomial time for restricted classes of planar graphs where the maximum degree is at most 3. Di Battista et al. [12] gave an $O(n^5 \log n)$time algorithm to find an orthogonal drawing of a planar graph $G(\Delta \leq 3)$ with the minimum number of bends.

Rahman et al. [13] gave a linear-time algorithm for determining whether a subdivision of a planar triconnected cubic graph has a rectilinear drawing and for finding out a drawing if it exists. Chang and Yen [14] gave an algorithm for a min-bend orthogonal drawing of a planar graph of maximum degree three if it exists that runs in $O(n^{17/7})$ time. One can find a no-bend orthogonal drawing of a planar graph G of maximum degree 3, if G has one, in $O(n^2)$ time using the algorithm by Didimo et al. [15] for finding a bend-minimum orthogonal drawing of a planar graph of maximum degree 3.

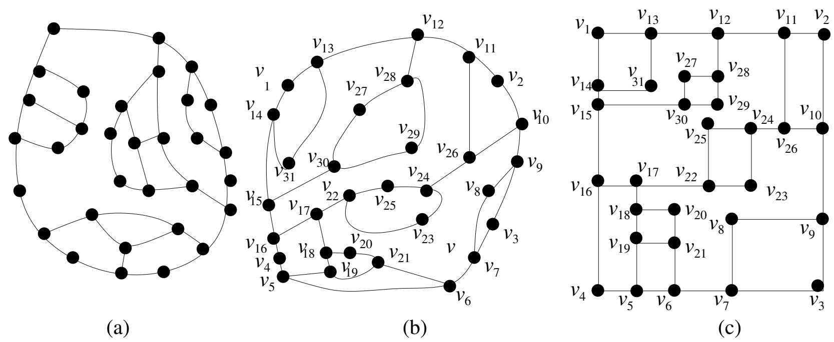

FIGURE 2. (a) A plane graph for which there is no rectilinear orthogonally convex drawing, (b) a plane graph Γ for which a rectilinear orthogonally convex drawing exists, and (c) corresponding rectilinear orthogonally convex drawing of Γ.

In a rectilinear drawing of a plane graph Γ, each inner face of Γ is drawn as rectilinear polygon. A rectilinear polygon P is called *orthogonally convex* if every horizontal or vertical segment connecting two points in P lies totally within P. An orthogonally convex drawing is an orthogonal drawing where each inner face is an orthogonally convex polygon as illustrated in Figure 2(c). Chang and Yen [16] gave a necessary and sufficient condition for a biconnected plane graph of maximum degree 3 to have a rectilinear orthogonally convex drawing and gave a linear-time algorithm to find such a drawing if it exists.

A planar graph is said to have a rectilinear orthogonally convex drawing if at least one of its plane embeddings has a rectilinear orthogonally convex drawing. For the plane embeddings Γ_1 and Γ_2, as illustrated in Figures 3(a) and 3(b) of a planar graph G there is no rectilinear orthogonally convex drawing. But for the plane embedding Γ_3 in Figure 3(c) of the same planar graph G, there exists a rectilinear orthogonally convex drawing, as illustrated in Figure 3(d), and hence G has a rectilinear orthogonally convex drawing.

Since a planar graph G may have an exponential number of planar embeddings, determining whether G has a rectilinear orthogonally convex drawing or not using the algorithm of Chang and Yen [16] by checking each planar embedding of G takes exponential time. Thus to develop an efficient algorithm to examine whether a planar graph G has a rectilinear orthogonally convex drawing or not is a non-trivial problem. Fortunately, a subdivision of a planar triconnected cubic graph has $O(n)$ embeddings and it is straightforward to enumerate them in $O(n^2)$ time. Therefore, determining whether a subdivision of planar triconnected cubic graph G has a rectilinear orthogonally convex

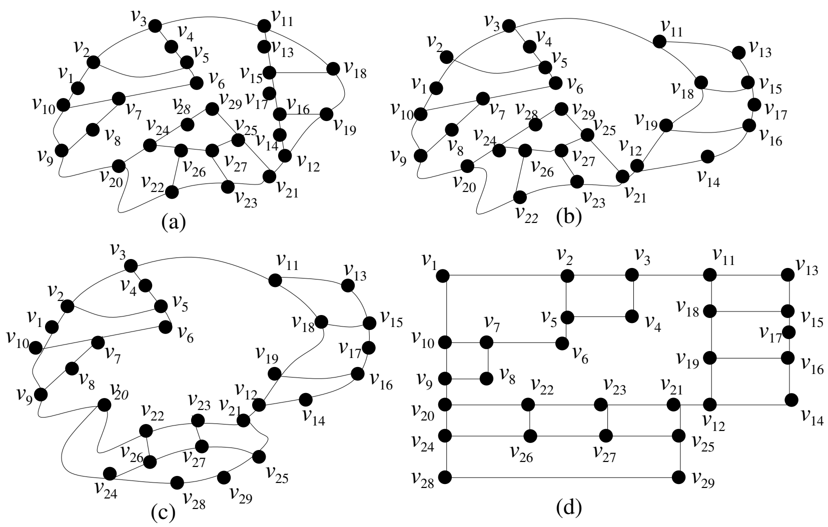

FIGURE 3. (a)-(b) Two plane embeddings of a planar graph G which have no rectilinear orthogonally convex drawing, (c) a planar embedding Γ of G which has a rectilinear orthogonally convex drawing, and (d) a rectilinear orthogonally convex drawing of Γ.

drawing or not using the algorithm of Chang and Yen [16] by checking each planar embedding of G takes $O(n^2)$ time. In addition to the rectilinear constraints, researchers also focused on restricting the shapes of the faces [16, 17]. Hasan and Rahman presented an algorithmic outline to extend the idea for biconnected planar graphs, but the technique is quite involved as it is based on linear-time planarity testing [18]. Very recently Didimo et al. [19] have developed a linear-time algorithm for bend-minimum orthogonal drawings of planar graphs but that does not ensure to find orhogonally convex drawings. In this paper we give a linear-time algorithm to determine whether a subdivision of a planar triconnected cubic graph G has a rectilinear orthogonally convex drawing or not, and to find a rectilinear orthogonally convex drawing of G, if it exists. We also show that, if such a graph G has a rectilinear drawing, then G has a rectilinear orthogonally convex drawing.

The rest of this paper is organized as follows. In Section 2, we give some terminologies and previous results. In Section 3, we describe a necessary and sufficient condition for a subdivision of a planar triconnected cubic graph G to have a rectilinear orthogonally convex drawing leads to a linear-time algorithm to find a drawing, if such one exists. We also show that if G has a rectilinear drawing, then G has a rectilinear orthogonally convex drawing. Finally, Section 4 concludes the paper with some future work.

2. Preliminaries

In this section, we give some definitions that will be used throughout the paper and present some preliminary results.

Graphs and Degrees of Vertices: Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. The *degree* $d(v)$ of a vertex v is the number of neighbors of v in G. We call a vertex of degree k in G a *degree-k vertex* of G. We denote the maximum degree of a graph G by $\Delta(G)$ or simply by Δ. A graph G is called *cubic* if $d(v) = 3$ for every vertex v.
Paths, Subdivisions, and Connectivity: A *path* in a graph G is a finite sequence $P = v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$ of alternating vertices and edges with no repeated vertex (except end vertices) such that, for $1 \leq i \leq k-1$, the edge e_i has ends v_i and v_{i+1}. For $V' \subseteq V$, $G - V'$ denotes a graph obtained from G by deleting all vertices in V' together with all edges incident to them. For a subgraph G' of G, we denote by $G - G'$ the graph obtained from G by deleting all vertices in G'. *Subdividing an edge* (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \ldots, w_k, v(= w_{k+1})$ passing through new vertices $w_1, w_2, \ldots, w_k$, $k \geq 1$, of degree 2. A graph G is called a *subdivision* of a graph G' if G is obtained from G' by subdividing some of the edges of G' as illustrated in Figure 4. The *connectivity* $\kappa(G)$ of a

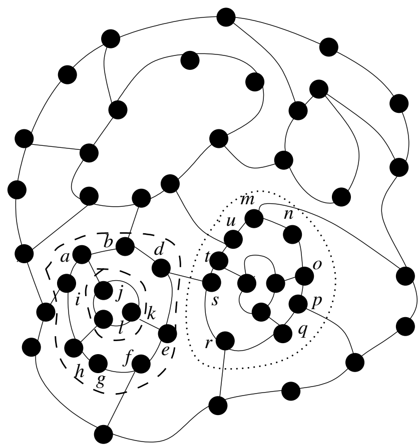

FIGURE 4. A subdivision of a planar triconnected cubic graph.

graph G is the minimum number of vertices whose deletion results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$. A subdivision of a triconnected cubic graph is biconnected, and the degree of any vertex is either 2 or 3. Figure 4 presents a subdivision of a planar triconnected cubic graph.
Faces and Cycles: A planar drawing Γ of a planar graph G, where there is no edge crossing, divides the plane into a set of open connected regions, called *faces*. The unbounded region is called the *the outer face* $F_o(\Gamma)$ and the regions that are bounded are called *inner faces* of Γ. There exists exactly one outer face in Γ. A subdivision of a planar triconnected cubic graph satisfies the following fact regarding faces [20].

Fact 1. Let G be a subdivision of a planar triconnected cubic graph. Let Γ_1 and Γ_2 be two different arbitrary plane embeddings of G. Then every face in Γ_1 is a face in Γ_2 and vice versa.

A sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1(= v_{k+1})$ of vertices and edges, where e_i connects v_i and $v_{(i+1)}$ and no vertex or edge appears more than once except the endpoint $v_1(=$

v_{k+1}), is called *a cycle*. The *contour* of a face F of Γ is the cycle formed by vertices and edges along the boundary of F. Such a cycle is also called a *facial cycle*. A facial cycle is sometimes called a face for simplicity. Sometimes we refer to the contour of the outer face F_o as the *boundary contour* and denote it by C_o. If G is biconnected, then all facial cycles are simple cycles. Let G be a planar graph, and Γ be an arbitrary plane embedding of G. For a cycle C of Γ, we call the plane subgraph of Γ inside C (including C) the *inner subgraph* $\Gamma_I(C)$ *for* C, and call the plane subgraph of Γ outside C (including C) the *outer subgraph* $\Gamma_O(C)$ *for* C. Any face of Γ is either in $\Gamma_I(C)$ or in $\Gamma_O(C)$. If an inner face or a cycle contains a vertex on F_o, then we call it a *boundary face* or a *boundary cycle*. If an inner face or a cycle does not contain a vertex on F_o, then we call it a *non-boundary face* or a *non-boundary cycle*.

Legs, Hands, Tracks, Regular Cycles, and Bad Cycles: An edge e is called a *leg* of a cycle C if e is located outside of C and exactly one endpoint of e is included in C. The vertex of C to which a leg is incident to is called a *leg-vertex* of C. We refer to e as a *hand* of C if e is located inside of C and exactly one endpoint of e is included in C. The vertex of C to which a hand is incident to is called a *hand-vertex* of C.

A cycle C in Γ is called a *k-legged cycle* of Γ if C has exactly k legs and satisfies an additional condition that there is no edge which joins two vertices on C in the outer subgraph $\Gamma_O(C)$. We define a *k-handed cycle* C symmetrically, i.e., C has exactly k hands and there is no edge which joins two vertices on C in the inner subgraph $\Gamma_I(C)$. The cycle $mnopqrstu$ in Figure 4 is a 3-handed cycle, whereas the same cycle is 5-legged in the embedding. A k-legged cycle (similarly, k-handed cycle) C in Γ is called *regular* if the plane graph $\Gamma - \Gamma_I(C)$ (similarly, $\Gamma - \Gamma_O(C)$) has a cycle. A k-legged cycle (similarly, k-handed cycle) is called *bad* if it does not have any degree-2 vertex.

A *track* of a cycle C is a path P on C such that P includes exactly two leg-vertices x and y of C, and x and y are the two endpoints of P. Therefore, each k-legged cycle has exactly k tracks. If a track (of a cycle which is not necessarily C_o) intersects (i.e., shares one or more edges) with the boundary contour, we call it *boundary track*. In fact, each boundary track is a sub-path of C_o.

We say that cycles C and C' in Γ are *independent* if $\Gamma_I(C)$ and $\Gamma_I(C')$ do not have any common vertex. A set S of cycles is *independent* if every pair of cycles in S is independent. In Figure 4, the cycles $abdefghi$, jkl (enclosed by long-dashed lines) are independent of the cycle $mnopqrstu$ (enclosed by dotted line) and vice versa.

Chains and Supports: Each track P of C is incident to exactly one face, denoted as $F_{C,P}$, in the outer region of C. Let $P = w_0, w_1, w_2, \ldots, w_{k+1}, k \geq 1$, be a path of G such that $d(w_0) \geq 3, d(w_1) = d(w_2) = \cdots = d(w_k) = 2$, and $d(w_{k+1}) \geq 3$. Then we call the subpath $P' = w_1, w_2, \ldots, w_k$ of P *a chain* of G, and we call vertices w_0 and w_{k+1} the *supports* of the chain P'. A degree-2 vertex belongs to exactly one chain in G.

Vertex Smoothing and Graph Homeomorphism: Two graphs G_1 and G_2 are called *homeomorphic* if they are subdivisions of the same graph. We now give a definition of smoothing degree-2 vertices from a graph. We often construct a new graph from a graph as follows. Let v be a degree-2 vertex in a connected graph G. We replace the two edges u_1v and u_2v incident to v with a single edge u_1u_2 and delete v. We call the operation above an *smoothing operation* and say that the vertex v is *smoothed out*. If all degree-2 vertices are smoothed out from a subdivision of a planar triconnected cubic graph G, then the resulting graph will be a planar triconnected cubic graph G', and removal of any two edges from a planar triconnected cubic graph G' does not disconnect G'.

Rahman et al. [11] gave a linear-time algorithm for a plane graph to have a rectilinear drawing and to find out a drawing if such one exists, as in the following lemma.

Lemma 2.1. *Assume that Γ is a biconnected plane graph with $\Delta \leq 3$ and there are four or more degree-2 vertices of Γ on $C_o(\Gamma)$. Then Γ has a rectilinear drawing if and only if every 2-legged cycle contains at least two degree-2 vertices of Γ and every 3-legged cycle contains at least one degree-2 vertex of Γ.*

Chang and Yen [16] gave a necessary and sufficient condition for a plane graph to have a rectilinear orthogonally convex drawing, as in the following lemma.

Lemma 2.2. *A biconnected plane graph Γ with $\Delta \leq 3$ has a rectilinear orthogonally convex drawing if and only if Γ satisfies all of the following conditions.*

(oc1) There are four or more degree-2 vertices of Γ on $C_o(\Gamma)$,
(oc2) every 2-legged cycle contains at least two degree-2 vertices,
(oc3) every 3-legged cycle contains at least one degree-2 vertex,
(oc4) every non-boundary 2-legged cycle contains at least one degree-2 vertex on each of its tracks, and
(oc5) every boundary 2-legged cycle contains at least one degree-2 vertex on its boundary track.

We also need the following observations regarding subdivisions of planar triconnected cubic graphs described in [13] stated in the following lemma.

Lemma 2.3. *Let G be a subdivision of a planar triconnected cubic graph, and let Γ be an arbitrary plane embedding of the planar graph G. Then the following (a) and (b) hold.*

(a) For any 2-legged cycle C of Γ, the set of all degree-2 vertices not in $\Gamma_I(C)$ induces a chain of G on $F_o(\Gamma)$.
(b) For any chain P on $F_o(\Gamma)$, the outer face of the plane graph $\Gamma - P$ is a 2-legged cycle in Γ.

The following lemma [13] gives a necessary and sufficient condition for a subdivision of a planar triconnected cubic graph G to have a rectilinear drawing.

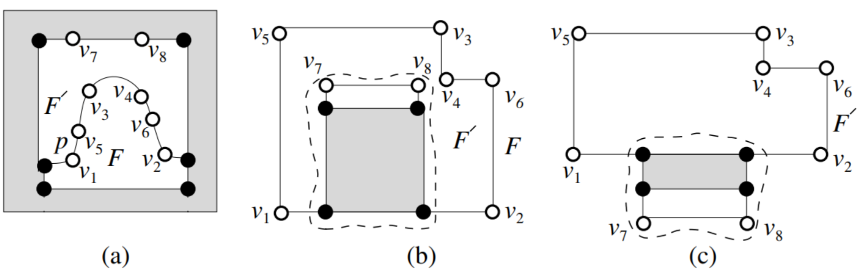

FIGURE 5. (a) Illustration of ($nb4$) in Lemma 2.4: degree-2 vertices are drawn by white circles, (b) a rectilinear orthogonal drawing taken the face F as outer face, and (c) a rectilinear orthogonal drawing taken the face F' as outer face.

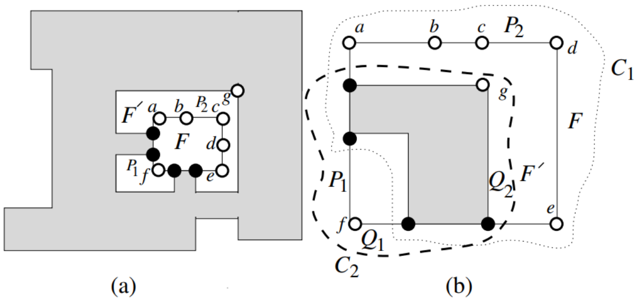

FIGURE 6. Illustration of $(nb5)$ in Lemma 2.4 and of (iv) in Theorem 3.1: degree-2 vertices are drawn by white circles.

Lemma 2.4. *Let G be a subdivision of a planar triconnected cubic graph, and let Γ be an arbitrary plane embedding of G. Then the planar graph G has a rectilinear drawing if and only if Γ has a face F satisfying the following conditions $(nb1) - (nb5)$.*

- *(nb1) There are at least four degree-2 vertices on F,*
- *(nb2) F is contained in $\Gamma_I(C)$ for any bad 3-legged cycle C in Γ,*
- *(nb3) F is contained in $\Gamma_O(C)$ for any bad 3-handed cycle C in Γ,*
- *(nb4) if there is exactly one chain P on F, then the face F' which contains P and is different from F contains at least two degree-2 vertices which are not on P (see Figure 5); and*
- *(nb5) if there are exactly two chains P_1 and P_2 on F, and one of them, say P_1, contains exactly one vertex, then the face F' which contains P_2 and is different from F contains at least one degree-2 vertex which is not on P_2 (see Figure 6).*

3. Rectilinear Orthogonally Convex Drawings

In this section, we give a necessary and sufficient condition for a subdivision of a planar triconnected cubic graph to have a rectilinear orthogonally convex drawing.

Theorem 3.1. *Let G be a subdivision of a planar triconnected cubic graph, and let Γ be an arbitrary plane embedding of G. G has a rectilinear orthogonally convex drawing if and only if Γ has a face F satisfying the following conditions $(i) - (iv)$.*

- *(i) There are at least four degree-2 vertices on F,*
- *(ii) F is contained in $\Gamma_I(C)$ for any bad 3-legged cycle C in Γ,*
- *(iii) F is contained in $\Gamma_O(C)$ for any bad 3-handed cycle C in Γ, and*
- *(iv) there are at least two chains on the facial cycle for F in Γ. If there are exactly two chains P_1 and P_2 on F, and any of them, say P_1, contains exactly one vertex, then the face F' which contains P_2 and is different from F contains at least one degree-2 vertex which is not on P_2 (illustrated in Figure 6).*

To prove Theorem 3.1 we need the following two lemmas which were stated as facts in [13] without proofs. Here we include the proofs for completeness.

Lemma 3.2. *Let G be a subdivision of a planar triconnected cubic graph and let Γ_1 and Γ_2 be two different arbitrary plane embeddings of G. Let F be the face of Γ_1 such that*

$F = F_o(\Gamma_2)$. *Assume that* C *is a* 3*-legged cycle in* Γ_1 *and* $\Gamma_I(C)$ *of* Γ_1 *contains* F. *Then* C *is a* 3*-handed cycle in* Γ_2*, and vice versa.*

Proof. Let G be a subdivision of a planar triconnected cubic graph. Let Γ_1 and Γ_2 be two different arbitrary plane embeddings of G. Since G be a subdivision of a planar triconnected cubic graph, by Fact 1, every face in Γ_1 is a face in Γ_2 and vice versa. Let F be a face in Γ_1 which is the outer face in Γ_2.

Assume that C is a 3-legged cycle or a 3-handed cycle in Γ_1 and $\Gamma_I(C)$ contains F in Γ_1.

If C is a 3-legged cycle in Γ_1, then the subgraph of Γ_1 outside C is folded inside of C in Γ_2, and hence C is 3-handed cycle in Γ_2. To observe this more formally, consider a path P that contains exactly one vertex from C and exactly one vertex from F, and lies entirely inside C in Γ_1. Since F is an outer face in Γ_2, the path must be inside F. If P remains outside of C in Γ_2, then the required property is satisfied. If P enters the interior of C and then exists C, then it must contain two vertices of C, which contradicts our initial assumption.

Similarly, if C is a 3-handed cycle in Γ_1, then the subgraph of Γ_1 inside C but outside of F, is folded outside of C in Γ_2, and hence C is 3-legged cycle in Γ_2. ∎

Lemma 3.3. *Let* G *be a subdivision of a planar triconnected cubic graph and let* Γ_1 *and* Γ_2 *be two different arbitrary plane embeddings of* G. *Let* F *be the face of* Γ_1 *such that* $F = F_o(\Gamma_2)$. *Assume that* C *is a cycle in* Γ_1*, and* $\Gamma_I(C)$ *of* Γ_1 *does not contain* F. *If* C *is a* 3*-legged cycle in* Γ_1*, then* C *is a* 3*-legged cycle in* Γ_2. *If* C *is a* 3*-handed cycle in* Γ_1*, then* C *is a* 3*-handed cycle in* Γ_2.

Proof. Let G be a subdivision of a planar triconnected cubic graph. Let Γ_1 and Γ_2 be two different arbitrary plane embeddings of G. Since G be a subdivision of a planar triconnected cubic graph, by Fact 1, every face in Γ_1 is a face in Γ_2 and vice versa. Let F be a face in Γ_1 which is the outer face in Γ_2.

Assume that, C is either a 3-legged cycle or a 3-handed cycle in Γ_1 and $\Gamma_I(C)$ does not contain F in Γ_1.

If C is a 3-legged cycle in Γ_1, then all the three legs remain same in Γ_2, induces a 3-legged cycle C in Γ_2. If C is a 3-handed cycle in Γ_1, then all the three hands remain same in Γ_2, induces a 3-handed cycle C in Γ_2. ∎

We now prove another lemma before giving a proof of Theorem 3.1.

Lemma 3.4. *Let* G *be a subdivision of a planar triconnected cubic graph and let* Γ *be an arbitrary plane embedding of* G. *Then the following properties hold.*

- *(pr1)* Γ *does not contain any regular* 2*-legged cycle.*
- *(pr2) Every* 2*-legged cycle in* Γ *has an edge on* $C_o(\Gamma)$.
- *(pr3)* Γ *does not contain two independent* 2*-legged cycles.*
- *(pr4) If* Γ *has three or more chains on* $C_o(\Gamma)$*, then every* 2*-legged cycle has at least two degree-*2 *vertices on* $C_o(\Gamma)$.

Proof. Let G be a subdivision of a planar triconnected cubic graph and Γ be an arbitrary plane of G.

(pr1) Let G' be the graph obtained by smoothing out all degree-2 vertices of G. G' is a planar triconnected cubic graph, and G and G' are homeomorphic to each other. If Γ has a regular 2-legged cycle C, then by the definition of regular

2-legged cycles, the deletion of the two legs of C, i.e., the edges incident to C but not their end vertices, results in a disconnected graph with two connected components having cycles. Then G' cannot be a planar triconnected cubic graph.

(pr2) Assume for a contradiction that any 2-legged cycle C does not have any edge on $C_o(\Gamma)$. Then removal of the two legs of C induces two connected components in Γ, each of them has cycle, and hence G would not be a subdivision of a planar triconnected cubic graph, a contradiction.

(pr3) Since there is no regular 2-legged cycle C in Γ, Γ cannot contain two independent 2-legged cycles.

(pr4) By property pr(1), every 2-legged cycle in Γ has an edge on $C_o(\Gamma)$, and by property pr(3), Γ has exactly one independent 2-legged cycle. By Lemma 2.3(b), for any chain P on $F_o(\Gamma)$, the outer face of the plane graph $\Gamma - P$ is a 2-legged cycle in Γ. Hence the induced 2-legged cycle for P has two or more chains, thus at least two degree-2 vertices on $C_o(\Gamma)$.

Since all the properties are necessary, the proof of the lemma is now complete. ∎

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1

Necessity: Let G be a subdivision of a planar triconnected cubic graph and Γ be an arbitrary plane embedding of G. Assume that a planar embedding Γ' of G has a rectilinear orthogonally convex drawing. Let F be the face of Γ such that $F = F_o(\Gamma')$. We now show that F satisfies (i) to (iv) of Theorem 3.1, as follows.

(i) Lemma 2.2 gives a necessary and sufficient condition for a biconnected plane graph with $\Delta \leq 3$ to have a rectilinear orthogonally convex drawing. Since a subdivision of a triconnected cubic plane graph is a subclass of biconnected plane graphs with $\Delta \leq 3$, the conditions in Lemma 2.2 are necessary and sufficient conditions for a subdivision of triconnected cubic plane graph to have a rectilinear orthogonally convex drawing. Since Γ' has a rectilinear orthogonally convex drawing, by $(oc1)$ of Lemma 2.2, the outer face F of Γ' has at least four degree-2 vertices. By Fact 1, F is also face in Γ and hence the face F for Γ must contain at least four degree-2 vertices.

(ii) Assume for a contradiction that there exists a bad 3-legged cycle C such that F is not contained in $\Gamma_I(C)$ in Γ, i.e. Γ has a 3-legged cycle C such that $\Gamma_I(C)$ does not contain F and C does not contain any degree-2vertex. Then by Lemma 3.3, in Γ', C is a 3-legged cycle without any degree-2 vertex, a contradiction with the $(oc3)$ of Lemma 2.2.

(iii) Assume for a contradiction that there exists a bad 3-handed cycle C such that F is not contained in $\Gamma_O(C)$ in Γ, i.e. Γ has a 3-handed cycle C and $\Gamma_I(C)$ contains F but C does not contain any degree-2 vertex. Then by Lemma 3.2, in Γ', C is a 3-legged cycle without any degree-2 vertex, a contradiction with the $(oc3)$ of Lemma 2.2.

(iv) Assume for a contradiction that there is exactly one chain P on F in Γ. The chain P on F induces a 2-legged cycle $F(C) - P$ on F. The 2-legged cycle $F(C) - P$ does not contain any degree-2 vertex on F. Since $F_o(\Gamma') = F$, the 2-legged cycle $F(C) - P$ does not contain any degree-2 vertex on its boundary track, a contradiction with the $(oc5)$ of Lemma 2.2. We now assume that F has exactly two chains P_1 and P_2, and P_1 contains exactly one vertex, as illustrated in Figure 6. Since $F_o(\Gamma') = F$ contains at least four degree-2 vertices, P_2 contains at least three degree-2 vertices. By Lemma 2.3(b), Γ' has two 2-legged cycles $C_1 = F_o(\Gamma' - P_1)$ and $C_2 = F_o(\Gamma' - P_2)$ as illustrated in Figure 6, where C_1 is indicated by small dotted lines and C_2 is indicated by big dotted lines. By Lemma 2.3, Γ' does not have any boundary 2-legged cycle other than C_1 and C_2. By $(oc2)$ of Lemma 2.2, C_2 contains at least two degree-2 vertices. The maximal subpath Q_1 of C_2 that is on $F_o(\Gamma')$ contains exactly one degree-2 vertex, and hence the maximal subpath Q_2 of C_2 that is not on $F_o(\Gamma')$ contains at least one degree-2 vertex. Clearly Q_2 is on F' and is not on P_2, and hence F' contains at least one degree-2 vertex which is not on P_2.

Sufficiency: Assume that Γ has a face F satisfying conditions $(i) - (iv)$ of Theorem 3.1. Let Γ' be an embedding of G such that $F = F_o(\Gamma')$. It is sufficient to prove that Γ' satisfies Conditions $(oc1) - (oc5)$ in Lemma 2.2.

(i) By Condition (i), $F_o(\Gamma') = F$ contains at least four degree-2 vertices, and hence Γ' satisfies condition $(oc1)$ in Lemma 2.2.

(ii) By Lemma 3.4, every 2-legged cycle has an edge on the outer face of Γ' and the number of independent 2-legged cycles in Γ' is exactly one. By condition (iv), $F_o(\Gamma') = F$ contains at least two chains. Assume there are exactly two chains P_1 and P_2 on $F_o(\Gamma') = F$. By Condition (iv), if any of P_1 and P_2, say P_1 contains exactly one vertex, then the face F' which contains P_2 and is different from F contains at least one degree-2 vertex which is not on P_2. The 2-legged cycle $C_o(\Gamma') - P_2$ has two degree-2 vertices, one is on $F_o(\Gamma')$ and another is on F'. The 2-legged cycle $C_o(\Gamma') - P_1$ has two degree-2 vertices, both are on $F_o(\Gamma')$. Thus Γ' satisfies $(oc2)$ in Lemma 2.2.

(iii) By Conditions (ii) and (iii), F is contained in $\Gamma_I(C)$ for any bad 3-legged cycle C in Γ, and F is contained in $\Gamma_O(C)$ for any bad 3-handed cycle C in Γ, i.e. every 3-legged cycle C in Γ whose $\Gamma_I(C)$ does not contain F, has at least one egree-2 vertex and every 3-handed cycle C in Γ whose $\Gamma_I(C)$ contains F, has at least one degree-2 vertex. Since $F_o(\Gamma') = F$, then by Lemma 3.2 and by Lemma 3.3, Γ' satisfies the condition $(oc3)$ of Lemma 2.2.

(iv) By property $(pr2)$ of Lemma 3.4, all 2-legged cycles have edges on the outer face of Γ'. Condition $(oc4)$ is void for Γ'.

(v) By properties $(pr2)$ and $(pr3)$ of Lemma 3.4, every 2-legged cycle has an edge on the outer face of Γ' and the number of independent 2-legged cycles in Γ' is exactly one. By condition (iv), there are at least two chains on F. Since $F_o(\Gamma') = F$, every boundary 2-legged cycle contains at least one degree-2 vertex on its boundary track.

Since all the conditions are necessary and sufficient, the proof of the theorem is now complete. ■

Traversing the contours of all faces in Γ, one can check in linear time whether G satisfies the conditions $(i) - (iv)$ in Theorem 3.1. We refer the readers to [13, 21–23] for contour traversal approaches to check whether a planar embedding Γ of a planar graph G has a face F that satisfies a similar set of conditions. A planar embedding Γ' can be computed from Γ making F as outer face in linear time [20]. Thus, if G satisfies the conditions $(i) - (iv)$ of Theorem 3.1 then G has a planar embedding Γ' which satisfies the conditions in Lemma 2.2, and hence using the drawing algorithm of Chang and Yen [16], one can also find a rectilinear orthogonally convex drawing of the planar embedding Γ' of G in linear time. Thus the following theorem holds.

Theorem 3.5. *Let G be a subdivision of a planar triconnected cubic graph. Then one can determine in linear time whether G has a rectilinear orthogonally convex drawing or not and find a drawing of G, if it exists.*

The following theorem shows an interesting relationship between rectilinear drawings and rectilinear orthogonally convex drawings of subdivisions of planar triconnected cubic graphs.

Theorem 3.6. *Let G be a subdivision of a triconnected cubic planar graph. G has a rectilinear drawing if and only if G has a rectilinear orthogonally convex drawing.*

Proof. Since all rectilinear orthogonally convex drawings are rectilinear drawings, sufficiency of the theorem is satisfied. Hence here we prove the necessity only. Let G be a subdivision of a planar triconnected cubic graph and Γ be an arbitrary planar embedding of G. Assume that G has a rectilinear drawing. Then Γ has a face F satisfying conditions $(nb1) - (nb5)$ of Lemma 2.4. We now show that Γ has a face F^* satisfying conditions $(i) - (iv)$ of Theorem 3.1. We need to consider the following three cases.
case 1: F has three or more chains.
F satisfies conditions $(nb1) - (nb3)$ of Lemma 2.4. Since conditions $(i) - (iii)$ of Theorem 3.1 are same to $(nb1) - (nb3)$ of Lemma 2.4 respectively, F satisfies conditions $(i) - (iii)$ of Theorem 3.1. Condition (iv) of Theorem 3.1 is void. Hence $F^* = F$.
case 2: F has exactly two chains.
F satisfies conditions $(nb1) - (nb3)$ and $(nb5)$ of Lemma 2.4. In this case, F satisfies conditions $(i) - (iv)$ of Theorem 3.1. Therefore $F^* = F$.
case 3: F has exactly one chain.
F satisfies conditions $(nb1) - (nb4)$ of Lemma 2.4 as illustrated in Figure 5(a). Corresponding rectilinear drawing for the plane embedding of Γ' is illustrated in Figure 5(b), where $F = F_o(\Gamma')$. In this case $F \neq F^*$. In Figure 5(b), every 3-legged cycles in the sub-graph shown by shaded region has at least one degree-2 vertex. Hence, there is no bad 3-legged cycle in Γ'. Every bad 3-handed cycle is in the shaded region, if any bad 3-handed cycle exists. One can observe that, the face F induces another face F' that satisfies conditions $(i) - (iv)$ of Theorem 3.1. F' has at least two chains and if F' has exactly two chains then every chain has at least two vertices. The orthogonal drawing taken the face F' as outer face is orthogonally convex as illustrated in Figure 5(c). In this case $F^* = F'$. ∎

4. Conclusions

In this paper we have presented a necessary and sufficient condition for a subdivision of a triconnected cubic planar graph G to have a rectilinear orthogonally convex drawing. Our condition leads to a linear-time algorithm to examine whether G satisfies the conditions and to find a rectilinear orthogonally convex drawing, if it exists. We have shown an interesting relationship that such a graph G has a rectilinear drawing if and only if G has a rectilinear orthogonally convex drawing. Finding linear-time algorithms for rectilinear orthogonally convex drawings for remaining larger classes of planar graphs may be a nice future work. We have given an outline of a linear algorithm for no-bend orthogonal drawings and no-bend orthogonally convex drawings of general planar graphs of maximum degree three [18]. However, the algorithm presented in this paper is a simpler one for a special class of graphs.

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