Thai Journal of Mathematics
Volume 21 Number 4 (2023)
Pages 775–784

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Discrete and Computational Geometry, Graphs, and Games

Biased Domination Games

Tharit Sereekiatdilok and Panupong Vichitkunakorn*

Division of Computational Science, Faculty of Science, Prince of Songkla University, Songkhla 90110, Thailand

e-mail : tharit.yoii@gmail.com (T. Sereekiatdilok); panupong.v@psu.ac.th (P. Vichitkunakorn)

Abstract We introduce an extended version of a domination game on a graph, called a biased domination game, in which Dominator or Staller plays more than one move in each turn. We show some relations of biased game domination numbers for different biases.

MSC: 91A46; 05C57; 91A43

Keywords: biased domination game; biased game domination number; maximal move; minimal move

Submission date: 23.12.2021 / Acceptance date: 06.02.2023

1. Introduction

A domination game was introduced in [1] as a game of two players, called *Dominator* and *Staller*, on a graph. The players alternatively pick a move by choosing a vertex in the graph. A chosen vertex *dominates* all vertices in its closed neighborhood. A move u is legal if it creates at least one new dominated vertex. In other words, its closed neighborhood $N[u]$, consisting of all vertices adjacent to u and the vertex u itself, is not contained in the union of the closed neighborhood of vertices which have been chosen before. That is, for a sequence of previously picked moves (before picking the n-th move) $u_1, u_2, u_3, \dots, u_{n-1}$, the player can pick a move u_n if and only if $N[u_n] \nsubseteq \bigcup_{i=1}^{n-1} N[u_i]$. The game ends when all vertices in the graph are dominated. Dominator tries to end the game as soon as possible, while Staller tries to prolong the game. In the domination game, if Dominator starts the game, this game is said to be *Game* 1. Otherwise, it is said to be *Game* 2. If both players play optimally in a domination game on a graph G, the number of moves when the game ends is called the *game domination numbers*, denoted by $\gamma_g(G)$ and $\gamma_g'(G)$ in Game 1 and Game 2, respectively.

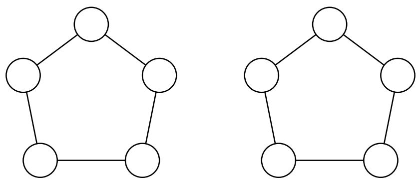
Many aspects of domination games have been studied. Game domination numbers on various graphs, such as trees [2], forests [3], paths and cycles [4], powers of cycles [5], disjoint union of paths and cycles [6], have been computed. Possible values of domination numbers of unions of graphs are studied in [7]. Bound on domination numbers have been studied, see [8–13] for example.

*Corresponding author.

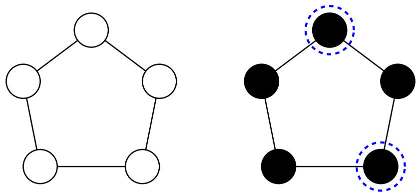
Published by The Mathematical Association of Thailand.

Some variations of the game have also been studied. The total domination game has been introduced in [14], in which a move u will dominate its open neighborhood $N(u)$ instead of its closed neighborhood $N[u]$. So a move is legal if and only if its open neighborhood is not contained in the union of open neighborhoods of all vertices chosen before. Similarly, bound on total domination numbers have been studied in [15–18], and total domination numbers themselves were computed for some families of graphs, such as cycles and paths [19] and a family of cyclic bipartite graphs [20]. Recently, some other variations based on the definition of legal moves have been proposed in [21].

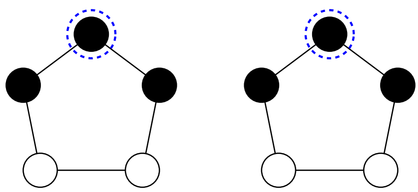
In this work, we introduce a variation of a domination game called a (δ, σ)-*biased domination game* or simply called a (δ, σ)-*biased game* where δ and σ are positive integers. Dominator and Staller must pick δ and σ moves for each turn (except possibly the last turn of the game), respectively. The (δ, σ)-biased game domination numbers are denoted by $\gamma_{(\delta,\sigma)}(G)$ for Game 1 and $\gamma'_{(\delta,\sigma)}(G)$ for Game 2. For example, we consider a graph $G = C_5 \sqcup C_5$. We will show that $\gamma_{(2,1)}(G) = 4$.



First, we notice that G has 10 vertices. Since each move can dominate at most 3 vertices, at least 4 moves are required to end the game. Hence, the biased game domination number is at least 4. Next, we can show that Dominator can force the game to end in 4 moves, which concludes that $\gamma_{(2,1)}(G) = 4$. In the first turn, Dominator starts the game by picking 2 moves. If Dominator decides to pick 2 moves in the same component as shown in the picture below, then Dominator can force Staller to dominate 3 new vertices on the other component and Dominator finishes the game in 4 moves. So, $\gamma_{(2,1)}(G) = 4$.



The choice of the first two moves by Dominator is important. If Dominator decides to pick 2 moves in separate components as shown in the picture below, then Staller can pick a move that dominates only one new vertex and force the game to end with more than 4 moves. Hence, Dominator will not choose this choice of the first two moves as it does not give the optimal result.



The main goal of this work is to compare, for given distinct $\delta \neq \delta'$ and $\sigma \neq \sigma'$, the biased game domination numbers $\gamma_{(\delta,\sigma)}(G)$ and $\gamma_{(\delta',\sigma)}(G)$, and similarly, $\gamma'_{(\delta,\sigma)}(G)$ and $\gamma'_{(\delta,\sigma')}(G)$ for Game 2 when certain special moves exist.

2. Preliminaries

In this section, we introduce some definitions and properties in domination games and biased domination game. First, we have the observation that follows from the observation in [1].

Lemma 2.1. *Let G be a graph and $\delta, \sigma \in \mathbb{N}$. Consider a (δ,σ)-biased game.*

(1) In Game 1 (Game 2), if Dominator has a strategy to make the game end within k moves when Staller plays optimally, then $\gamma_{(\delta,\sigma)}(G) \leq k$ ($\gamma'_{(\delta,\sigma)}(G) \leq k$).

(2) In Game 1 (Game 2), if Staller has a strategy to make the game end in at least k moves when Dominator plays optimally, then $\gamma_{(\delta,\sigma)}(G) \geq k$ ($\gamma'_{(\delta,\sigma)}(G) \geq k$).

Proof. The result follows from the definition of biased game domination numbers. ∎

By Lemma 2.1, if one player plays his optimal strategy while the other plays a certain (possibly not optimal) strategy, we get an inequality between the number of moves in such game and the biased game domination number. Next, we will often use this lemma for proving theorem in this work.

Definition 2.2 (Partially dominated graphs [8]). Let G be a graph and S be a subset of $V(G)$. A *partially dominated graph* $G|S$ is a graph G in which all vertices in S have already been dominated.

Theorem 2.3 (Continuation Principle [8]). *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_g(G|A) \geq \gamma_g(G|B)$ and $\gamma'_g(G|A) \geq \gamma'_g(G|B)$.*

We can extend Theorem 2.3 to biased domination games, using similar proof which is based on *imagination strategy* [1].

Theorem 2.4 (Continuation Principle of the Biased Domination Game). *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$ and $\gamma'_{(\delta,\sigma)}(G|A) \geq \gamma'_{(\delta,\sigma)}(G|B)$.*

Proof. Assume that $A \subseteq B \subseteq V(G)$. We will show that $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$ and $\gamma'_{(\delta,\sigma)}(G|A) \geq \gamma'_{(\delta,\sigma)}(G|B)$.

To show $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$, we let the real game be a biased domination game on $G|A$ where Dominator plays optimally, and let the imagined game be a biased domination game on $G|B$ imagined and optimally played by Staller. The number of moves in the real game and the imagined game when the games ended are denoted by R and I, respectively. Then $\gamma_{(\delta,\sigma)}(G|A) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G|B)$. Thus it is enough to show that $R \geq I$.

In each turn, when Dominator plays moves in the real game, Staller copies such moves to the imagined game. Staller then responds optimally in the imagined game and copies the moves back to the real game. Note that every Staller's move in the imagined game is always legal in the real game, but a Dominator's move in the real game is not necessary legal in the imagined game as $A \subseteq B$.

If all Dominator's moves in the real game are legal in the imagined game, then both games may end at the same time or there is some undominated vertices left in the real game. So $R \geq I$.

If there exists a move of Dominator in the real game that cannot be copied to the imagined game. This means that all vertices in the closed neighborhood of such move are already dominated in the imagined game. Staller then imagined that Dominator picks a random legal move in the imagined game and continues the game. The game continues until the imagined game ends or there is another move of Dominator in the real game which is not legal in the imagined game. In the later case, Staller then imagines another random legal move for Dominator in the imagined game.

We notice that at every turn in the game, the dominated vertices in the real game are also dominated in the imagined game. This means the real game cannot end before the imagined game. Hence $R \geq I$. Thus, $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$.

The proof above always works whether it is Dominator or Staller who plays the first move. Thus the same proof can be directly applied to Game 2. ∎

In [1], the authors considered a domination game such that Dominator (resp. Staller) is allowed, but not obligated, to skip exactly one move in the game. That is, there is at most one turn such that Dominator (resp. Staller) may decide to pass. After the game ends, the number of moves in the game where both players played optimally, is denoted by $\gamma_g^{dp}(G)$ (resp. $\gamma_g^{sp}(G)$). We call this game the *Dominator-pass game* (resp. *Staller-pass game*).

We define a version of Dominator-pass games and Staller-pass games for biased domination games as follows.

Definition 2.5. In a (δ, σ)-game on a graph G, if Staller is allowed to pass some moves in each turn (except the first move of each turn) in total of at most n moves per game, then we define such game as an *n-Staller-pass-(δ,σ)-game* or *$sp(n)$-(δ,σ)-game*. The number of moves in an $sp(n)$-(δ,σ)-game when both players play optimally are denoted by $\gamma_{sp(n),(\delta,\sigma)}(G)$ in game 1 and $\gamma'_{sp(n),(\delta,\sigma)}(G)$ in game 2. Similar notation, $dp(n)$, is used for *n-Dominator-pass* games.

We note that a *turn* in a (δ,σ)-game is comprised of δ *moves* for Dominator and σ *moves* for Staller. For pass games, since the order of moves in a turn by the same player does not matter, we can assume that the player plays a certain number of consecutive moves and then skip the rest. In order to prevent an empty turn, we forbid skipping the first move of any turn as shown in Definition 2.5.

The continuation principle (Theorem 2.4) also holds for pass games with the same proof.

We define two special types of moves in a biased domination game, which will play important roles in our main results.

Definition 2.6. We say that a move is *minimal* if it dominates exactly one new vertex.

Definition 2.7. We say that a move u is *maximal* if it dominates at least one new vertex w such that if u' is another move dominating w then all new vertices dominated by u' (instead of choosing u) can also be dominated by u. So if A is the set of all vertices chosen by previous moves, as the set of all dominated vertices by previous moves is $N[A]$ (the union of closed neighborhoods of all elements in A), we have the following relation on *closed neighborhoods* of u, u' and A. For all u' dominating w,

$$N[u'] \setminus N[A] \subseteq N[u] \setminus N[A]. \tag{2.1}$$

In other words, u is always one of the best among all moves dominating w.

3. Main Results

In this section, we consider a biased game on a graph in which a minimal move or a maximal move are always available except possibly at the first move of the game.

3.1. A Biased Game and Minimal Moves

Theorem 3.1. *For any graph G, if Staller can always make a minimal move, then*

$$\gamma_{sp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Proof. Using the imagination strategy in [1], we consider a situation where Staller is playing Game 1 of an $sp(1)$-(δ,σ)-biased domination game (Real Game: RG) with an optimal strategy while Dominator imagines and plays Game 1 of a (δ,σ)-biased domination game (Imagined Game: IG) optimally. Let the real game and the imagined game end in R and I moves, respectively. By Lemma 2.1, $R \geq \gamma_{sp(1),(\delta,\sigma)}(G)$ and $I \leq \gamma_{(\delta,\sigma)}(G)$. So it is enough to prove that $R \leq I$.

In the real game, Staller has to play σ moves at each turn, except possibly at one turn where either $\sigma - 1$ moves (a pass of the last move) or σ moves is allowed. Whenever Staller plays σ moves at a turn, Dominator copies each move of Staller to the imagined game, responds optimally in the imagined game, and copies the moves back to the real game. If Staller does not skip a move until the game ends, the sequences of moves are formed as the following sequence, where d_i^j and s_i^j denote the j-th move in the i-th turn of Dominator and Staller, respectively.

$$\text{RG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, d_2^1 \ldots, d_2^\delta, s_2^1, \ldots, s_2^\sigma, \ldots$$
$$\text{IG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, d_2^1 \ldots, d_2^\delta, s_2^1, \ldots, s_2^\sigma, \ldots$$

This means that both games are played with the same sequence of moves. Thus $R = I$.

If Staller decides to play only $\sigma - 1$ moves at turn k, Dominator copies each move of Staller up to such move to the imagined game. Then Dominator imagined that Staller makes the σ-th move s_k^* that dominates exactly one new vertex v_k. This is a minimal move, which is always available by the assumption. Now the sequences of moves are formed as follows.

$$\text{RG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^{\sigma-1}, \times$$
$$\text{IG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^{\sigma-1}, s_k^*$$

Dominator then responses optimally in the imagined game and copies the moves back to the real game.

The game continues with Staller playing exactly σ moves at each turn. Note that all moves by Dominator in the imagined game are always legal in the real game. On the other hand, a move by Staller in the real game may not be legal in the imagined game.

If there is an illegal move at turn $m > k$ in the imagined game, says s_m^θ for some $1 \leq \theta \leq \sigma$. First, we consider when the move is s_m^1. We have the following sequences of moves.

$$\begin{aligned}\text{RG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, \times, \dots, d_m^\delta, s_m^1 \\ \text{IG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, s_k^*, \dots, d_m^\delta\end{aligned}$$

Since s_m^1 is not legal in the imagined game, we have

$$\emptyset \neq N[s_m^1] \setminus N[C] \subseteq N[s_k^*] \setminus N[C] = \{v_k\}$$

where C is the set of all vertices played before s_m^1 in the real game, i.e.,

$$C = \left\{ \bigcup_{j=1}^{m-1} \left(\bigcup_{i=1}^{\delta} \{d_j^i\} \cup \bigcup_{i=1}^{\sigma} \{s_j^i\} \right) \cup \bigcup_{i=1}^{\delta} \{d_m^i\} \right\} \setminus \{s_k^\sigma\}.$$

Hence $N[s_m^1] \setminus N[C] = \{v_k\}$. That is s_m^1 dominates only one new vertex v_k. This means both games now have the same set of dominated vertices.

When Staller picks the rest of the moves in the turn, Dominator skips s_m^1 and copies these moves to the imagined game, and also imagines that Staller picks another minimal move s_m^* (newly dominating a vertex v_m) as the last move. Thus the sequences of moves in the both games are as follows.

$$\begin{aligned}\text{RG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, \times, \dots, d_m^\delta, s_m^1, s_m^2, s_m^3, \dots, s_m^{\sigma-1}, s_m^\sigma \\ \text{IG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, s_k^*, \dots, d_m^\delta, s_m^2, s_m^3, s_m^4, \dots, s_m^\sigma, s_m^*\end{aligned}$$

Note that $s_m^2, s_m^3, s_m^4, \dots, s_m^\sigma$ are all legal in the imagined game since

$$N[s_m^1] \setminus N[C] = N[s_k^*] \setminus N[C] = \{v_k\}.$$

The same computation is applied when it is the move s_m^θ for $\theta > 1$ which is not legal in the imagined game. The vertex v_k is the only new vertex dominated by s_m^θ, and the set of all dominated vertices in both games are now the same. Dominator then imagines a minimal move s_m^*. So we get the following sequences of moves.

$$\begin{aligned}\text{RG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, \times, \dots, d_m^\delta, \dots, s_m^{\theta-1}, s_m^\theta, s_m^{\theta+1}, \dots, s_m^{\sigma-1}, s_m^\sigma \\ \text{IG: } & d_1^1, \dots, d_1^\delta, s_1^1, \dots, s_1^\sigma, \dots, s_k^1, \dots, s_k^*, \dots, d_m^\delta, \dots, s_m^{\theta-1}, s_m^{\theta+1}, s_m^{\theta+2}, \dots, s_m^\sigma, s_m^*\end{aligned}$$

We can always repeat the same procedure if there is a move in the real game that is not legal in the imagined game. At the end, both games end at the same move or the imagined game ends before the real game. In the first case, we have $R = I - 1$. In the second case, since v_m is the only vertex left undominated by not playing s_m^*, it is this unique vertex left undominated in the real game when the imagined game has ended. So $R = I$.

From both cases, we have $R \leq I$. Since $\gamma_{sp(1),(\delta,\sigma)}(G) \leq R$ and $I \leq \gamma_{(\delta,\sigma)}(G)$, we have $\gamma_{sp(1),(\delta,\sigma)}(G) \leq \gamma_{(\delta,\sigma)}(G)$.

We can consider $\gamma_{(\delta,\sigma)}(G)$ as the number of moves that Dominator plays optimally and Staller decides not to skip any move on an $sp(1)$-(δ,σ)-biased domination game. By Lemma 2.1, $\gamma_{(\delta,\sigma)}(G) \leq \gamma_{sp(1),(\delta,\sigma)}(G)$. Therefore $\gamma_{sp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G)$. ∎

Theorem 3.2. *For any graph G and $i \geq 0$, if Staller can always make a minimal move, then*

$$\gamma_{sp(i+1),(\delta,\sigma)}(G) = \gamma_{sp(i),(\delta,\sigma)}(G).$$

Proof. Let the real game be the $sp(i+1)$-(δ,σ)-game with an optimal strategy of Staller and the imagined game be the $sp(i)$-(δ,σ)-game imagined by Dominator and played with his optimal strategy. Dominator copies all the moves of Staller from the real game to the imagined game, up to the i-th time Staller skipped the move. At the $(i+1)$-th skip, Dominator imagines a random minimal move as in the proof of Theorem 3.1. The same analysis can then be directly applied. ∎

From Theorem 3.1 and Theorem 3.2, we immediately get the following corollary.

Corollary 3.3. *For any graph G and $i \geq 0$, if Staller can always make a minimal move, then*

$$\gamma_{sp(i),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Using Corollary 3.3, we can now compare the biased game domination numbers with different σ.

Theorem 3.4. *For any graph G, if Staller can always make a minimal move, then*

$$\gamma_{(\delta,j)}(G) \leq \gamma_{(\delta,\sigma)}(G)$$

for all $j \leq \sigma$.

Proof. Let $j \leq \sigma$. We can consider a (δ, j)-biased game as a situation in a Staller-pass (δ,σ)-game where Staller passes $\sigma - j$ moves in every of his turn until the game ends. Let k be a number a lot larger than the possible total number of all passed moves by Staller in this game. Then $\gamma_{(\delta,j)}(G)$ is a number of moves in the $sp(k)$-(δ,σ)-game where Dominator plays optimally and Staller passes $\sigma - j$ moves for each turn until the game ends. This implies that $\gamma_{(\delta,j)}(G) \leq \gamma_{sp(k),(\delta,\sigma)}(G)$. By Corollary 3.3, we have $\gamma_{(\delta,j)}(G) \leq \gamma_{(\delta,\sigma)}(G)$. ∎

3.2. A Biased Game and Maximal Moves

Similar to the previous section, we want to compare biased game domination numbers on biased games with different δ. We first consider Dominator-pass games.

Theorem 3.5. *For any graph G, if Dominator can always make a maximal move (except possibly at the first move of the game), then*

$$\gamma_{dp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Proof. Let the real game (RG) be a $dp(1)$-(δ,σ)-game with an optimal strategy of Dominator and the (δ,σ)-game be a game imagined by Staller (IG) and played with an optimal strategy of Staller. The number of moves in the real game and the imagined game when the games end are denoted by R and I, respectively. Then $\gamma_{dp(1),(\delta,\sigma)}(G) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G)$. We claim that $R \geq I$.

If Dominator decides not to pass a move in the real game, then Staller copies all of Dominator's moves from the real game to the imagined game, responds optimally and copies the moves back to the real game. Since the two games are identical, we have $R = I$.

If Dominator decides to pass a move at turn k in the real game, Staller imagines that Dominator picks a maximal move d_k^* in the imagined game. So it dominates a vertex

w_k such that all legal moves dominating w_k in the real game are illegal in the imagined game, see Equation (2.1). The sequences of moves are formed as follows.

$$\begin{aligned} \text{RG: } & d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, d_k^1, \ldots, d_k^{\delta-1}, \times \\ \text{IG: } & d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, d_k^1, \ldots, d_k^{\delta-1}, d_k^* \end{aligned}$$

The game continues by Staller playing optimally in the imagined game then copying the moves back to the real game. Note that all moves by Staller in the imagined game are always legal in the real game. However, a move by Dominator in the real game may not be legal in the imagined game.

If there is no illegal move until one of the games ends, we know that the vertex w_k still remains undominated in the real game. (All legal moves dominating w_k in the real game are illegal in the imagined game.) So the imagined game ends, while the real game has at least one vertex w_k remaining. Thus the real game needs at least one extra move to finish the game. Including the move d_k^* imagined by Staller, we have $R \geq I$.

Whenever there is an illegal copying from the real game to the imagined game, Staller imagines that Dominator picks a new maximal move instead of such illegal move. Assume that in the last illegal copying, Staller imagined a maximal move which dominates w such that all legal moves moves dominating w in the real game are illegal in the imagined game.

We know when the imagined game ends, the real game must have at least one undominated vertex w. Thus the real game needs at least an extra move to finish the game. Excluding the skip, we have $R \geq I$.

Hence $R \geq I$ in every case. Since $\gamma_{dp(1),(\delta,\sigma)}(G) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G)$, we have $\gamma_{dp(1),(\delta,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$.

We consider the biased game domination number $\gamma_{(\delta,\sigma)}(G)$ as the number of moves that Staller plays optimally and Dominator decides not to skip any move on an $dp(1)$-(δ,σ)-biased domination game. By Lemma 2.1, $\gamma_{dp(1),(\delta,\sigma)}(G) \leq \gamma_{(\delta,\sigma)}(G)$. Therefore $\gamma_{dp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G)$. ∎

Theorem 3.6. *For any graph G and $i \geq 0$, if Dominator can always make a maximal move (except possibly the first move of a game), then*

$$\gamma_{dp(i+1),(\delta,\sigma)}(G) = \gamma_{dp(i),(\delta,\sigma)}(G).$$

Proof. Let the $dp(i+1)$-(δ,σ)-game be the real game with an optimal strategy of Dominator and the $dp(i)$-(δ,σ)-game be the imagined game by Staller with an optimal strategy. Staller copies all the moves of Dominator from the real game to the imagined game, up to the i-th time Dominator skipped the move. The rest of the proof is similar to Theorem 3.2. ∎

Corollary 3.7. *For any graph G and $i \geq 0$, if Dominator can always make a maximal move (except possibly the first move of a game), then*

$$\gamma_{dp(i),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Theorem 3.8. *For any graph G, if Dominator can always make a maximal move (except the first move of a game), then*

$$\gamma_{(j,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$$

for all $j \leq \delta$.

Proof. Let $j \leq \delta$. We can consider a (j, σ)-biased game as a situation in a Dominator-pass game where Dominator passes $\delta - j$ moves at every turn until the game ends. Let k be a number a lot larger than the possible total number of all passed moves by Dominator in this game. Then $\gamma_{(j,\sigma)}(G)$ is the number of moves in the $dp(k)$-(δ, σ)-game where Staller plays optimally and Dominator passes $\sigma - j$ moves for each turn until the game ends. This gives $\gamma_{(j,\sigma)}(G) \geq \gamma_{dp(k),(\delta,\sigma)}(G)$. By Corollary 3.7, we have $\gamma_{(j,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$. ∎

All results in this section also hold for Game 2 with the same proofs. We see that for Game 1, the condition that Dominator can always make a maximal move will be relaxed at the first move of the game. Similarly, the condition that Staller can always make a minimal move will be relaxed at the first move of Game 2. Examples of graphs which a minimal move and a maximal move are always available are paths and cycles, see [5] and [4] for example.

4. Conclusion and Discussion

In this paper, we have introduced a biased version of domination games. Under the condition that a minimal move (resp. a maximal move) is always available, we can compare the biased game domination number of two biased games with different number of moves for Staller (resp. Dominator) in each turn.

The property that a graph always has a minimal move or a maximal move available is still rather strong. So it is interesting to know what other collections of graphs have this property, or what other conditions give the same results as in Theorem 3.4 and Theorem 3.8.

Acknowledgements

The authors are grateful to Division of Computational Science, Faculty of Science, Prince of Songkla university and Development and Promotion of Science and Technology Talents Project (DPST) for their support.

References

[1] B. Brešar, S. Klavžar, D. Rall, Domination game and an imagination strategy, SIAM Journal on Discrete Mathematics 24 (2010) 979–991.

[2] B. Brešar, S. Klavžar, D. Rall, Domination game played on trees and spanning subgraphs, Discrete Mathematics 313 (2013) 915–923.

[3] C. Bujtás, Domination game on forests, Discrete Mathematics 338 (2015) 2220–2228.

[4] G. Košmrlj, Domination game on paths and cycles, ARS Mathematica Contemporanea 13 (2017) 125–136.

[5] N. Chantarachada, C. Worawannotai, Domination game on powers of cycles, Thai Journal of Mathematics 2020 (2020) 261–268.

[6] W. Ruksasakchai, K. Onphaeng, C. Worawannotai, Game domination numbers of a disjoint union of paths and cycles, Quaestiones Mathematicae 42 (2019) 1357–1372.

[7] P. Dorbec, G. Košmrlj, G. Renault, The domination game played on unions of graphs, Discrete Mathematics 338 (2015) 71–79.

[8] W. Kinnersley, D. West, R. Zamani, Extremal problems for game domination number, SIAM Journal on Discrete Mathematics 27 (2013) 2090–2107.

[9] B. Brešar, S. Klavžar, G. Košmrlj, D. Rall, Domination game: Extremal families of graphs for 3/5-conjectures, Discrete Applied Mathematics 161 (2013) 1308–1316.

[10] C. Bujtás, On the game domination number of graphs with given minimum degree, Electronic Journal of Combinatorics 22 (2015).

[11] M. Henning, W. Kinnersley, Domination game: A proof of the 3/5-conjecture for graphs with minimum degree at least two, SIAM Journal on Discrete Mathematics 30 (2016) 20–35.

[12] M. Henning, C. Löwenstein, Domination game: Extremal families for the 3/5-conjecture for forests, Discussiones Mathematicae - Graph Theory 37 (2017) 369–381.

[13] C. Bujtás, General upper bound on the game domination number, Discrete Applied Mathematics 285 (2020) 530–538.

[14] M. Henning, S. Klavžar, and D. Rall, Total version of the domination game, Graphs and Combinatorics 31 (2015) 1453–1462.

[15] C. Bujtás, M. Henning, Z. Tuza, Transversal game on hypergraphs and the 3/4-conjecture on the total domination game, SIAM Journal on Discrete Mathematics 30 (2016) 1830–1847.

[16] M. Henning, D. Rall, Progress towards the total domination game 3/4-conjecture, Discrete Mathematics 339 (2016) 2620–2627.

[17] M. Henning, S. Klavžar, D. Rall, Game total domination critical graphs, Discrete Applied Mathematics 250 (2018) 28–37.

[18] C. Bujtás, On the game total domination number, Graphs and Combinatorics 34 (2018) 415–425.

[19] P. Dorbec, M. Henning, Game total domination for cycles and paths, Discrete Applied Mathematics 208 (2016) 7–18.

[20] Y. Jiang, M. Lu, Game total domination for cyclic bipartite graphs, Discrete Applied Mathematics 265 (2019) 120–127.

[21] B. Brešar, C. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, T. Marc, B. Patkós, Z. Tuza, M. Vizer, The variety of domination games, Aequationes Mathematicae 93 (2019) 1085–1109.