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Graph of uv-Paths in Connected Graphs

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Abstract For a connected graph G and vertices u, v of G we define an abstract graph $\mathcal{P}(G_{uv})$ whose vertices are the paths joining u and v in G, where paths S and T are adjacent if T is obtained from S by replacing a subpath S_{xy} of S with an internally disjoint subpath T_{xy} of T. Let $\mathcal{C}$ be a set of cycles of G; the *uv-path graph of G defined by $\mathcal{C}$* is the spanning subgraph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ of $\mathcal{P}(G_{uv})$ in which two paths S and T are adjacent if and only if the unique cycle σ contained in $S \cup T$ lies in $\mathcal{C}$. We prove that $\mathcal{P}(G_{uv})$ is always connected and give a necessary condition and a sufficient condition for a graph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ to be connected.

MSC: 05C38; 05C12
Keywords: uv-path; path graph: flipping subpaths

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1. Introduction

For any vertices x, y of a path L, we denote by L_{xy} the subpath of L that joins x and y. Let G be a connected graph and u and v be vertices of G. The *uv-path graph of G* is the graph $\mathcal{P}(G_{uv})$ whose vertices are the paths joining u and v in G, where two paths S and T are adjacent if T is obtained from S by replacing a subpath S_{xy} of S with an internally disjoint subpath T_{xy} of T. See Fig. 1 for a small example.

Let P and f be a polytope and a linear functional f in $\mathbb{R}^d$ which is nonconstant on every edge of P. Let x and y be the vertices of P in which f achieves its minimum and maximum, respectively. An f-monotone path on P is a sequence $x = v_0, v_1, \ldots, v_m = y$ of vertices of P such that for $i = 0, 1, \ldots, m-1$, $v_i v_{i+1}$ is an edge of P with $f(v_i) < f(v_{i+1})$.

The uv-path graph $\mathcal{P}(G_{uv})$ is closely related to the graph $G(P, f)$ of f-monotone paths on a polytope P (see C. A. Athanasiadis *et al* [1, 2]), whose vertices are the f-monotone paths on P and where two paths S and T are adjacent if there is a 2-dimensional face F of P such that T is obtained from S by replacing an f-monotone subpath of S contained in F with the complementary f-monotone subpath of T contained in F.

In Section 2 we show that the graphs $\mathcal{P}(G_{uv})$ are always connected as is the case for the graphs $G(P, f)$.

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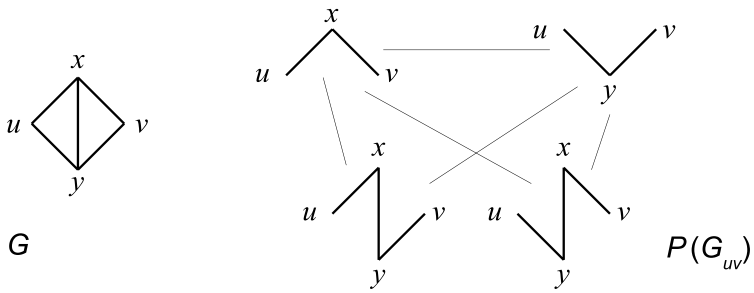

FIGURE 1. A graph G and the corresponding uv-path graph $\mathcal{P}(G_{uv})$

If S and T are adjacent vertices in a uv-path graph $\mathcal{P}(G_{uv})$, then $S \cup T$ is a subgraph of G consisting of a unique cycle σ joined to u and v by disjoint paths P_u and P_v. See Fig. 2.

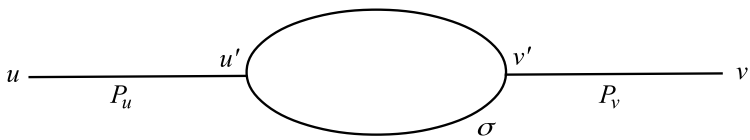

FIGURE 2. $S \cup T$

Let $\mathcal{C}$ be a set of cycles of G; the *uv-path graph of G defined by $\mathcal{C}$* is the spanning subgraph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ of $\mathcal{P}(G_{uv})$ where two paths S and T are adjacent if and only if the unique cycle σ which is contained in $S \cup T$ lies in $\mathcal{C}$. A graph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ may be disconnected.

The uv-path graph $\mathcal{P}(G_{uv})$ is also related to the well-known *tree graph* $\mathcal{T}(G)$ of a connected graph G, studied by R. L. Cummins [3], in which the vertices are the spanning trees of G and the edges correspond to pairs of trees S and R which are obtained from each other by a single edge exchange. As in the uv-path graph, if two trees S and R are adjacent in $\mathcal{T}(G)$, then $S \cup R$ is a subgraph of G containing a unique cycle. X. Li *et al* [5] define, in an analogous way, a subgraph $\mathcal{T}_{\mathcal{C}}(G)$ of $\mathcal{T}(G)$ for a set of cycles $\mathcal{C}$ of G and give a necessary condition and a sufficient condition for $\mathcal{T}_{\mathcal{C}}(G)$ to be connected. In Section 3 and Section 4, respectively, we show that the same conditions apply to uv-path graphs $\mathcal{P}_{\mathcal{C}}(G_{uv})$.

Similar results are obtained by A. P. Figueroa *et al* [4] with respect to the *perfect matching graph* $\mathcal{M}(G)$ of a graph G where the vertices are the perfect matchings of G and in which two matchings L and M are adjacent if their symmetric difference is a cycle of G. Again, if L and M are adjacent matchings in $\mathcal{M}(G)$, then $L \cup M$ contains a unique cycle of G.

For any subgraphs F and H of a graph G, we denote by $F \Delta H$ the subgraph of G induced by the set of edges $(E(F) \setminus E(H)) \cup (E(H) \setminus E(F))$.

2. Preliminary Results

In this section we prove that the uv-path graph is connected for any connected graph G and give an upper bound for the diameter of a graph $\mathcal{P}(G_{uv})$.

Theorem 2.1. *Let G be a connected graph. The uv-path graph $\mathcal{P}(G_{uv})$ is connected for every pair of vertices u, v of G.*

Proof. For any different uv paths Q and R in G denote by $n(Q,R)$ the number of consecutive initial edges Q and R have in common. Assume the result is false and choose two uv paths $S : u = x_0, x_1, \ldots, x_s = v$ and $T : u = y_0, y_1, \ldots, y_t = v$ in different components of $\mathcal{P}(G_{uv})$ for which $n^* = n(S,T)$ is maximum.

Since edges $x_{n^*}x_{n^*+1}$ and $y_{n^*}y_{n^*+1}$ are not equal, $x_{n^*+1} \neq y_{n^*+1}$. Let $j = min\{i : x_{n^*+i} \in V(T)\}$ and $k = min\{i : y_{n^*+i} \in V(S)\}$ and let l and m be integers such that $y_l = x_{n^*+j}$, $x_m = y_{n^*+k}$. Consider the path:

$$S' : u = x_0, x_1, \ldots, x_{n^*}, y_{n^*+1}, y_{n^*+2}, \ldots, y_{n^*+k}, x_{m+1}, x_{m+2}, \ldots, x_s = v$$

Paths S and S' are adjacent in $\mathcal{P}(G_{uv})$ since S' is obtained from S by replacing the subpath $x_{n^*}, x_{n^*+1}, \ldots, x_m$ of S with the subpath $y_{n^*}, y_{n^*+1}, \ldots, y_{n^*+k}$ of S'. Notice that $n(S',T) \geq n(S,T) + 1$ since $x_0x_1, x_1x_2, \ldots, x_{n^*-1}x_{n^*}, x_{n^*}y_{n^*+1} \in E(S') \cap E(T)$. By the choice of S, and T, paths S' and T are connected in $P(G_{uv})$. This implies that S and T are also connected in $\mathcal{P}(G_{uv})$ which is a contradiction. ∎

For any two vertices u and v of a connected graph G we denote by $d_G(u,v)$ the distance between u and v in G, that is the length of a shortest uv path in G. The diameter of a connected graph G is the maximum distance among pairs of vertices of G. For a path P, we denote by $l(P)$ the length of P.

Theorem 2.2. *Let u and v be vertices of a connected graph G. The diameter of the graph $\mathcal{P}(G_{uv})$ is at most $2d_G(u,v)$.*

Proof. Let S and T be uv paths in G and let P be a shortest uv path in G. From the proof of Theorem 2.1 one can see that there are two paths Q_S and Q_T in $\mathcal{P}(G_{uv})$, each with length at most $l(P)$, joining S to P and T to P, respectively. Clearly $Q_S \cup Q_T$ contains a path joining S and T in $\mathcal{P}(G_{uv})$ with length at most $2l(P) = 2d_G(u,v)$. ∎

In Fig. 3 we show a connected graph G and paths S and T joining vertices u and v of G such that $d_G(u,v) = 2$ and $d_{\mathcal{P}(G_{uv})}(S,T) = 4$.

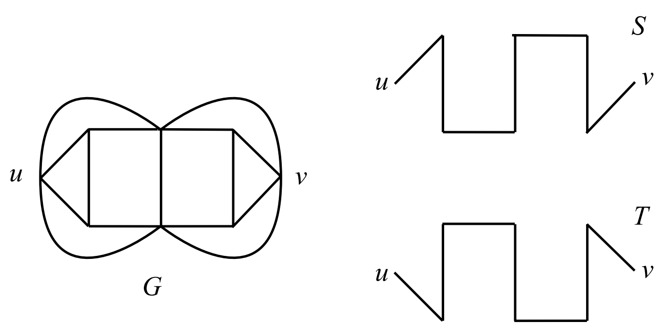

FIGURE 3. Graph G and paths S and T.

For any positive integer $k > 2$ the graph G can be extended as in Fig. 4 to a graph G_k such that $d_{G_k}(u,v) = k$, while the diameter of the corresponding uv-path graph is $2k$. This shows that Theorem 2.2 is tight.

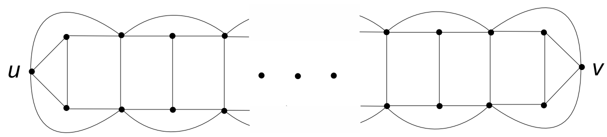

Figure 4. Graph G_k.

3. Necessary Condition

Let u and v be vertices of a connected graph G and S and T be two uv paths adjacent in $\mathcal{P}(G_{uv})$. Since T is obtained from S by replacing a subpath S_{xy} of S with an internally disjoint subpath T_{xy} of T, the graph $S\Delta T$ is the cycle $S_{xy} \cup T_{xy}$.

An *even subgraph* of a graph G is a subgraph of G with the property that each of its vertices has even degree. The *cycle space* of G is the set of all even subgraphs of G, together with the symmetric difference operator.

Theorem 3.1. *Let G be a connected graph, u and v be vertices of G and $\mathcal{C}$ be a set of cycles of G. If the graph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected, then $\mathcal{C}$ spans the cycle space of G.*

Proof. Let σ be a cycle of G. Since G is connected, there are two disjoint paths P_u and P_v joining, respectively, u and v to σ. Denote by u' and v' the unique vertices of P_u and P_v, respectively, that lie in σ. Vertices u' and v' partition cycle σ into two internally disjoint paths Q and R. Let $S = P_u \cup Q \cup P_v$ and $T = P_u \cup R \cup P_v$. Clearly S and T are two different uv paths in G such that $S\Delta T = \sigma$.

Since $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected, there are uv paths $S = W_0, W_1, \ldots, W_k = T$ such that for $i = 1, 2, \ldots, k$, paths W_{i-1} and W_i are adjacent in $\mathcal{P}_{\mathcal{C}}(G_{uv})$. For $i = 1, 2, \ldots k$ let $\alpha_i = W_{i-1}\Delta W_i$. Then $\alpha_1, \alpha_2, \ldots, \alpha_k$ are cycles in $\mathcal{C}$ such that:

$$\alpha_1\Delta\alpha_2\Delta\cdots\Delta\alpha_k = (W_0\Delta W_1)\Delta(W_1\Delta W_2)\Delta\cdots\Delta(W_{k-1}\Delta W_k) = W_0\Delta W_k = \sigma$$

Therefore $\mathcal{C}$ spans σ. ∎

Let G be a complete graph with four vertices u, x, y, v and let $\mathcal{C} = \{\alpha, \beta, \delta\}$, where $\alpha = uxv$, $\beta = uyv$ and $\delta = uxyv$. Set $\mathcal{C}$ spans the cycle space of G but the graph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is not connected since the uv path $uyxv$ is an isolated vertex of $\mathcal{P}_{\mathcal{C}}(G_{uv})$, see Fig 5. This shows that the condition in Theorem 3.1 is not sufficient for $P_{\mathcal{C}}(G_{uv})$ to be connected.

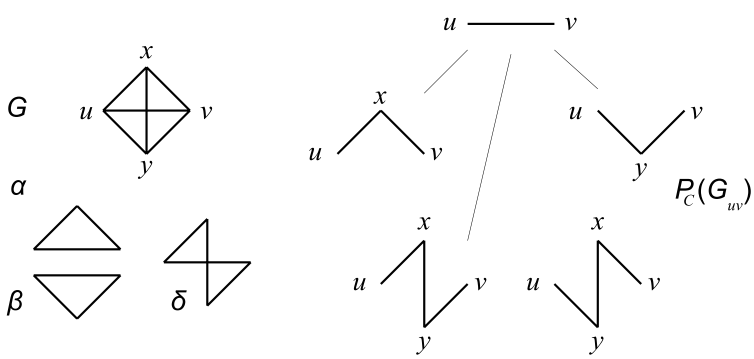

Figure 5. Graph G, set $\mathcal{C} = \{\alpha, \beta, \delta\}$ and graph $\mathcal{P}_C(G_{uv})$.

4. Sufficient Condition

A *unicycle* of a connected graph G is a spanning subgraph $\mathcal{U}$ of G that contains a unique cycle. Let u and v be vertices of a connected graph G. A *uv-monocle* of G is a subgraph of G that consists of a cycle σ and two disjoint paths P_u and P_v that join, respectively u and v to σ, see Fig. 2. Clearly for each uv-monocle $\mathcal{M}$ of a connected graph G, there is a unicycle $\mathcal{U}$ of G that contains $\mathcal{M}$.

Let $\mathcal{C}$ be a set of cycles of G. A cycle σ of G has *Property* Δ^* with respect to $\mathcal{C}$ if for every unicycle $\mathcal{U}$ containing σ there is an edge e of G, not in $\mathcal{U}$ and two cycles $\alpha, \beta \in \mathcal{C}$, contained in $\mathcal{U} + e$, such that $\sigma = \alpha \Delta \beta$.

Lemma 4.1. *Let G be a connected graph and u and v be vertices of G. Also let $\mathcal{C}$ be a set of cycles of G and σ be a cycle having Property Δ^* with respect to $\mathcal{C}$. The graph $\mathcal{P}_{\mathcal{C}\cup\{\sigma\}}(G_{uv})$ is connected if and only if $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected.*

Proof. If $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected, then $\mathcal{P}_{\mathcal{C}\cup\{\sigma\}}(G_{uv})$ is connected since the former is a subgraph of the latter.

Assume now $\mathcal{P}_{\mathcal{C}\cup\{\sigma\}}(G_{uv})$ is connected and let S and T be uv paths in G which are adjacent in $\mathcal{P}_{\mathcal{C}\cup\{\sigma\}}(G_{uv})$. We show next that S and T are connected in $\mathcal{P}_{\mathcal{C}}(G_{uv})$ by a path of length at most 2.

If $\omega = S \Delta T \in \mathcal{C}$, then S and T are adjacent in $\mathcal{P}_{\mathcal{C}}(G_{uv})$. For the case $\omega = \sigma$ denote by $\mathcal{M}$ the uv-monocle given by $S \cup T$.

Let $\mathcal{U}$ be a unicycle of G containing $\mathcal{M}$. Since σ has Property Δ^* with respect to $\mathcal{C}$, there exists an edge $e = xy$ of G, not in $\mathcal{U}$, and two cycles $\alpha, \beta \in \mathcal{C}$ contained in $\mathcal{U} + e$ such that $\sigma = \alpha \Delta \beta$.

Let x' and y' denote the vertices in $\mathcal{M}$ which are closest in $\mathcal{U}$ to x and y, respectively. Then there exists a path $R_{x'y'}$ in G, with edges in $E(\mathcal{U} + e) \setminus E(\mathcal{M})$ joining x' and y' and such that cycles α and β are contained in $\mathcal{M} \cup R_{x'y'}$. We analyze several cases according to the location of x' and y' in $\mathcal{M}$.

Denote by P_u and P_v the unique paths, contained in $\mathcal{M}$, that join u and v to σ and by u' and v' the vertices where P_u and P_v, respectively, meet σ.

Case 1.- $x' \in V(P_u)$, $y' \in V(P_v)$. Without loss of generality we assume $\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = T_{x'y'} \cup R_{y'x'}$, see Fig. 6.

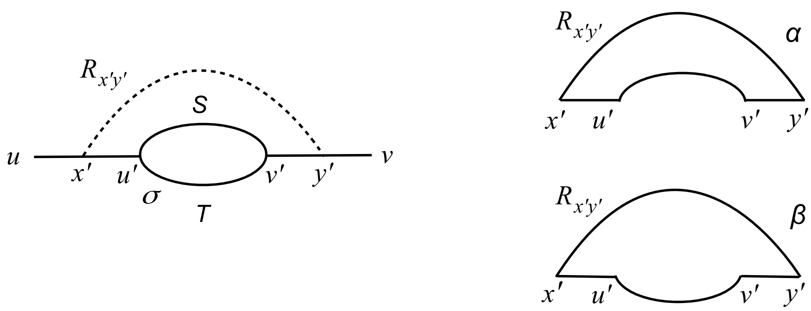

Figure 6. Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles α and β.

Let Q be the uv-path obtained from S by replacing $S_{x'y'}$ with $R_{x'y'}$. Notice that Q can also be obtained from T by replacing $T_{x'y'}$ with $R_{x'y'}$.

Case 2.- $x' \in V(P_u)$, $y' \in S \cap \sigma$. Without loss of generality we assume $\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = T_{x'v'} \cup S_{v'y'} \cup R_{y'x'}$, see Fig. 7.

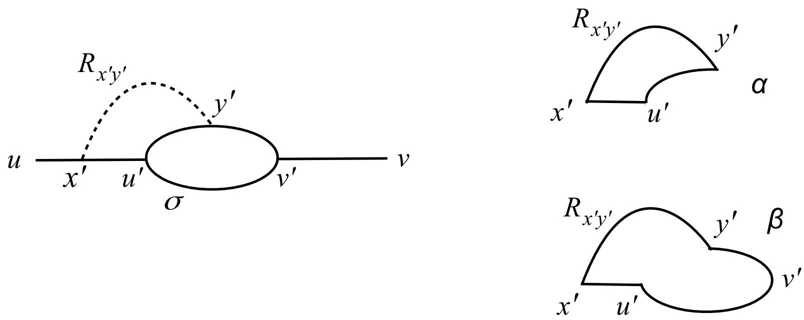

FIGURE 7. Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles α and β.

Again let Q be the uv-path obtained from S by replacing $S_{x'y'}$ with $R_{x'y'}$. In this case, Q can also be obtained from T by replacing $T_{x'v'}$ with $R_{x'y'} \cup S_{y'v'}$.

Case 3.- $x', y' \in S \cap \sigma$. Without loss of generality we assume $\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = S_{u'x'} \cup R_{x'y'} \cup S_{y'v'} \cup T_{v'u'}$, see Fig. 8.

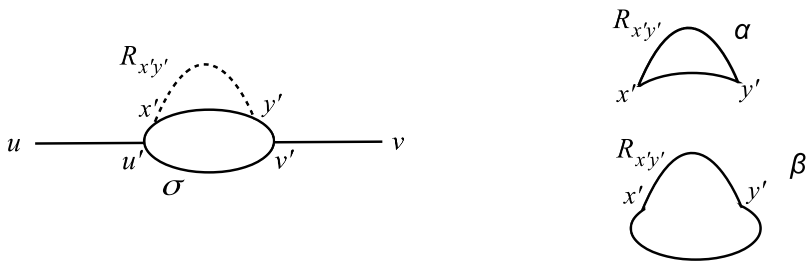

FIGURE 8. Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles α and β.

Let Q be the uv-path obtained from S by replacing $S_{x'y'}$ with $R_{x'y'}$. Path Q is also obtained from T by replacing $T_{u'v'}$ with $S_{u'x'} \cup R_{x'y'} \cup S_{y'v'}$.

Case 4.- $x' \in S \cap \sigma$ and $y' \in T \cap \sigma$. Without loss of generality we assume $\alpha = S_{u'x'} \cup R_{x'y'} \cup T_{y'u'}$ and $\beta = T_{y'v'} \cup S_{v'x'} \cup R_{x'y'}$. , see Fig. 9.

Let Q be the uv-path obtained from S by replacing $S_{u'x'}$ with $T_{u'y'} \cup R_{y'x'}$. Now Q can also be obtained from T by replacing $T_{y'v'}$ with $R_{y',x'} \cup S_{x'v'}$

In each case $S \Delta Q = \alpha$ and $Q \Delta T = \beta$. Since $\alpha, \beta \in \mathcal{C}$, path S is adjacent to Q and path Q is adjacent to T in $P_{\mathcal{C}}(G_{uv})$. Therefore S and T are connected in $\mathcal{P}_{\mathcal{C}}(G_{uv})$ by a path with length at most 2.

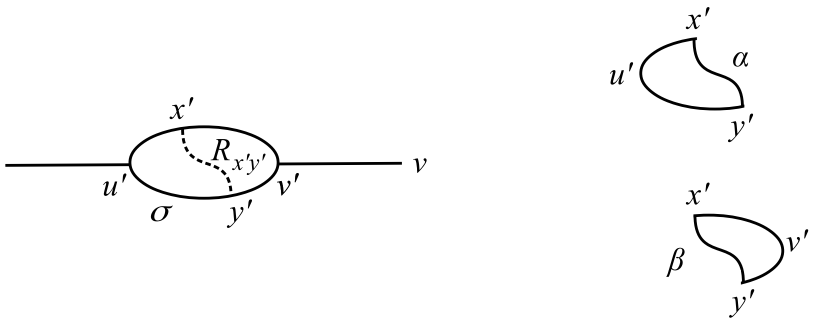

FIGURE 9. Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles α and β.

All remaining cases are analogous to either Case 2 or to Case 3. ∎

Consider a connected graph G with two specified vertices u and v and let $\mathcal{C}$ be a set of cycles of G. Construct a sequence of sets of cycles $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$ as follows: If there is a cycle σ_1 not in $\mathcal{C}_0$ that has Property Δ^* with respect to $\mathcal{C}_0$ add σ_1 to $\mathcal{C}_0$ to obtain $\mathcal{C}_1$. At step t add to $\mathcal{C}_t$ a new cycle σ_{t+1} (if it exists) that has Property Δ^* with respect to $\mathcal{C}_t$ to obtain $\mathcal{C}_{t+1}$. Stop at a step k where there are no cycles, not in $\mathcal{C}_k$, having Property Δ^* with respect to $\mathcal{C}_k$. We denote by $Cl(\mathcal{C})$ the final set obtained with this process. Li *et al* [5] proved that the final set of cycles obtained is independent of which cycle σ_t is added at each step in the case of multiple possibilities.

A set of cycles of G is Δ^*-*dense* if $Cl(\mathcal{C})$ is the whole set of cycles of G.

Theorem 4.2. *If $\mathcal{C}$ is Δ^*-dense, then $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected.*

Proof. Since $\mathcal{C}$ is Δ^*-dense, $Cl(\mathcal{C})$ is the set of cycles of G and therefore $\mathcal{P}_{Cl(\mathcal{C})}(G_{uv}) = \mathcal{P}(G_{uv})$ which is connected by Theorem 2.1.

Let $\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k = Cl(\mathcal{C})$ be a sequence of sets of cycles obtained from $\mathcal{C}$ as above. By Lemma 4.1, all graphs $\mathcal{P}_{Cl(\mathcal{C})}(G_{uv}) = \mathcal{P}_{\mathcal{C}_k}(G_{uv}), \mathcal{P}_{\mathcal{C}_{k-1}}(G_{uv}), \ldots, \mathcal{P}_{\mathcal{C}_0}(G_{uv}) = \mathcal{P}_{\mathcal{C}}(G_{uv})$ are connected. ∎

Li *et al* [5] proved the following:

Theorem 4.3. *If G is a plane connected graph and $\mathcal{C}$ is the set of internal faces of G, then $\mathcal{C}$ is Δ^*-dense.*

Theorem 4.4. *If G is a connected graph and $\mathcal{C}$ is the set of cycles that contain a given edge e of G, then $\mathcal{C}$ is Δ^*-dense.*

We end this section with the following immediate corollaries.

Corollary 4.5. *Let u and v be vertices of a connected plane graph G. If $\mathcal{C}$ is the set of internal faces of G, then $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected.*

Proof. By Theorem 4.3, $\mathcal{C}$ is Δ^*-dense and by Theorem 4.2, $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected. ∎

Corollary 4.6. *Let u and v be vertices of a connected graph G. If $\mathcal{C}$ is the set of cycles of G that contain a given edge e, then $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected.*

Proof. By Theorem 4.4, $\mathcal{C}$ is Δ^*-dense and by Theorem 4.2, $\mathcal{P}_{\mathcal{C}}(G_{uv})$ is connected. ■

Corollary 4.7. *Let u and v be vertices of a connected graph G. If $\mathcal{C}_u$ is the set of cycles of G that contain vertex u, then $\mathcal{P}_{\mathcal{C}_u}(G_{uv})$ is connected.*

Proof. Let e be an edge of G incident with vertex u. Clearly the set $\mathcal{C}(e)$ of cycles that contain edge e is a subset of the set $\mathcal{C}_u$. Therefore $\mathcal{P}_{\mathcal{C}(e)}(G_{uv})$ is a subgraph of $\mathcal{P}_{\mathcal{C}_u}(G_{uv})$. By Corollary 4.6, the graph $\mathcal{P}_{\mathcal{C}(e)}(G_{uv})$ is connected. ■

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