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Feedback Game on Eulerian Graphs

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Abstract In this paper, we introduce a two-player impartial game on graphs, called *feedback game*, which is a variant of generalized geography. Feedback game can be regarded as undirected edge geography with an additional rule that the first player who goes back to the starting vertex wins the game. We consider feedback game on an Eulerian graph since the game ends only by going back to the starting vertex. We first show that it is PSPACE-complete in general to determine the winner of the feedback game on Eulerian graphs even if its maximum degree is at most 4. In the latter half of the paper, we discuss the feedback game on two subclasses of Eulerian graphs, i.e., triangular grid graphs and toroidal grid graphs.

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1. Introduction

All graphs considered in this paper are finite, loopless, and undirected unless otherwise mentioned. A graph G is *Eulerian* if each vertex of G has even degree. For other basic terminology in graph theory, we refer to [6].

In combinatorial game theory, impartial games have been well studied for a long time, where a game is *impartial* if the allowable moves depend only on the position and not on which of the two players is currently moving. So far, many interesting impartial games have been found; e.g., Nim [4], Kayles [8] and Poset game [16]. The most famous result in this area is the Sprague-Grundy theorem [12, 17] stating that every impartial game (under the normal play convention) is equivalent to a one-heap game of Nim. There are also many interesting games played on graphs as for example; Vertex Nim [7], Ramsey game [9] and Voronoi game [18]. For more details and other topics, we refer the reader to survey several books and articles [1–3, 5].

One of the most popular impartial games on graphs is generalized geography. *Generalized geography* is a two-player game played on a directed graph D whose vertices are

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words and $xy \in A(D)$ if and only if the end character of a word x is the first one of y, where $A(D)$ is the set of arcs of D. For example, if x is "Japan" and y is "Netherlands", then $xy \in A(D)$ but $yx \notin A(D)$. In this setting, the game begins from some starting word and both players alternately extend a directed path using unused words. The first player unable to extend the directed path loses. It is PSPACE-complete to determine the winner of generalized geography [14]. Moreover, several variants of generalized geography have been considered, e.g., planar generalized geography [14], edge geography [15] and undirected geography [11]. It is also known that for each of above variants is PSPACE-complete to determine which player wins the game except undirected vertex geography; we can determine the winner in polynomial time.

In this paper, we consider a new impartial game on a graph, called *feedback game*, which is a variant of undirected edge geography. (We sometimes call it a *game* for the sake of simplicity.)

Definition 1 (Feedback game)**.** There are two players; Alice and Bob, starting with Alice. For a given connected graph G with a starting vertex s, a token is put on s. They alternately move the token from a vertex u to a neighbor v of u and then delete an edge uv. The first player who moves the token back to s or to an isolated vertex (after removing the edge used by the last move) wins the game.

We can find that feedback game is the same as edge geography when the starting vertex s has degree 2. But these two games are quite different when the degree of s is more than 2. For example, on a butterfly graph like as Fig. 1 and the start vertex is s, Alice clearly wins feedback game. On the other hand, Alice never wins edge geography on the same graph and start vertex. One can observe that the difference between these games lies on the choice of the moves: players cannot take a move to neighbouring vertices of s on feedback game. Due to this difference, we can not directly apply existing results on edge geography to feedback game.

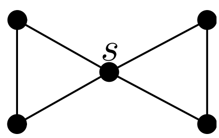

FIGURE 1. Alice wins feedback game, but loses edge geography.

In this paper, we investigate feedback game on Eulerian graphs. Note that if a given connected graph G is Eulerian, then the game does not end until the token goes back to the starting vertex s, and further observe that Bob always wins feedback game on any connected bipartite Eulerian graph (cf. [11]): Let G be a connected bipartite Eulerian graph, and so, all vertices of G are properly colored by two colors, black and white. Without loss of generality, we may suppose that the starting vertex is colored by black. Throughout the game on G, a token is always moved to a white (resp., black) vertex by Alice (resp., Bob). Thus, Bob necessarily wins the game.

For a given connected Eulerian graph G, it is PSPACE-complete to determine which player wins feedback game on G even if the maximum degree of G is at most 4 (Theorem 3). Therefore, a main study on feedback game is to determine the winner of the game on a connected Eulerian graph with more additional restrictions.

The remaining of the paper is organized as follows. In the next section, we prove the PSPACE-completeness of feedback game. In Section 3, we introduce an *even kernel* (resp., an *even kernel graph*), first introduced in [11], which is a useful subset (resp., subgraph) guaranteeing the existence of a winning strategy of the second player. In Sections 4 and 5, focusing on *triangular grid graphs* and *toroidal grid graphs*, we determine the winner of feedback game on several subclasses of them.

2. Complexity of Feedback Game

Because feedback game can be seen as a variant of undirected edge geography, it is a simple idea to construct a reduction from undirected edge geography to feedback game.

Definition 2 (Undirectred/Directed edge geography)**.** There are two players; Alice and Bob, starting with Alice. For a given connected undirected/directed graph G with a starting vertex s, a token is put on s. They alternately move the token from a vertex u to a neighbor/out-neighbor v of u and then delete an edge/arc uv. The first player who moves the token to an isolated vertex (after removing the edge/arc used by the last move) wins the game.

Directed edge geography is known to be PSPACE-complete [15] via a reduction from TQBF, and undirected edge geography is also known as PSPACE-complete [11] via a reduction from directed edge geography. Here TQBF (true quantified Boolean formula) is, given a quantified formula, the determination of whether there exists an assignment to the input variables such that the formula is true.

Feedback game is different from these edge geographies on the winning rule. Since a player wins when a token reaches the starting vertex, it is difficult to reduce from an instance of undirected edge geography to that of feedback game. To avoid this difficulty, we use the same idea about reduction from TQBF to directed edge geography and add a gadget before making the graph undirected.

Theorem 3. *It is* PSPACE*-complete to determine whether there exists a winning strategy for the first player in feedback game, even if the given graph is Eulerian.*

Proof. We can see that this determination is in PSPACE, since we can check the winner using a DFS-like algorithm that recurs $O(|E|)$ times and uses $O(n)$ spaces on each recursion.

Now we reduce any instance of TQBF to an instance of determining the winner on feedback game. The first step is the same as the famous reduction from TQBF to directed edge geography [15] and we obtain a graph H as an instance. Note that, applying the same reduction in [14], $\Delta(H) = 3$, where $\Delta(H)$ denotes the maximum degree of H, and that the obtained graph H has only one vertex s with in-degree 0. We also note the out-degree of s is 2.

By the definition of the feedback game, the winner can also win in the view of the "directed version" of the feedback game on H. (Note that a player can win when the opponent cannot move anymore.) From now on, as shown in Figure 2, we use pseudo-arcs to make a reduction to the "undirected version" of feedback game [11].

We make the undirected graph H' obtained as above be Eulerian. Let $D = \{x_1, x_2, \ldots, x_{2p}\}$ for $p \geq 1$ be the set of vertices in $V(H')$ of odd degree. First, we add a path abc and two edges as and cs, that is, $sabc$ forms a 4-cycle. Note that the first player does not use the edge sa nor sc at the start of the game; that immediately leads to a suicide. Next, for

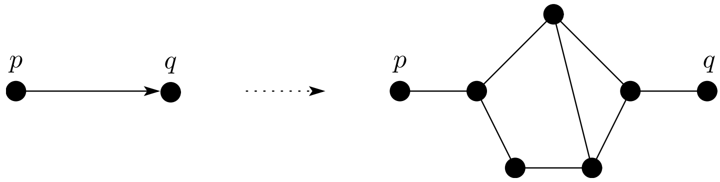

FIGURE 2. Replacing an arc pq with a pseudo-arc

each x_i where $1 \leq i \leq 2p$, we make a path $P_i = x_i y_i z_i$ of length 2 with adding two vertices y_i and z_i. Finally, we add edges $z_1 a$, $z_2 a$, $z_{2i-1} y_{2i-3}$ and $z_{2i} y_{2i-3}$, where $2 \leq i \leq p$. Then, in the resulting graph G, we can see the degree of a is 4, the degree of b, c, z_i is 2, the degree of y_i is 2 or 4, and the degree of x_i is greater 1 than itself in H'. Therefore, clearly the resulting graph G is Eulerian. Furthermore, it is not difficult to see that the winner of feedback game on G is the same as that of H'; note that the player who moves the token from a vertex x_i (which is an odd in H') to y_i loses because that move makes one way journey and commits suicide. ■

Note that, a graph we obtain from these reductions has no vertex degree greater than 3. When we discuss Eulerian graphs, by suitably modifying the addition of vertices and edges, we can make the graph have vertices degree only 2 or 4. Thus, we obtain the following corollary.

Corollary 4. *It is* PSPACE-*complete to determine whether there exists a strategy that the first player wins a feedback game, even if the given graph is a connected graph with maximum degree at most* 3 *or a connected Eulerian graph with maximum degree at most* 4.

3. Even Kernel Graph

We recall that Bob wins feedback game on every connected bipartite Eulerian graph. Focusing on this fact, Fraenkel et al. [11] introduced a good concept, called an *even kernel.*

Definition 5 (Even kernel)**.** Let G be a connected graph with a starting vertex s. An *even kernel* of G with respect to s is a non-empty subset $B \subseteq V(G)$ such that

(1) $s \in B$,
(2) no two elements of B are adjacent, and
(3) every vertex not in B is adjacent to an even number (possibly 0) of vertices in B.

It is known in [10] that finding an even kernel of a given graph is NP-complete even if the graph is bipartite or its maximum degree is at most 3. Based on an even kernel, we define a good subgraph of graphs, called an *even kernel graph.* For a graph G and two disjoint subsets $A, B \subseteq V(G)$, $E_G(A, B)$ denotes the set of edges between A and B (i.e., one of ends of the edge in the set belongs to A and the other belongs to B).

Definition 6 (Even kernel graph)**.** Let B be an even kernel of a connected Eulerian graph G with a starting vertex s. An *even kernel graph* with respect to s is a bipartite subgraph H_s with the bipartition $V(H_s) = B \cup W$ and $E(H_s) = E_G(B, W)$, where $W \subseteq V(G) \setminus B$ is a arbitrary superset of the set $N_G(B) = \{v \in V(G) \setminus B : v \text{ is adjacent to some vertex in } B\}$.

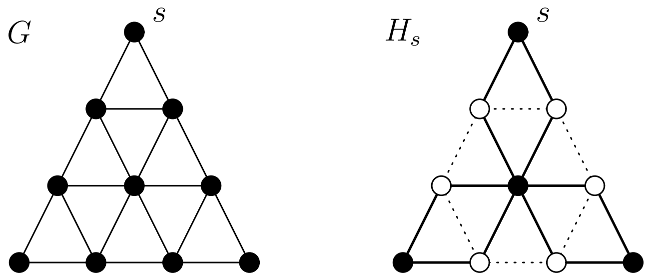

FIGURE 3. An even kernel graph H_s of a connected Eulerian graph G

For example, see Figure 3. The right of the figure, the graph H_s, is an even kernel graph of the graph G with a starting vertex s. The bold lines are edges of H_s and dotted lines are ones in $E(G) \setminus E(H_s)$, and black vertices in B (where $s \in B$) and white ones in W. Observe that for every vertex $v \in B$, all edges incident to v in G belong to $E(H_s)$.

Remark. If G has an even kernel, then G always has an even kernel graph. In Figure 3, H_s is a spanning subgraph of G, but an even kernel graph is not necessarily spanning in general. Furthermore, the existence of even kernel graphs depends on the position of a starting vertex s. It is easy to see that the graph G shown in Figure 3 has no even kernel graph if its starting vertex is of degree 4. Moreover, an even kernel graph is not unique for a given even kernel B since the partite set W may have a vertex of degree 0 in H_S.

By the definition, we see the existence of an even kernel (graph) of a connected Eulerian graph G guaranteeing that Bob wins feedback game on G.

Lemma 7 ([11]). *Let G be a connected Eulerian graph with a starting vertex s. If G has an even kernel with respect to s, then Bob can win feedback game on G.*

We conclude this section with showing that the converse of Lemma 7 is not true even if G is Eulerian, that is, a connected Eulerian graph G does not necessarily have an even kernel graph even if Bob can win the game on G.

Proposition 8. *There exist infinitely many connected Eulerian graphs without an even kernel graph on which Bob wins feedback game (with respect to a prescribed starting vertex).*

Proof. We first give a construction of desired connected Eulerian graphs. Prepare two even cycles $C_{2k} = u_0u_1u_2 \cdots u_{2k-1}$ and $C_{4k} = v_0v_1v_2 \cdots v_{4k-1}$ for some $k \geq 2$. Add edges u_iv_{2i} and u_iv_{2i+1} for any $i \in \{0, 1, \ldots, 2k-1\}$. Finally, we add a starting vertex s so that s and v_j are adjacent for any $j \in \{0, 1, \ldots, 4k-1\}$. The resulting graph is denoted by G_k; for example, see Figure 4.

We next show that Bob can win the game on G_k. Without loss of generality, we may suppose that Alice first moves the token to v_0 and that Bob moves it from v_0 to u_0. If Alice moves the token to v_1, then Bob wins the game. Thus, we may assume that Alice moves it to u_1, and then Bob moves it to u_2. After that, Alice (resp., Bob) moves the token from u_{2i} to u_{2i+1} (resp., from u_{2i+1} to u_{2i+2}), where subscripts are modulo $2k$. Therefore, Bob finally moves the token to u_0, that is, Alice has to move it to v_1. Thus, Bob wins the game on G_k.

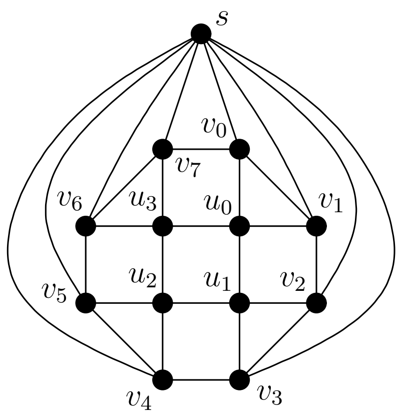

FIGURE 4. The graph G_2

Finally, we claim that G_k has no even kernel graph with respect to s. Suppose to the contrary that G_k has an even kernel graph H_s with bipartite sets B and W where $s \in B$. By the definition of an even kernel graph, $sv_i \in E(H_s)$ for all $i \in \{0, 1, \ldots, 4k-1\}$, that is, $v_i \in W$. Since H_s is bipartite, $v_iv_{i+1} \notin E(H_s)$ where subscripts are modulo $4k$. Thus all edges between two cycles C_{4k} and C_{2k} belong to $E(H_s)$, and hence, $u_j \in B$ for any $j \in \{0, 1, \ldots, 2k-1\}$. However, u_0 and u_1 must be adjacent in H_s, which contradicts the bipartiteness of H_s. ∎

4. Triangular Grid Graphs

At first, we give a recursive definition of triangular grid graphs.

Definition 9 (Triangular grid graph)**.** A *triangular grid graph* T_n with $n \geq 0$ is recursively constructed as follows.

- $T_0\,(= P^0)$ consists of an isolated vertex v_0^0 and no edge.
- T_n with $n \geq 1$ is obtained from T_{n-1} by adding a path $P^n = v_0^n v_1^n \cdots v_n^n$ and edges $v_0^n v_0^{n-1}$, $v_n^n v_{n-1}^{n-1}$, $v_i^n v_{i-1}^{n-1}$ and $v_i^n v_i^{n-1}$ for any $i \in \{1, \ldots, n-1\}$.

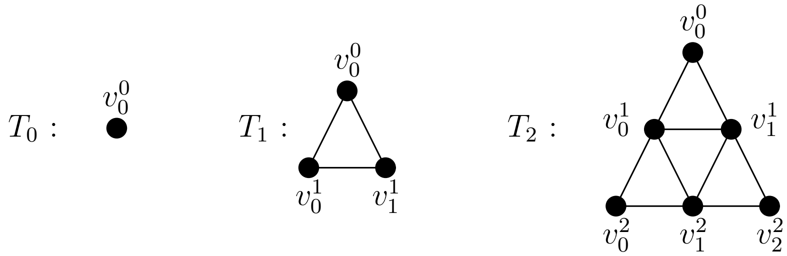

FIGURE 5. Triangular grid graphs T_0, T_1 and T_2

For example, see Figure 5. It is easy to see that every triangular grid graph is connected and Eulerian and that its maximum degree is at most 6. Moreover, it has high symmetry as we know. Thus the class of triangular grid graphs seems to be a reasonable subclass of connected Eulerian graphs for considering feedback game.

For triangular grid graphs, we have the following setting v_0^0 as a starting vertex (where note that the vertex v_0^0 can be regarded as v_0^n and v_n^n by symmetry).

Theorem 10. *If $n \neq 2^m - 3$ with $m \geq 2$, then Bob wins the game on the triangular grid graph T_n with a starting vertex v_0^0.*

Proof. We prove the theorem by induction on the height of the triangular grid graph. For the base case, we can easily find that all T_2 (the left of Figure 7), T_3 (the right of Figure 3), T_4, T_6 (Figure 6) have at least one even kernel graph, i.e., Bob wins the game on these triangular grid graphs by Lemma 7.

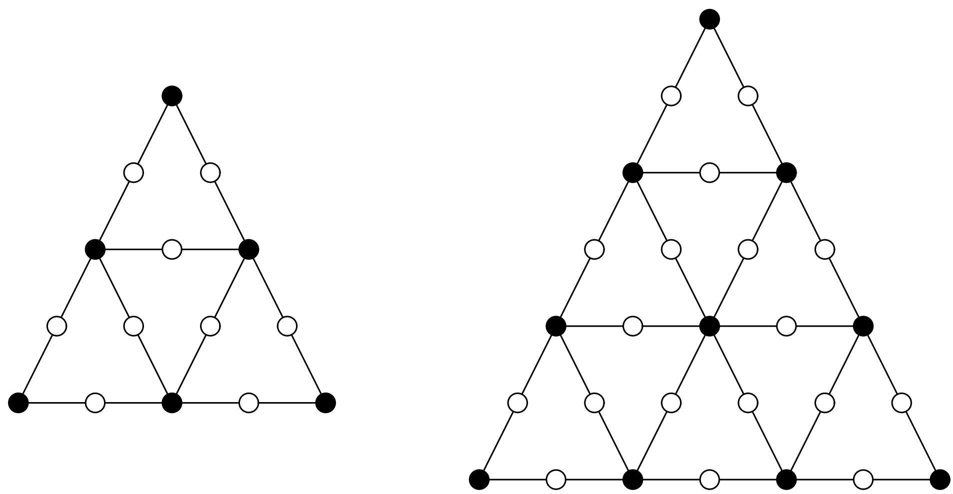

FIGURE 6. Even kernel graphs of T_4 and T_6

For an induction rule, we assume that all $T_{2^i-2}, T_{2^i-1}, \ldots, T_{2^{i+1}-4}, T_{2^{i+1}-2}$ have at least one even kernel graph. Here we construct even kernel graphs on triangular grid graphs using those even kernel graphs. Using four even kernel graphs on T_α, we can construct an even kernel graph on $T_{2\alpha+3}$; for example, see Figure 7.

From the assumption and this fact, each of $T_{2^{i+1}-1}, T_{2^{i+1}+1}, \ldots, T_{2^{i+2}-5}, T_{2^{i+2}-1}$ has at least one even kernel graph. For triangular grid graphs $T_{2^{i+1}-2}, T_{2^{i+1}}, \ldots, T_{2^{i+2}-4}, T_{2^{i+2}-2}$, it is clear that they have an even kernel graph with bipartite sets $B = \{v_k^j : j \equiv k \equiv 0 \pmod 2\}$ and $W = \{v_k^j : j \equiv 1 \pmod 2$ or $k \equiv 1 \pmod 2\}$ since their height is even (as shown in Figure 6); note that in every even kernel graph constructed above, all vertices of degree 2 are in the same partite set. Then, all triangular grid graphs $T_{2^{i+1}-2}, T_{2^{i+1}-1}, \ldots, T_{2^{i+2}-4}, T_{2^{i+2}-2}$ have at least one even kernel graph. By induction, all triangular grid graph T_n has at least one even kernel graph when $n \neq 2^m - 3$. This together with Lemma 7 leads to that Bob wins the game on T_n when $n \neq 2^m - 3$. ∎

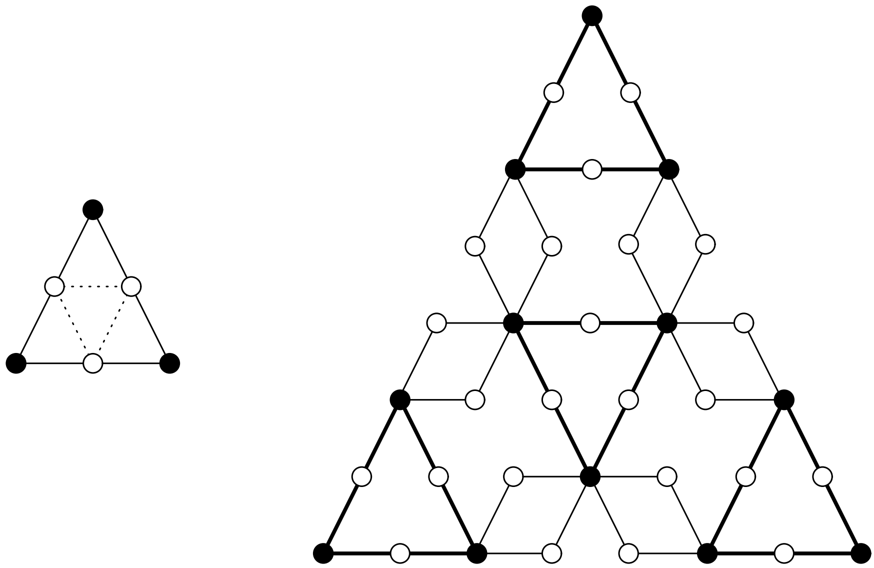

FIGURE 7. An even kernel graph H of T_2 and that of T_7 based on H

Theorem 10 shows that Bob can win the game when the starting vertex is v_0^0. Indeed, we show below that every even kernel graph of T_n must include v_0^0.

Lemma 11. *There is no even kernel of T_n that does not include v_0^0 when $n > 1$.*

Proof. We prove the lemma by contradiction and induction on the distance between v_0^0 and an arbitrary other vertex. Let B be an even kernel not including v_0^0, and let H be an even kernel graph with bipartition $B \cup W$ such that every vertex in W has degree at least 2 in H. By assumption of B, neither v_0^1 nor v_1^1 is contained in $V(H)$ by the definition of vertices in W. Therefore, all vertices whose distance from v_0^0 is 1 must not be in $V(H)$.

Assume that no vertex whose distance from v_0^0 is at most k is in $V(H)$, we can see that any $v_i^{k+1}(0 \leq i \leq k+1)$ cannot be in B by definition; because any $v_j^k(0 \leq j \leq k)$ is not a member of W from the assumption. If v_i^{k+1} is a member of W, by definition and assumption of W, v_i^{k+1} must have two or four edges in H. This condition and local restrictions show that both v_i^{k+2} and v_{i+1}^{k+2} are a member of B. This violates the definition for B. Therefore, v_i^{k+1} cannot be a member of W.

By induction on k, any vertex is not a member of $V(H)$, a contradiction. Therefore, all even kernels of T_n must include v_0^0. ∎

For the case when $n = 2^m - 3$ with $m \geq 2$, we have checked that Alice wins the game on T_n with a starting vertex v_0^0 for small cases when $n = 1$ and $n = 5$.

Theorem 12. *Alice wins feedback game on T_1 and T_5 with a starting vertex v_0^0.*

Proof. Since it is clear that Alice wins feedback game on T_1, we shall prove that Alice wins the game on T_5 with starting vertex $s = v_0^0$.

Without loss of generality, Alice first moves the token to v_0^1, and then Bob moves it to either (i) v_0^2 or (ii) v_1^2. In the case (i) (resp., (ii)), Alice next moves the token to v_0^3 (resp., v_2^2). For the case (i), as shown the left of Figure 8, we can construct a "good" bipartite subgraph for Alice; note that Alice can move the token to a black vertex in the remaining game as in the argument of the even kernel. Therefore, Alice can finally move the token back to the starting vertex s.

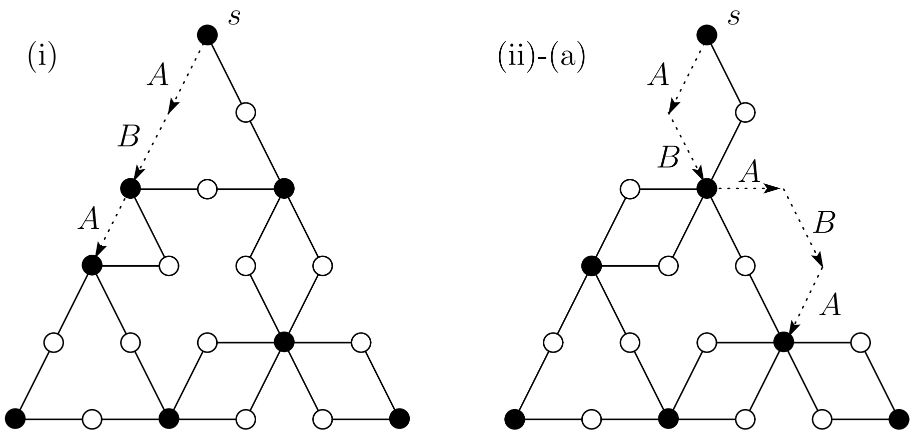

FIGURE 8. Good bipartite subgraphs for the cases (i) and (ii)-(a)

We divide the case (ii) to two subcases; (a) Bob moves the token to v_3^3, or (b) he moves the token to v_2^3. In the former case, as shown in the right of Figure 8, Alice wins the game similarly to the case (i). In the latter case, Alice moves the token to v_3^4. If Bob moves the token to v_4^4 or v_4^5, then Alice can move it back to v_3^4 along a 4-cycle $v_3^4 v_4^4 v_5^5 v_4^5$. Moreover, if Bob moves the token to v_3^3, then Alice can win the game by moving it to v_2^2 (since Bob must move it to v_1^1 in his next move). Such a vertex u (to which a player loses the game by moving the token) is called a *dead vertex* (see Figure 9; a dead vertex is marked by 'd' and colored by gray). Thus, Bob moves the token from v_3^4 to either (1) v_3^5 or (2) v_2^4.

The proof of the case (1): Alice moves the token to v_2^5. Observe that v_2^4 and v_1^5 are dead; since if Bob moves the token to v_2^4, then by moving the token back to v_3^4 Alice can force Bob move it to the dead vertex v_3^3, and if Bob moves the token to v_1^5, then by the following sequence ($\to_A$ (resp., $\to_B$) means a move of the token by Alice (resp., Bob));

$$v_1^5 \to_A v_0^5 \to_B v_0^4 \to_A v_1^5 \to_B v_1^4 \to_A v_2^5 \to_B v_2^4,$$

Alice can force Bob move it to v_2^4 (after that, Bob must move it to the dead vertex v_3^3 similarly to the above). Thus, Bob must move the token to v_1^4, and then Alice moves it to v_0^3. Since v_0^4 is also dead now, Bob moves the token to v_1^3. Therefore, Alice can force Bob to move the token to a dead vertex, by moving it from v_1^3 to v_1^4.

The proof of the case (2): Alice moves the token to v_2^5. Similarly to the previous case, Bob must move the token to v_1^4 since v_1^5 and v_3^5 are dead. After that, Alice can force Bob to move the token to a dead vertex by using one of the following two patterns:

- $v_1^4 \to_A v_2^4 \to_B v_1^3 \to_A v_1^2 \to_B v_2^3 \to_A v_2^4$
- $v_1^4 \to_A v_2^4 \to_B v_2^3 \to_A v_1^2 \to_B v_1^3 \to_A v_2^4$

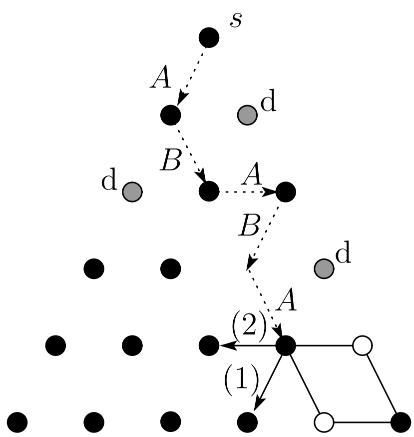

FIGURE 9. The case (ii)-(b)

Therefore, Alice wins feedback game on the triangular grid graph T_5. ■

Furthermore, we confirm that there exists no even kernel graph of T_n if $n = 2^m - 3$ with $m \geq 2$, as follows.

Theorem 13. *If $n = 2^m - 3$ with $m \geq 2$, then the triangular grid graph T_n has no even kernel graph.*

Proof. Suppose to the contrary that T_n has an even kernel graph H_n. From Lemma 11, any even kernel of T_n contains v_0^0, v_0^n and v_n^n. Then these three vertices are in $B \subset V(H_n)$, which is a subset containing a starting vertex.

By symmetry, let i be the smallest number such that $v_0^{2i} \notin B$, i.e., if $v_{2j}^{2j} \notin B$ with $j < i$, then we relabel $v_0^k, v_1^k, \ldots, v_k^k$ as $v_k^k, v_{k-1}^k, \ldots, v_0^k$ for any $k \in \{1, 2, \ldots, n\}$. Then for every $j = 1, \ldots, 2i-1$, $v_0^j \in W$ (resp. $v_0^j \in B$) if j is odd (resp. even). Moreover, by definition and local restrictions, $v_j^k \in B$ for $j, k < 2i$ when j, k is even, otherwise $v_j^k \in W$. (Since the degree of a vertex in W may be zero, every vertex of T_n can be a member in $V(H_n)$.)

Here we define a closed-packed triangle to discuss the situation of layers below i.

Definition 14 (Close-packed)**.** Let $\triangle abc$ $(a, b, c \in V(T_n))$ denotes a triangular grid graph T_p for some $p \in \{0, 1, \ldots, n\}$ which is contained in T_n as a subgraph. The triangular grid subgraph $\triangle abc$ is *close-packed* (or $\triangle abc$ is *a close-packed triangle*) if $v_j^k \in B \cap V(\triangle abc)$ for $j, k \in \{0, 1, \ldots, n\}$ when j, k are even, otherwise $v_j^k \in W \cap V(\triangle abc)$.

See Figure 10. Since v_0^{2i-1}, v_1^{2i-1} and v_0^{2i} are in W and $v_0^{2i-2} \in B$, we have $v_1^{2i} \in B$, and this leads to $v_1^{2i+1} \in W$ and $v_0^{2i+1} \in B$. Observe that $v_{2i}^{2i} \notin B$, since otherwise, v_2^{2i} is also in B by the observation in the second paragraph, which contradicts $v_1^{2i} \in B$. Moreover, with the similar observation, $v_{2i-1}^{2i}, v_{2i+1}^{2i+1} \in B$ and $v_{2i}^{2i+1} \in W$.

Two black vertices v_1^{2i} and v_{2i-1}^{2i} and white vertices v_j^{2i-1}, where $0 \leq j \leq 2i-1$, force that all $v_j^{2i} \in B$ if j is odd and all $v_j^{2i} \in W$ if j is even. By local restrictions, $\triangle v_1^{2i} v_{2i-1}^{2i} v_{2i-1}^{4i-2}$ is close-packed (see Figure 11); note that $v_2^{2i+2}, v_{2i}^{2i+2} \notin B$, since if $v_2^{2i+2} \in$

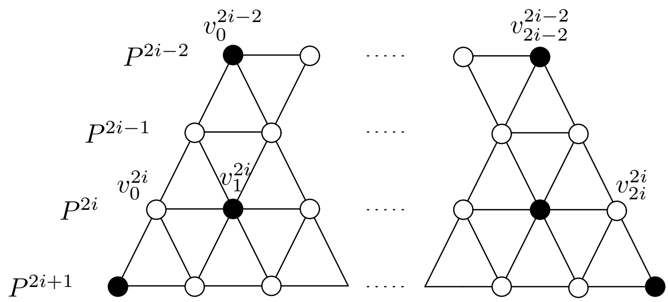

FIGURE 10. The situation around v_0^{2i} and v_{2i}^{2i}; the numbers listed in the left denote the superscript numbers of v_k^j lying on the same column and black (resp. white) vertices denote those in B (resp. W).

B (resp., $v_{2i}^{2i+2} \in B$), then v_1^{2i+1} (resp., v_{2i}^{2i+1}) in W must be of degree 3 in H_n, a contradiction. Furthermore, $\triangle v_1^{2i} v_{2i-1}^{2i} v_{2i-1}^{4i-2}$ forces v_j^{2i+j}, where $0 \le j \le 2i-1$ and v_{2i}^{2i+j}, where $0 \le j \le 2i-1$ are in W. Note that whether v_{2i}^{4i} is in B or W is not revealed yet under the above discussions.

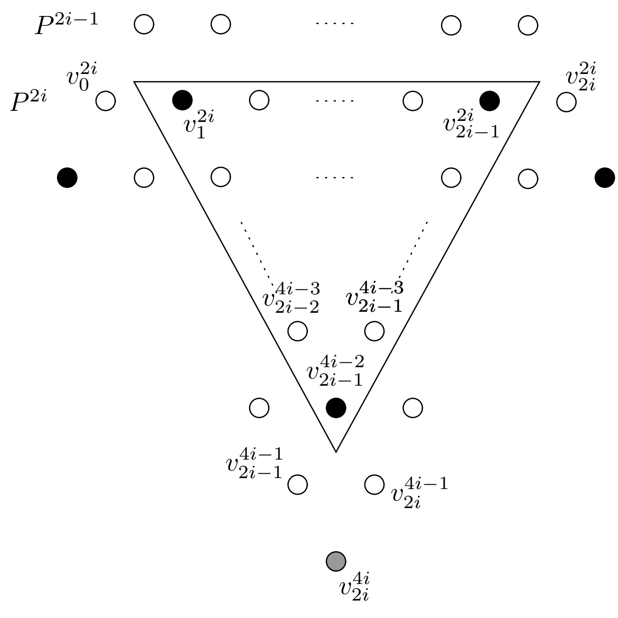

FIGURE 11. The close-packed triangle $\triangle v_1^{2i} v_{2i-1}^{2i} v_{2i-1}^{4i-2}$ (surrounded by three lines forming a triangle); for the sake of simplicity, we omit edges and the gray vertex means that it is not decided yet whether the vertex is in B or W.

Now similar discussion leads us the following facts.

(1) White vertices v_j^{2i+j}, where $0 \le j \le 2i-1$, and black one v_0^{2i+1} generate a close-packed triangle $\triangle v_0^{2i+1} v_0^{4i-1} v_{2i-2}^{4i-1}$.

(2) White vertices v_{2i}^{2i+j}, where $0 \le j \le 2i-1$, and black one v_{2i+1}^{2i+1} generate a close-packed triangle $\triangle v_{2i+1}^{2i+1} v_{2i+1}^{4i-1} v_{4i-1}^{4i-1}$.

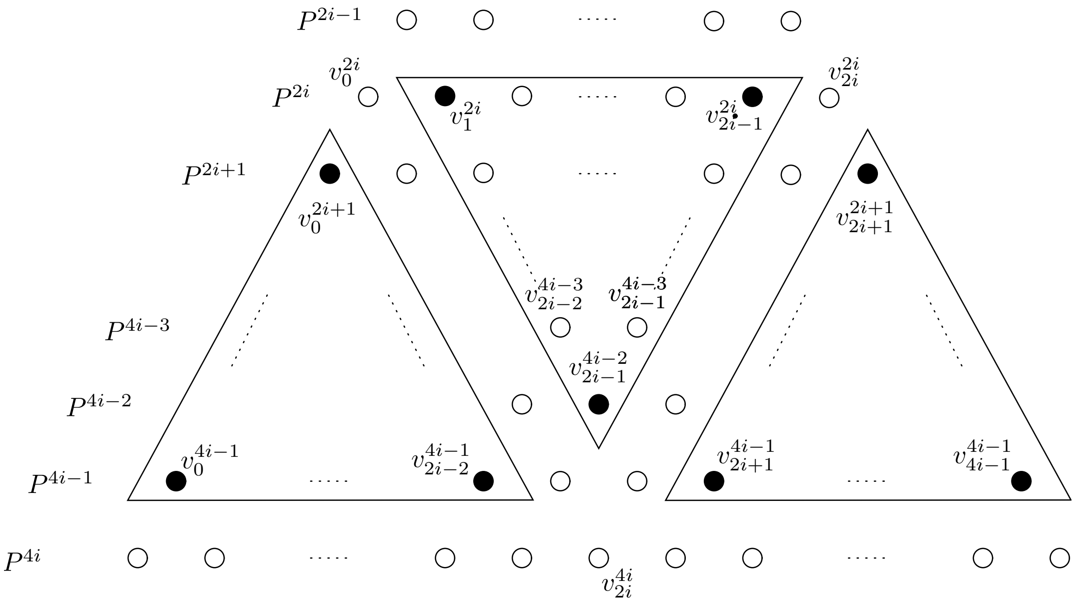

FIGURE 12. The three close-packed triangles $\triangle v_1^{2i} v_{2i-1}^{2i} v_{2i-1}^{4i-2}$, $\triangle v_0^{2i+1} v_0^{4i-1} v_{2i-2}^{4i-1}$ and $\triangle v_{2i+1}^{2i+1} v_{2i+1}^{4i-1} v_{4i-1}^{4i-1}$

The situation is depicted in Figure 12. These successive generation can stop if $n = 4i-1$. If $n = 4i-1$, H_n is constructed by four close-packed triangles. If $n < 4i-1$, the above generation are not satisfied. Therefore, there does not exist such i under that n. If $n > 4i-1$, the above generation must continue as follows:

Two close-packed triangles $\triangle v_0^{2i+1} v_0^{4i-1} v_{2i-2}^{4i-1}$ and $\triangle v_{2i+1}^{2i+1} v_{2i+1}^{4i-1} v_{4i-1}^{4i-1}$ force that $v_j^{4i} \in W$, where $0 \le j \le 4i$ and $j \ne 2i$. We next focus on the fact that $v_{2i-1}^{4i-1}, v_{2i}^{4i-1}, v_{2i-1}^{4i}$ and v_{2i+1}^{4i} must be in W. This fact implies v_{2i}^{4i} must be of degree 0 in H_n (i.e., it is in W) since local constrains force $v_{2i-1}^{4i+1}, v_{2i+2}^{4i+1}, v_{2i+1}^{4i+2} \in B$. These new black vertices generate new three close-packed triangles $\triangle v_1^{4i+1} v_{2i-1}^{4i+1} v_{2i-1}^{6i-1}, \triangle v_{2i+1}^{4i+2} v_{2i+1}^{6i} v_{4i-1}^{6i}$ and $\triangle v_{2i+2}^{4i+1} v_{4i}^{4i+1} v_{4i}^{6i-1}$, and these new close-packed triangles force two extra close-packed triangles $\triangle v_0^{4i+2} v_0^{6i} v_{2i-2}^{6i}, \triangle v_{4i+2}^{4i+2} v_{4i+2}^{6i} v_{6i}^{6i}$. In this case, these successive generation can stop if $n = 6i$, and also this discussion can continue recursively if $n > 6i$.

Let r be the number of recursion on the above discussion, i.e., H_n contains r^2 close-packed triangles. (Note that $\triangle v_0^0 v_0^{2i-2} v_{2i-2}^{2i-2}$ is also a close-packed triangle.) By the hypothesis, there can exist such i on T_n if $n = r(2i+1)-3$, where $i, r \ge 1$, which implies that only if n can be represented as $n = r(2i+1)-3$, where $i, r \ge 1$, then H_n can exist. Therefore, by the assumption that $n = 2^m - 3$, there must not exist an even kernel graph for T_n when $m > 1$ since 2^{m-j} cannot be represented as $2i+1$ for any $i \ge 1$ and $j \le m$, a contradiction. ∎

Thus, we propose the following conjecture which implies that for every triangular grid graph T_n with a starting vertex v_0^0, Bob wins the game on T_n if and only if T_n contains an even kernel graph with respect to v_0^0.

Conjecture 15. *If $n = 2^m - 3$ with $m \geq 2$, then Alice wins feedback game on the triangular grid graph T_n with a starting vertex v_0^0.*

In the end of this section, we describe feedback game on T_n in which the starting vertex is not v_0^0. In general, changing the starting vertex of T_n changes the winner of the game. For example, Bob wins the game on T_2 with starting vertex v_0^0, but it is easy to check that if the starting vertex v_0^1, then Alice wins. It is clear that if T_n has an even kernel graph with partite sets B and W, where B contains the starting vertex, then Bob wins the game on T_n with every starting vertex $s \in B$. However, it is not clear whether Alice wins the game on T_n with every starting vertex $s \in W$.

5. Toroidal Grid Graphs

In this section, we investigate feedback game on toroidal grid graphs. Undirected edge geography on a grid graph (which is the Cartesian product of two paths) is completely solved [11], and directed edge geography on a directed toroidal grid graph is also investigated in [13].

Definition 16 (Toroidal grid graph)**.** A *toroidal grid graph* $Q(m, n)$ is the Cartesian product of two cycles $C_m = u_0u_1 \cdots u_{m-1}$ and $C_n = v_0v_1 \cdots v_{n-1}$ with $m \geq 2$ and $n \geq 2$, that is,

- $V(Q(m, n)) = \{(u_i, v_j) : i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1, \ldots, n-1\}\}$.
- $(u_i, v_j)(u_{i'}, v_{j'}) \in E(Q(m, n))$ if and only if
  - $i = i'$ and $v_jv_{j'} \in E(C_n)$ or
  - $j = j'$ and $u_iu_{i'} \in E(C_m)$.

In other words, $Q(m, n)$ is a 4-regular *quadrangulation* embedded on the torus, which is a graph on a surface with each face quadrangular. For example, see Figure 13; by identifying the top and bottom (resp., right and left) sides along the direction of arrows, we have the toroidal grid graph $Q(3, 4)$. Note that $Q(m, n)$ is vertex-transitive, that is, there exists an automorphism of the graph mapping a vertex into any other vertex. Thus, feedback game on $Q(m, n)$ does not depend on the choice of a starting vertex, and hence, toroidal grid graphs seem to be a reasonable subclass of connected Eulerian graphs with maximum degree at most 4 for considering feedback game.

For several combinations of m and n, we have determined the winner of the game as follows. In particular, if the greatest common divisor of m and n, denoted by $\gcd(m, n)$, is bigger than one, then Bob can win the game on $Q(m, n)$, and otherwise it seems to be that Alice can win the game.

Theorem 17. *If $\gcd(m, n) = c > 1$, then Bob can win feedback game on $Q(m, n)$.*

Proof. By the assumption, let $m = ck$ and $n = ck'$ for some positive integers k and k'. The toroidal grid graph $Q(c, c)$ with a starting vertex $s = (u_0, v_0)$ has an even kernel graph H^c with partite sets B and W such that $(u_i, v_i) \in B$, $(u_i, v_{i+1}), (u_{i+1}, v_i) \in W$ and edges $(u_i, v_i)(u_i, v_{i+1}), (u_i, v_i)(u_{i+1}, v_i), (u_i, v_{i+1})(u_{i+1}, v_{i+1})$ and $(u_{i+1}, v_i)(u_{i+1}, v_{i+1})$ are in $E(H^c)$ for any $i \in \{0, 1, \ldots, c-1\}$, where subscripts are modulo c (see Figure 14).

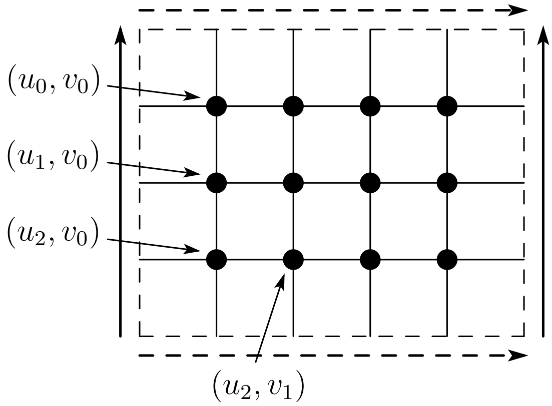

FIGURE 13. The toroidal grid graph $Q(3, 4)$

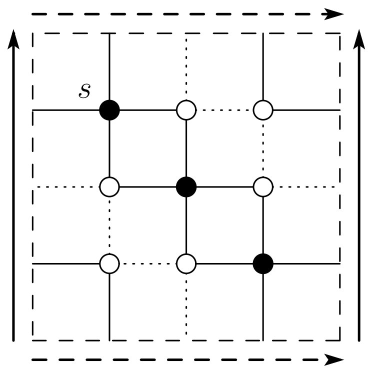

FIGURE 14. An even kernel graph of $Q(3, 3)$

Note that $Q(m, n)$ can be covered by $Q(c, c)$'s, and hence, we can obtain an even kernel graph of $Q(m, n)$ by combining that of $Q(c, c)$, as shown in Figure 15. (Figure 15 represents $Q(6, 9)$ covered by six $Q(3, 3)$'s with an even kernel graph shown in Figure 14.) Therefore, the theorem holds by Lemma 7. ∎

Theorem 18. *If* $\gcd(2, n) = 1$*, then Alice can win feedback game on* $Q(2, n)$*.*

Proof. Without loss of generality, we set (u_0, v_0) be a starting vertex. Since $\gcd(2, n) = 1$, n is odd. Alice first moves the token to (u_0, v_1). After that, Alice plays the game according to Bob's move as follows:

(i) If Bob moves the token to (u_1, v_i) through an edge $(u_0, v_i)(u_1, v_i)$, Alice moves it to (u_0, v_i) using $(u_1, v_i)(u_0, v_i)$.

(ii) If Bob moves the token to (u_0, v_{i+1}), then Alice moves it to (u_0, v_{i+2}), where subscripts modulo n.

Observe that the strategy (i) can be always applied and that after the strategy (i) is applied, Bob must move the token from (u_0, v_i) to (u_0, v_{i+1}). Note that the token lies on (u_0, v_j) for some odd $j < n - 1$ after Alice uses the strategy (ii). Therefore, since n is

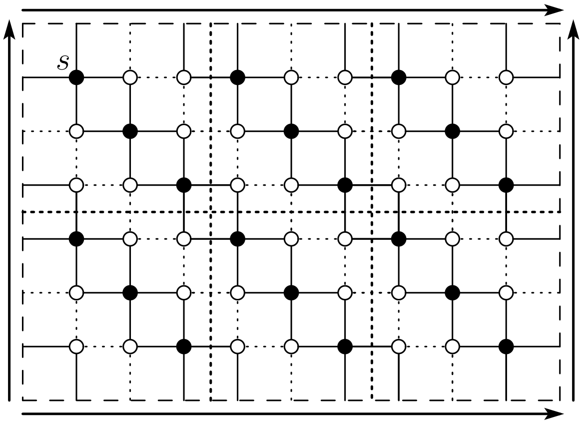

FIGURE 15. The toroidal grid graph $Q(6,9)$ covered by $Q(3,3)$'s with even kernel graphs

odd, Alice finally moves the token from (u_0, v_{n-1}) to (u_0, v_0), that is, she wins the game. ■

Theorem 19. *If* $\gcd(3, n) = 1$*, then Alice can win feedback game on* $Q(3, n)$*.*

Proof. Without loss of generality, we may assume a starting vertex s is (u_0, v_0). Moreover, by Theorem 18, we may assume that $n \geq 4$.

Alice first moves the token to (u_1, v_0). If Bob moves it to (u_2, v_0), then Alice wins the game. Thus, Bob moves the token to (u_1, v_1) by symmetry and then Alice moves it to (u_2, v_1). Next, Bob has to move the token to (u_2, v_2) (otherwise Alice can move it back to s) and then Alice moves it to (u_0, v_2). After that, Alice plays the game according to Bob's move until the token is moved to (u_i, v_{n-2}) for some $i \in \{0, 1, 2\}$ by herself, as follows (where the subscripts of u_i and v_j in the following are modulo 3 and n, respectively):

(i) If Bob moves the token on (u_i, v_j) to (u_i, v_{j-1}), then Alice moves it to (u_i, v_{j-2}).
(ii) If Bob moves the token on (u_i, v_j) to (u_{i+1}, v_j), then Alice moves it to (u_{i+1}, v_{j-1}).
(iii) If Bob moves the token on (u_i, v_j) to (u_i, v_{j+1}) then Alice moves it to (u_{i+1}, v_{j+1}).

Observe that in the above beginning moves from the starting vertex to (u_0, v_2), Alice applies only the strategy (iii) twice except her first move.

In the strategy (i), after Alice's move, (u_i, v_{j-2}) is incident to the unique edge $(u_{i-1}, v_{j-2})(u_i, v_{j-2})$ unless $(u_i, v_{j-2}) = (u_0, v_0)$, since two edges $(u_i, v_{j-3})(u_i, v_{j-2})$ and $(u_i, v_{j-2})(u_{i+1}, v_{j-2})$ are used by the moves in (iii). Similarly, for the strategy (ii), (u_{i+1}, v_{j-1}) is incident to the unique edge $(u_i, v_{j-1})(u_{i+1}, v_{j-1})$. Thus, after Alice's move by the strategy (i) (resp., (ii)), Bob must move the token to (u_{i-1}, v_{j-2}) (resp., (u_i, v_{j-1})). Hereafter, Alice moves the token to (u_{i-1}, v_{j-3}) (resp., (u_i, v_{j-2})) and then the same situation occurs at the current vertex. Hence, by applying the above move repeatedly, the token is finally carried to s from (u_0, v_1) by Alice.

Therefore, we may suppose that until Alice moves the token to (u_i, v_{n-2}) for some $i \in \{0, 1, 2\}$ by herself, she always applies the strategy (iii), that is, two indices i and j of a current vertex (u_i, v_j) are alternately increased one by one by Alice and Bob, respectively. Therefore, we may assume that Alice finally moves the token to (u_0, v_{n-2})

(resp., (u_1, v_{n-2})) from (u_2, v_{n-2}) (resp., (u_0, v_{n-2})) depending on n; otherwise, i.e., if Alice finally moves the token (u_2, v_{n-2}) from (u_1, v_{n-2}), then $n-2 \equiv 1 \pmod 3$ and hence $n \equiv 0 \pmod 3$, which contradicts $\gcd(3, n) = 1$.

Thus the token is now put on (u_0, v_{n-2}) or (u_1, v_{n-2}). In the former case, Bob moves to (u_0, v_{n-1}) and then Alice wins the game by moving it back to s. In the latter case, Bob moves to (u_1, v_{n-1}) and then Alice moves it to (u_1, v_0). After that, since Bob must move the token to (u_2, v_0), Alice wins the game by moving it from (u_2, v_0) to s. Therefore, the theorem holds. ∎

Theorem 20. *If* $\gcd(m, n) = 1$, *then there exists no even kernel graph of* $Q(m, n)$.

Proof. Suppose to the contrary that $Q(m, n)$ with $\gcd(m, n) = 1$ has an even kernel graph. Let $\mathrm{Ev}(m, n) \subseteq Q(m, n)$ be an even kernel graph of $Q(m, n)$. From the definition, any vertex in the white part of $\mathrm{Ev}(m, n)$, denoted by $W(m, n)$, has two or four neighbours (a vertex in $W(m, n)$ can have no neighbour, but in this case we can remove it from $\mathrm{Ev}(m, n)$) and they are in the black part of $\mathrm{Ev}(m, n)$, denoted by $B(m, n)$. A *stopgap* of $\mathrm{Ev}(m, n)$ is a vertex in $W(m, n)$ of degree 2 such that its neighbours lie on the same row or column. When we ignore all stopgaps, $\mathrm{Ev}(m, n)$ has several components surrounded by vertices in $W(m, n)$. Note that any vertex in $B(m, n)$ cannot be adjacent to vertices not in $W(m, n)$. We denote a component and stopgaps which are its neighbours (if exist) as a *cluster* (see Figure 16). In Figure 16, black vertices are in $B(m, n)$, gray vertices with bold circle are in $W(m, n)$, and gray vertices without edges are not in $\mathrm{Ev}(m, n)$.

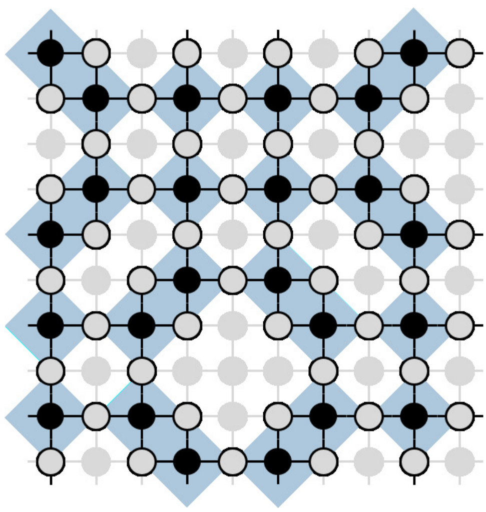

FIGURE 16. An even kernel graph $\mathrm{Ev}(10, 10)$ of $Q(10, 10)$ and its clusters denoted by shaded regions

Every cluster looks a rectangle rotated 45 degrees. This means that a cluster has four sides consisting of diagonally consecutive vertices in $W(m, n)$. For clusters, we have following claims.

Claim 1. *Every clusters are rectangles unless* $\mathrm{Ev}(m, n) = Q(m, n)$.

Proof. Assume that a cluster C is not a rectangle. Then there must exist a vertex in $W(m,n) \subset C$ which is not a stopgap, and is adjacent to a vertex not in $\mathrm{Ev}(m,n)$ and odd number of vertices in $B(m,n)$ (since all vertices in $B(m,n)$ are of degree 4). This contradicts the definition of $\mathrm{Ev}(m,n)$. ■

From claim 1, if there exists $\mathrm{Ev}(m,n) \subsetneq Q(m,n)$ when $\gcd(m,n) = 1$, because any straight line rotated 45 degrees on $Q(m,n)$ contains all of $V(Q(m,n))$, any stop gap in arbitrary cluster in $\mathrm{Ev}(m,n)$ succeeds on every vertices in $Q(m,n)$. This is contradiction because all vertices are in stop gap, this means $W(m,n) = V(Q(m,n))$ and $B(m,n)$ has no vertex. Therefore, $\mathrm{Ev}(m,n)$ must be $Q(m,n)$ when $\gcd(m,n) = 1$ However, $Q(m,n)$ is not bipartite when $\gcd(m,n) = 1$, which contradicts the definition of $\mathrm{Ev}(m,n)$. Therefore, the theorem holds. ■

Under the results obtained above, we conclude the paper with proposing the following conjecture which implies that Alice can win feedback game on $Q(m,n)$ if and only if $\gcd(m,n) = 1$.

Conjecture 21. *Alice can win feedback game on $Q(m,n)$ if $\gcd(m,n) = 1$.*

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References

[1] B. Allen, M.A. Nowak, Games on graphs, Eur. Math. Soc. 1 (2014) 113–151.

[2] C. Berge, Combinatorial games on a graph, Discrete Math. 151 (1996) 59–65.

[3] E.R. Berlekamp, J.H. Conway, R.K. Guy, Winning Ways for Your Mathematical Plays, A K Peters/CRC Press, 4 vols., 1982–2004.

[4] C.L. Bouton, Nim, a game with a complete mathematical theory, Ann. of Math. 3 (1901) 35–39.

[5] J.H. Conway, On Numbers and Games, A K Peters/CRC Press, 2000.

[6] R. Diestel, Graph Theory (fifth edition), Graduate Texts in Mathematics, 173, Springer, 2016.

[7] E. Duchêne, G. Renault, Vertex Nim played on graphs, Theoret. Comput. Sci. 516 (2014) 20–27.

[8] H.E. Dudeney, The Canterbury puzzles, Dover, puzzle 73 (2002) 118–119 (originally published in 1908).

[9] P. Erdös, J.L. Selfridge, On a combinatorial game, J. Combin. Theory Ser. A 14 (1973) 298–301.

[10] A.S. Fraenkel, Even kernels, Electron. J. Combin. 1 (1994) #R5.

[11] A.S. Fraenkel, E.R. Scheinerman, D. Ullman, Undirected edge geography, Theoret. Comput. Sci. 112 (1993) 371–381.

[12] P.M. Grundy, Mathematics and games, Eureka 2 (1939) 6–9.

[13] M.S. Hogan, D.G. Horrocks, Geography played on an n-cycle times a 4-cycle, Integers 3 (2003) #G02.

[14] D. Lichtenstein, M. Sipser, GO is polynomial-space hard, J. ACM 27 (1980) 393–401.

[15] T.J. Schaefer, On the complexity of some two-person perfect-information games, J. Comput. System Sci. 16 (1978) 185–225.

[16] M. Soltys, C. Wilson, On the complexity of computing winning strategies for finite poset games, Theory Comput. Syst. 48 (2011) 680–692.

[17] R.P. Sprague, Über mathematische Kampfspiele, Tohoku Math. J. 41 (1935) 438–444.

[18] S. Teramoto, E.D. Demaine, R. Uehara, The Voronoi game on graphs and its complexity, J. Graph Algorithms Appl. 15 (2011) 485–501.