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Discrete and Computational Geometry, Graphs, and Games

Maximum Nim and Chocolate Bar Games

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Abstract The authors studied maximum Nim wherein each player can remove at most $f(m)$ stones from a pile of m stones, where $f(m) = \lceil \frac{m}{d} \rceil$ or $f(m) = \lfloor \frac{m+1}{d} \rfloor$ with a positive number d that is larger than 1, and present function h_d such that $\mathcal{G}(h_d(x)) = \mathcal{G}(x)$, where $\mathcal{G}(x)$ is the Grundy number of the pile of x stones. The authors apply function h_d to the study of chocolate bar games, wherein two players take turns and cut a chocolate bar in a straight line and eat the pieces. These games can be considered the sum of two maximum Nim with a restriction on the size of the chocolate bar that can be eaten by the players. Therefore, using h_d, the authors devise formulas to calculate the winning position of the previous player for a rectangular chocolate bar game with a restriction on the size of the chocolate bar piece to be eaten. In addition, the authors present conjectures about rectangular chocolate bar games with some missing parts.

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Keywords: combinatorial game; maximum Nim; Grundy number

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1. Introduction

The authors present formulas for the Grundy numbers of Maximum Nim, and they apply these formulas to calculate the winning position of the previous player in rectangular chocolate bar games with restrictions on the size of the chocolate bar piece to be eaten.

The classic game of Nim is played with stone piles. A player can remove any number of stones from any one pile during their turn; the player who takes the last stone is considered the winner.

There are many variants of the classical game of Nim. In Maximum Nim, we place an upper bound $f(n)$ on the number of stones that can be removed in terms of the number n of stones in the pile (see [1]). In this study, we investigate the case wherein $f(m) = \lceil \frac{m}{d} \rceil$, and present function h_d such that $\mathcal{G}(h_d(x)) = \mathcal{G}(x)$, where $\mathcal{G}(x)$ is the Grundy number of

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the pile of x stones. A part of this research has already been presented by [2], [3], and [4].

Chocolate bar games are variants of the CHOMP game [5]. The first two-dimensional chocolate bar was presented by Robin [6]; it is a rectangular array of squares with a black square representing a bitter block located at some part in the bar. Each player breaks the bar in a straight line along the grooves into two pieces and eats the piece without the bitter part, at his/her turn. The player that manages to leave the opponent with a single bitter block (black block) is the winner. This is a variation of the classical Nim with two heaps; the chocolate bar game shown in Figure 1 is mathematically the same as that of the classical Nim with two piles shown in Figure 2. Many variants of chocolate bar games have been published [7] and [8], with some using two-dimensional chocolate bars where some squares are removed (Figure 3).

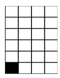

FIGURE 1. Original chocolate bar with a bitter part

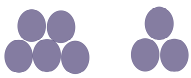

FIGURE 2. Traditional Nim with two piles

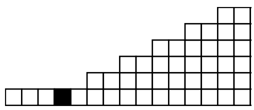

FIGURE 3. Chocolate bar with a bitter part, where some squares are removed

In this article, the authors study a rectangular chocolate bar game with restrictions on the size of the chocolate bar piece to be eaten. We manage to play without a bitter part owing to the restriction on the size of the chocolate bar to be eaten (Figure 4).

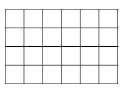

FIGURE 4. Chocolate bar without any bitter part

Given this restriction on the size of the chocolate bar piece to be eaten, the chocolate bar game considered in this study is the sum of two Maximum Nim. Therefore, the research on Maximum Nim can be applied to this chocolate bar game.

2. Maximum Nim

Let $Z_{\geq 0}$ and N represent the sets of non-negative integers and natural numbers, respectively.

We consider maximum Nim as follows.

Suppose there is a pile of n stones, and two players take turns to remove stones from the pile. At each turn, the player is allowed to remove at least one and at most $f(m)$ stones if the number of stones is m. The player who removes the last stone or stones is the winner. Here, $f(m)$ represents a function whose values are non-negative integers for $m \in Z_{\geq 0}$ such that $0 \leq f(m) - f(m-1) \leq 1$ for any natural number m. We refer to f as the rule function.

We briefly review some necessary concepts in combinatorial game theory (see [9] for more details).

We deal with impartial games with no draws, and therefore, there are only two outcome classes.

(a) A position is called a $\mathcal{P}$-*position* if it is a winning position for the previous player (the player who just moved), as long as he/she plays correctly at every stage.

(b) A position is called an $\mathcal{N}$-*position* if it is a winning position for the next player, as long as he/she plays correctly at every stage.

In combinatorial game theory, "nim-sum is an important concept that indicates addition without carry overs for numbers expressed in base 2. Nim-sum is identical to the "exclusive or operation in computer science.

Let x, $y \in Z_{\geq 0}$. We represent $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$. We define the nim-sum $x \oplus y$ by $x \oplus y = \sum\limits_{i=0}^{n} w_i 2^i$, where $w_i = x_i + y_i \pmod 2$.

The Grundy number is one of the most important tools in research on combinatorial game theory, and we need "move for the definition of the Grundy number.

(i) For any position $\mathbf{p}$ of a game $\mathbf{G}$, there is a set of positions that can be reached by making precisely one move in $\mathbf{G}$, which is denoted by $move(\mathbf{p})$.

(ii) The *minimum excluded value* (*mex*) of set S of non-negative integers is the least non-negative integer, not in S.

(iii) Let $\mathbf{p}$ be the position of an impartial game. The associated Grundy number is denoted by $\mathcal{G}(\mathbf{p})$ and is recursively defined by $\mathcal{G}(\mathbf{p}) = mex\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

For the maximum Nim of x stones with rule function $f(x)$,
$move(x) = \{x - u : 1 \leq u \leq f(x) \text{ and } u \in N\}$. The Grundy number is important for two reasons. The first is its relation to the $\mathcal{P}$-position presented in Theorem 2.1, and the second is its application to the sum of games.

Theorem 2.1. *Let $\mathcal{G}$ represent the Grundy number of the combinatorial game $\mathbf{G}$. Then, for any position $\mathbf{p}$ of $\mathbf{G}$, we have $\mathcal{G}(\mathbf{p}) = 0$ if and only if $\mathbf{p}$ represents a $\mathcal{P}$-position.*

For the proof, please see [9].

The sum of two games is an important concept in combinatorial game theory.

If $\mathbf{G_1}$ and $\mathbf{G_2}$ are combinatorial games, their sum, denoted by $\mathbf{G_1} + \mathbf{G_2}$, represents a game in which each player alternately chooses one among $\mathbf{G_1}$ and $\mathbf{G_2}$, and plays a move in the selected game; the player who cannot play loses the game. This scenario occurs only when $\mathbf{G_1}$ and $\mathbf{G_2}$ have reached a terminal position.

There is a simple relation for the Grundy number of the sum of two games.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ represent the Grundy numbers of $\mathbf{G_1}$ and $\mathbf{G_2}$. Then, the Grundy number of a position $\{\mathbf{g}, \mathbf{h}\}$ in the game $\mathbf{G_1} + \mathbf{G_2}$ is $\mathcal{G}_1(\mathbf{g}) \oplus \mathcal{G}_1(\mathbf{h})$.

Given this background, the authors present new results from this point forward.

Lemma 2.2. *Let $\mathcal{G}$ represent the Grundy number of the maximum Nim with the rule function $f(x)$. Then, we have the following properties:*
(i) For any $x \in Z_{\geq 0}$ $\mathcal{G}(x) \leq f(x)$.
(ii) If $f(x) = f(x-1)$, $\mathcal{G}(x) = \mathcal{G}(x - f(x) - 1)$.
(iii) If $f(x) > f(x-1)$, then the following hold:
$(iii.1)$ $\mathcal{G}(x) = f(x)$.
$(iii.2)$ Let $m = \mathcal{G}(x)$. Then, x represent the smallest integer such that $\mathcal{G}(x) = m$.

Proof. Properties (ii) and $(iii.1)$ are proved in Lemma 2.1 of [1]. We prove property (i) using mathematical induction. We assume that $\mathcal{G}(x) \leq f(x)$ for $x = 0, 1, \cdots, n$. If $f(n+1) = f(n)$, then by (ii), $\mathcal{G}(n+1) = \mathcal{G}(n+1-f(n+1)-1) = \mathcal{G}(n-f(n+1))$ $\leq f(n - f(n+1)) \leq f(n) = f(n+1)$.

If $f(n+1) > f(n)$, then by $(iii.1)$ $\mathcal{G}(n+1) = f(n+1)$. Therefore, we have (i).

Next, we prove $(iii.2)$. Suppose that $m = \mathcal{G}(x) = f(x)$. If there exists x' such that $x' < x$ and $m = \mathcal{G}(x')$, then by (i), we have $m = \mathcal{G}(x') \leq f(x') \leq f(x-1) < f(x) = m$, which leads to a contradiction. Therefore, we have $(iii.2)$. ∎

For any real number x, the ceiling and floor of x denoted by $\lceil x \rceil$ and $\lfloor x \rfloor$ represent the least integer greater than or equal to x and the greatest integer less than or equal to x, respectively.

Remark 2.3. When $f(x) = f(x-1)$, we have $\mathcal{G}(x) = \mathcal{G}(x-f(x)-1)$ by Lemma 2.2; this equation presents a relation between the Grundy number $\mathcal{G}(x)$ of the pile of x stones and the Grundy number $\mathcal{G}(x - f(x) - 1)$ of the pile of $x - f(x) - 1$ stones. This relationship provides a position with a smaller number of stones that have the same Grundy number. In subsections 2.1, 2.2, 2.3 and 2.4, we devise a function h such that $\mathcal{G}(h(x)) = \mathcal{G}(x)$; this function provides a position with a larger number of stones that have the same Grundy number.

2.1. Case I: $f(m) = \lceil \frac{m}{d} \rceil$ for d such that $2 < d$

Hereafter, we assume that d represents a positive real number such that $d > 2$. We study maximum Nim with the rule function $f(m) = \lceil \frac{m}{d} \rceil$; $\mathcal{G}(n)$ represents the Grundy number of this maximum Nim with n stones.

The maximum Nim with $d > 2$ has a relatively simple mathematical structure compared to the case $0 < d \leq 2$ treated in Subsection 2.2. Especially for the structure of Grundy numbers, the value of d makes a big difference.

For $p \in Z_{\geq 0}$ and a positive real number q, there exist $s \in Z_{\geq 0}$ and a real number t such that $0 \leq t < q$, and $p = s \times q + t$. We denote t as $\mathrm{mod}(p, q)$.

Definition 2.4. We define a function $h_d(n)$ for a non-negative integer n as
(a) $h_d(n) = n + \left\lfloor \frac{n}{d-1} \right\rfloor + 2$, if $\mathrm{mod}(n, d-1) \leq d-2$.
(b) $h_d(n) = n + \left\lfloor \frac{n}{d-1} \right\rfloor + 3$ if $\mathrm{mod}(n, d-1) > d-2$.

Theorem 2.5. *For h_d defined in Definition 2.4, the following holds:*
(i) $h_d(n) = \left\lceil \frac{h_d(n)}{d} \right\rceil + n + 1$ *for* $n \in Z_{\geq 0}$.
(ii) $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ *for* $n \in Z_{\geq 0}$.
(iii) For any $m \in Z_{\geq 0}$, *there exists* $n_0 \in Z_{\geq 0}$ *such that* $\{n : \mathcal{G}(n) = m\} = \{h_d^p(n_0) : p \in Z_{\geq 0}\}$, *where* h_d^p *is the p-th functional power of* h_d.

Proof. Let

$$n = (d-1)k + s \tag{2.1}$$

for $k \in Z_{\geq 0}$ and $s = \text{mod}(n, d-1)$, respectively. Then, we have

$$\left\lfloor \frac{n}{d-1} \right\rfloor = k. \tag{2.2}$$

First, we prove (i) and (ii).
$[I]$. Suppose that

$$s = \text{mod}(n, d-1) \leq d-2. \tag{2.3}$$

Then, by Definition 2.4 and (2.2), we have

$$h_d(n) = n + k + 2, \tag{2.4}$$

and hence, by (2.1), we have

$$\frac{h_d(n)}{d} = \frac{n+k+2}{d} = \frac{(d-1)k+s+k+2}{d} = k + \frac{s+2}{d} \tag{2.5}$$

and

$$\frac{h_d(n)-1}{d} = k + \frac{s+1}{d}. \tag{2.6}$$

By (2.3),

$$0 < \frac{s+1}{d} < \frac{s+2}{d} \leq \frac{d}{d} = 1,$$

and hence, by (2.5) and (2.6),

$$\left\lceil \frac{h_d(n)-1}{d} \right\rceil = \left\lceil \frac{h_d(n)}{d} \right\rceil = k+1. \tag{2.7}$$

By (2.7) and (2.4),

$$h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil = n + k + 2 - (k+1) = n + 1, \tag{2.8}$$

and hence, we have (i) of this theorem.

By (2.7),

$$f(h_d(n) - 1) = f(h_d(n)),$$

and hence, by (ii) of Lemma 2.2,

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - f(h_d(n)) - 1) = \mathcal{G}(h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil - 1), \tag{2.9}$$

and by (2.8),

$$\mathcal{G}(h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil - 1) = \mathcal{G}(n+1-1) = \mathcal{G}(n). \tag{2.10}$$

Then, by (2.9) and (2.10), we have (ii) of this theorem.
$[II]$. Suppose that

$$s = \bmod(n, d-1) > d-2. \tag{2.11}$$

Then, by Definition 2.4 and (2.2),

$$h_d(n) = n + k + 3, \tag{2.12}$$

and hence, by $s < d-1$, $2 < d$, and (2.1),

$$\begin{aligned} \frac{h_d(n)}{d} &= \frac{n+k+3}{d} \\ &= \frac{(d-1)k + s + k + 3}{d} \\ &= k + \frac{s+3}{d} \\ &< k + \frac{d+2}{d} \\ &= k + 1 + \frac{2}{d} \\ &< k + 2. \end{aligned} \tag{2.13}$$

By (2.11) and (2.13),

$$\frac{h_d(n)-1}{d} = k + \frac{s+2}{d} > k+1. \tag{2.14}$$

By (2.13) and (2.14),

$$\left\lceil \frac{h_d(n)-1}{d} \right\rceil = \left\lceil \frac{h_d(n)}{d} \right\rceil = k+2, \tag{2.15}$$

and hence,

$$f(h_d(n)) = f(h_d(n)-1). \tag{2.16}$$

By (2.15) and (2.12),

$$h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil = n + k + 3 - (k+2) = n+1. \tag{2.17}$$

Therefore, we have (i).
From (2.16), (2.17), and (ii) of Lemma 2.2,

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil - 1) = \mathcal{G}(n).$$

Then, we have (ii) of this theorem.
(iii) For any m, there is the smallest $n \in Z_{\geq 0}$ such that $\mathcal{G}(n) = m$. We denote this

number by n_0. By (ii), $\{h_d^p(n_0) : p \in Z_{\geq 0}\} \subset \{n : \mathcal{G}(n) = m\}$, where $h_d^0(n_0) = n_0$ and $h_d^{p+1}(n_0) = h_d(h_d^p(n_0))$. Let $n' \in Z_{\geq 0}$, such that

$$h_d^p(n_0) < n' < h_d^{p+1}(n_0) \tag{2.18}$$

for some $p \in Z_{\geq 0}$.

From (i),

$$h_d^{p+1}(n_0) = h_d(h_d^p(n_0)) = \left\lceil \frac{h_d(h_d^p(n_0))}{d} \right\rceil + h_d^p(n_0) + 1,$$

and hence, from (2.18) and the definition of *move*,

$$\begin{aligned} n' \in &\{h_d^{p+1}(n_0) - 1, h_d^{p+1}(n_0) - 2, \cdots, h_d^p(n_0) + 1\} \\ =&\{h_d^{p+1}(n_0) - 1, h_d^{p+1}(n_0) - 2, \cdots, h_d^{p+1}(n_0) - \left\lceil \frac{h_d^{p+1}(n_0)}{d} \right\rceil\} \\ =&move(h_d^{p+1}(n_0)). \end{aligned}$$

Therefore, from the definition of the Grundy number, $\mathcal{G}(n') \neq \mathcal{G}(h_d^{p+1}(n_0)) = m$, and therefore, we have $\{h_d^p(n_0) : p \in Z_{\geq 0}\} = \{n : \mathcal{G}(n) = m\}$. ■

2.2. Case II: $f(m) = \lceil \frac{m}{d} \rceil$ for d such that $1 < d \leq 2$

Hereafter, we assume that d represents a positive real number such that $1 < d \leq 2$. We study the maximum Nim with the rule function $f(m) = \lceil \frac{m}{d} \rceil$; $\mathcal{G}(n)$ represents the Grundy number of this maximum Nim of n stones.

The maximum Nim with $0 < d \leq 2$ has a relatively complicated mathematical structure compared to the case of $d > 2$. Although the case of $d = 2$ is straightforward, we include this in the case $0 < d \leq 2$.

Lemma 2.6. *For $n \in Z_{\geq 0}$, there is a unique number $p \in Z_{\geq 0}$ and a unique real number m such that*

$$-1 < m \leq d - 2 \tag{2.19}$$

and

$$n = (d - 1)p + m. \tag{2.20}$$

Proof. Let $p_0 = \min\{p : n - (d - 1)p \leq d - 2\}$. Then we have

$$n - (d - 1)p_0 \leq d - 2 \tag{2.21}$$

and

$$n - (d - 1)(p_0 - 1) > d - 2. \tag{2.22}$$

Let $m = n - (d - 1)p_0$. Then by (2.21) we have

$$m \leq d - 2 \tag{2.23}$$

and

$$n = (d - 1)p_0 + m. \tag{2.24}$$

By (2.22) we have

$$m + (d - 1) > d - 2,$$

and hence

$$m > -1. \tag{2.25}$$

By (2.23), (2.24) and (2.25), we have p, d that satisfy (2.19) and (2.20).

Next, we prove the uniqueness of p and m in (2.20). Suppose that there are numbers $p_1, p_2 \in Z_{\geq 0}$ and real numbers m_1, m_2 such that $p_1 \leq p_2$ and $m_2 \leq m_1$, $n = (d-1)p_1 + m_1 = (d-1)p_2 + m_2$, and $-1 < m_1, m_2 \leq d-2$. Then, we have $0 \leq (d-1)(p_2 - p_1) = m_1 - m_2 < d - 2 - (-1) = d - 1$, and hence, $0 \leq p_2 - p_1 < 1$. Since $p_1, p_2 \in Z_{\geq 0}$, we have $p_1 = p_2$. Therefore, we also have $m_1 = m_2$. ∎

Definition 2.7. We define a function $h_d(n)$ for $n \in Z_{\geq 0}$ as
$h_d(n) = dp + m + 2$, where p and m represent numbers in Lemma 2.6.

Theorem 2.8. *For h_d defined in Definition 2.7, the following holds:*
(i) $h_d(n) = \left\lceil \frac{h_d(n)}{d} \right\rceil + n + 1.$
(ii) $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ *for* $n \in Z_{\geq 0}$.
(iii) For any $k \in Z_{\geq 0}$*, there exists* $n_0 \in Z_{\geq 0}$ *such that* $\{n : \mathcal{G}(n) = k\} = \{h_d^p(n_0) : p \in Z_{\geq 0}\}$.

Proof. Since $-1 < m \leq d-2$, by Definition 2.7,

$$\begin{aligned}
p <& \left\lceil \frac{dp + m + 1}{d} \right\rceil \\
=& \left\lceil \frac{h_d(n) - 1}{d} \right\rceil \\
\leq& \left\lceil \frac{h_d(n)}{d} \right\rceil \\
=& \left\lceil \frac{dp + m + 2}{d} \right\rceil \\
\leq& \left\lceil \frac{dp + d}{d} \right\rceil \\
=& p + 1.
\end{aligned}$$

Therefore,

$$\left\lceil \frac{h_d(n) - 1}{d} \right\rceil = \left\lceil \frac{h_d(n)}{d} \right\rceil = p + 1$$

and

$$f(h_d(n) - 1) = f(h_d(n)), \tag{2.26}$$

and hence, by Definition 2.7 and (2.20),

$$h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil = dp + m + 2 - (p+1) = (d-1)p + m + 1 = n + 1. \tag{2.27}$$

Therefore, we have (i).

From (2.26), (2.27), and (ii) of Lemma 2.2, we have

$$\mathcal{G}(h_d(n)) = \mathcal{G}(h_d(n) - \left\lceil \frac{h_d(n)}{d} \right\rceil - 1) = \mathcal{G}(n).$$

Therefore, we have (ii).
From the method that is the same as that used in the proof of (iii) of Theorem 2.5, we have (iii). ■

2.3. Case III: $f(m) = \lfloor \frac{m+1}{d} \rfloor$ for $d = \frac{st+1}{t}$ for natural number s, t.

We study the Maximum Nim with the rule function

$$f(m) = \left\lfloor \frac{m+1}{d} \right\rfloor \quad \text{for} \quad d = \frac{st+1}{t}, \text{ where } s, t \in N. \tag{2.28}$$

We use the floor function $\lfloor \ \rfloor$ instead of the ceiling function $\lceil \ \rceil$. We use the floor function here because it is required in Section 3.

In subsections 2.1 and 2.2, we have h_d such that

$$\mathcal{G}(h_d(n)) = \mathcal{G}(n) \text{ for } n \in Z_{\geq 0} \tag{2.29}$$

in Theorems 2.5 and 2.8, where the rule function is $f(m) = \lceil \frac{m}{d} \rceil$. If we use (2.28) as a rule function, h_d that satisfies (2.29) becomes quite complicated, even if we use $d = \frac{st+1}{t}$ for the natural number s, t.

Lemma 2.9. *The function $f(x)$ has the following properties.*
(i) For $k \in N$,

$$\begin{aligned} f(kst+k-2) < f(kst+k-1) = f(kst+k) = \cdots \\ = f(kst+k+s-1) = kt < kt+1 = f(kst+k+s). \end{aligned} \tag{2.30}$$

(ii) For $k \in Z_{\geq 0}$ and $u \in N$ such that $1 \leq u \leq t-1$,

$$\begin{aligned} f(kst+us+k-1) < & f(kst+us+k) = f(kst+us+k+1) = \cdots \\ = & f(kst+(u+1)s+k-1) = kt+u. \end{aligned} \tag{2.31}$$

Proof. (i) Since $d = \frac{st+1}{t} = s + \frac{1}{t}$, for $k \in N$,

$$ktd = kst + k, \tag{2.32}$$

$$(kt+1)d = kst + k + s + \frac{1}{t}. \tag{2.33}$$

and

$$(kt+2)d = kst + k + 2s + \frac{2}{t}. \tag{2.34}$$

Since

$$\begin{aligned} kst+k-1 < kst+k < kst+k+1 < \cdots \leq kst+k+s \\ < kst+k+s+\frac{1}{t} \leq kst+k+s+1 < kst+k+2s+\frac{2}{t}, \end{aligned}$$

by (2.32), (2.33), and (2.34), we have
$\lfloor \frac{kst+k-1}{d} \rfloor < \lfloor \frac{kst+k}{d} \rfloor = \cdots = \lfloor \frac{kst+k+s}{d} \rfloor = kt < kt+1 = \lfloor \frac{kst+k+s+1}{d} \rfloor$,
and hence, by (2.28), we have (2.30).
(ii) For $k \in Z_{\geq 0}$ and $u \in N$ such that $1 \leq u \leq t-1$,

$$(kt+u)d = kst + us + k + \frac{u}{t}, \tag{2.35}$$

$$(kt+u+1)d = kst+(u+1)s+k+\frac{u+1}{t}. \tag{2.36}$$

and

$$(kt+u+2)d = kst+(u+1)s+s+k+\frac{u+2}{t}. \tag{2.37}$$

$kst+us+k < kst+us+k+\frac{u}{t} < kst+us+k+1 < \cdots \leq kst+(u+1)s+k,$
$< kst+(u+1)s+k+\frac{u+1}{t} < kst+(u+1)s+s+k+\frac{u+2}{t}$, and hence, by (2.35), (2.36), and (2.37),

$$\left\lfloor \frac{kst+us+k}{d} \right\rfloor < \left\lfloor \frac{kst+us+k+1}{d} \right\rfloor = \cdots = \left\lfloor \frac{kst+(u+1)s+k}{d} \right\rfloor$$
$$= kt+u < kt+u+1 = \left\lfloor \frac{kst+(u+1)s+k+1}{d} \right\rfloor,$$

and thus, we have (2.31). ■

Lemma 2.10. *(i) For $k \in Z_{\geq 0}$ and $u \in N$ such that $1 \leq u \leq t$,*

$$\mathcal{G}(kst+k+us) = kt+u. \tag{2.38}$$

(ii) $kst+k+us = \min\{n : \mathcal{G}(n) = kt+u\}$.

Proof. (i) From (i) and (ii) of Lemma 2.9 and $(iii.1)$ of Lemma 2.2, we have (2.39) and (2.40): For $k \in N$

$$\mathcal{G}(kst+k-1) = kt \tag{2.39}$$

and $k \in Z_{\geq 0}$ and $u \in N$ such that $1 \leq u \leq t-1$,

$$\mathcal{G}(kst+k+us) = kt+u. \tag{2.40}$$

If we use $k+1$ instead of k in (2.39), for $k \in Z_{\geq 0}$, we have

$$\mathcal{G}(kst+k+st) = kt+t, \tag{2.41}$$

and hence, by (2.40) and (2.41), we have (2.38). (i) is direct from $(iii.2)$ of Lemma 2.2. ■

Lemma 2.11. *For $k, u \in N$ such that $1 \leq u \leq t-1$, we have*

$$\begin{aligned}
&\mathcal{G}(kst+k) = \mathcal{G}(k(s-1)t+k-1),\\
&\cdots, \mathcal{G}(kst+k+s-1) = \mathcal{G}(k(s-1)t+k+s-2) && (2.42)\\
&\mathcal{G}(kst+k+s+1) = \mathcal{G}(k(s-1)t+k+s-1)\\
&\cdots, \mathcal{G}(kst+k+2s-1) = \mathcal{G}(k(s-1)t+k+2s-3) && (2.43)\\
&\qquad\vdots\\
&\mathcal{G}(kst+k+us+1) = \mathcal{G}(k(s-1)t+k+us-u)\\
&\cdots, \mathcal{G}(kst+k+(u+1)s-1) = \mathcal{G}(k(s-1)t+k+(u+1)s-u-2) && (2.44)\\
&\qquad\vdots\\
&\mathcal{G}(kst+k+(t-1)s+1) = \mathcal{G}(k(s-1)t+k+(t-1)s-t+1)\\
&\cdots, \mathcal{G}(kst+k+ts-1) = \mathcal{G}(k(s-1)t+k+ts-t-1). && (2.45)
\end{aligned}$$

Proof. By (i) of Lemma 2.9 and (ii) of Lemma 2.2, $\mathcal{G}(kst+k) = \mathcal{G}(kst+k-f(kst+k)-1) = \mathcal{G}(kst+k-kt-1) = \mathcal{G}(k(s-1)t+k-1)$. Similarly, we have $\mathcal{G}(kst+k+s-1) = \mathcal{G}(kst+k+s-1-f(kst+k+s-1)-1) = \mathcal{G}(kst+k+s-1-kt-1) = \mathcal{G}(k(s-1)t+k+s-2)$, and thus, we have (2.42). Similarly by using (ii) of Lemma 2.9 and (ii) of Lemma 2.2, we obtain (2.44). For $u = 1, \cdots, t-1$, we obtain (2.43) and (2.45) from (2.44). ∎

Lemma 2.12. *Let $k \in N$.*
For $x = k(s-1)t+k-1, k(s-1)t+k, \cdots, k(s-1)t+k+s-2$,

$$\mathcal{G}(x) = \mathcal{G}(x+kt+1).$$

For $x = k(s-1)t+k+s-1, k(s-1)t+k, \cdots, k(s-1)t+k+2s-3$,

$$\mathcal{G}(x) = \mathcal{G}(x+kt+2).$$

$$\vdots$$

For $x = k(s-1)t+k+(t-1)s-t+1, k(s-1)t+k+(t-1)s-t+2, \cdots, k(s-1)t+k+ts-t-1$,

$$\mathcal{G}(x) = \mathcal{G}(x+kt+t).$$

Proof. Lemma 2.12 is direct from Lemma 2.11. ∎

By Lemma 2.12, the function that satisfies $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ for $n \in Z_{\geq 0}$ depends on the value n, and therefore, it is considerably complicated even if we use $d = \frac{st+1}{t}$ for natural number s, t. Therefore, in Subsection 2.4, we study the case when $d = \frac{t+1}{t}$ with natural number t to make the situation more simple.

2.4. Case IV: $f(m) = \lfloor \frac{m+1}{d} \rfloor$ for $d = \frac{t+1}{t}$ with natural number t.

Let $f(m) = \lfloor \frac{m+1}{d} \rfloor$ for $d = \frac{t+1}{t}$ with the natural number t. Our aim is to find a simple function that satisfies $\mathcal{G}(h_d(n)) = \mathcal{G}(n)$ for $n \in Z_{\geq 0}$.

Lemma 2.13. *For the Grundy number of the maximum Nim with the rule function $f(m) = \lfloor \frac{m+1}{d} \rfloor$ for $d = \frac{t+1}{t}$ with the natural number t, we have the following equations (2.46) and (2.47).*
(i)

$$\mathcal{G}(k(t+1)) = \mathcal{G}(k-1). \tag{2.46}$$

for any $k \in N$.
(ii)

$$\mathcal{G}(k(t+1)+u) = kt+u \tag{2.47}$$

for any $k \in Z_{\geq 0}$ and $u \in N$, such that $u \leq t$.
(iii) $k(t+1)+u = \min\{n : \mathcal{G}(n) = kt+u\}$.

Proof. When $s = 1$, (i), (ii) and (iii) are directly from (2.42), and (i) and (ii) of Lemma 2.10. ∎

Lemma 2.14. *Let $k \in Z_{\geq 0}$ and $u \in N$ such that $u \leq t$; let $h(x) = (t+1)(x+1)$. Then, we obtain*

$$\{h^p(k(t+1)+u) : p \in Z_{\geq 0}\} = \{x : \mathcal{G}(x) = kt+u\}$$

and

$$\{h^p(0) : p \in Z_{\geq 0}\} = \{x : \mathcal{G}(x) = 0\}.$$

Proof. Let $x_0 = k(t+1)+u$. By (i) and (ii) of Lemma 2.13, x_0 represents the smallest number such that $\mathcal{G}(x_0) = kt+u$. If $n = k-1$, then $k(t+1) = (t+1)(n+1) = h(n)$. Therefore, by (2.46), we have for any $n \in N$,

$$\mathcal{G}(n) = \mathcal{G}(h(n)), \tag{2.48}$$

and hence,

$$\{h^p(x_0) : p \in Z_{\geq 0}\} \subset \{x : \mathcal{G}(x) = kt+u\}.$$

Suppose that $x \in Z_{\geq 0}$ and $h^p(x_0) < x < h^{p+1}(x_0)$ for some $p \in Z_{\geq 0}$. Since $f(h^{p+1}(x_0)) = \lfloor \frac{h^{p+1}(x_0)+1}{d} \rfloor = \lfloor \frac{h(h^p(x_0))+1}{d} \rfloor = \lfloor \frac{(t+1)(h^p(x_0)+1)+1}{d} \rfloor = \lfloor \frac{t(t+1)(h^p(x_0)+1)+t}{t+1} \rfloor = t(h^p(x_0)+1)$, by the definition of *move*

$$\begin{aligned} &move(h^{p+1}(x_0)) \\ &= \{h^{p+1}(x_0)-1, \cdots, h^{p+1}(x_0) - f(h^{p+1}(x_0))\} \\ &= \{h^{p+1}(x_0)-1, \cdots, (t+1)(h^p(x_0)+1) - t(h^p(x_0)+1)\} \\ &= \{h^{p+1}(x_0)-1, \cdots, h^p(x_0)+1\}. \end{aligned} \tag{2.49}$$

Therefore, $x \in move(h^{p+1}(x_0))$, and by the definition of the Grundy number $\mathcal{G}(x) \neq \mathcal{G}(h^{p+1}(x_0) = kt+u$. Therefore, we have $\{h^p(x_0) : p \in Z_{\geq 0}\} = \{x : \mathcal{G}(x) = kt+u\}$.

When $x_0 = 0$, we have $\mathcal{G}(x_0) = 0$ by the definition of Grundy numbers. When $k = 1$, by (i) of Lemma 2.13 we have

$$\mathcal{G}(h(0)) = \mathcal{G}(t+1) = \mathcal{G}(k(t+1)) = \mathcal{G}(k-1) = \mathcal{G}(0).$$

Then, by (2.48) we have $\{h^p(0) : p \in Z_{\geq 0}\} \subset \{x : \mathcal{G}(x) = 0\}$. By a method similar to the one used in the previous part of this proof, we prove $\{h^p(0) : p \in Z_{\geq 0}\} = \{x : \mathcal{G}(x) = 0\}$. ■

So far, we studied the case that $f(m) = \lfloor \frac{m+1}{d} \rfloor$ for fractions $d = \frac{st+1}{t}$ and $d = \frac{t+1}{t}$ with natural numbers s, t. The authors could not find any meaningful formulas of Grundy numbers for other types of fractions.

3. Chocolate Bar Game and Maximum Nim

Let $d = \frac{t+1}{t}$ for a natural number t, and let $f(m) = \lfloor \frac{m+1}{d} \rfloor$. We study a chocolate bar game with a restriction on the size of the chocolate bar piece to be eaten, and we devise formulas for the winning position of the previous player. We use the results in Subsection 2.4.

We define chocolate bar games with the restriction using Figure 5.

Definition 3.1. (i) A chocolate bar is a rectangular array of squares.
(ii) We obtain the coordinate system shown in Figure 5, and we describe the position of this chocolate bar as $\{x, y\}$. Then, the size of the chocolate bar is calculated by $(x+1)(y+1)$, where $(x+1)$ and $(y+1)$ represent the width and length of the chocolate.
(iii) Two players take turns and break the chocolate bar along any one of the horizontal or vertical lines into two pieces and eat one of the pieces. Each player can eat less than or equal to $\lfloor \frac{s}{d} \rfloor$ when the size of the chocolate bar is s.
(iv) The player who is left with a 1×1 piece of chocolate, and hence cannot make another

move, loses the game. Therefore the 1×1 piece of chocolate with the coordinates $\{0, 0\}$ in Figure 6 is the terminal position of the game.

We need the following lemma to study this chocolate bar game.

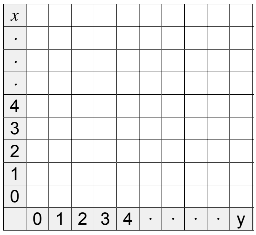

Figure 5. Rectangular chocolate bar with coordinates

Figure 6. The terminal position $\{0, 0\}$

Lemma 3.2. *For $p, q \in N$,*
$$\lfloor \frac{p}{d} \rfloor \times q \leq \lfloor \frac{pq}{d} \rfloor < (\lfloor \frac{p}{d} \rfloor + 1)q.$$

Proof. Let $p = ud + r$ with $u \in Z_{\geq 0}$ and $r \in R$ such that $0 \leq r < d$. Since
$$uq \leq \frac{p}{d} \times q,$$
we have
$$\lfloor \frac{p}{d} \rfloor \times q = uq \leq \lfloor \frac{pq}{d} \rfloor.$$
Since
$$\frac{pq}{d} = \frac{ud + r}{d} \times q < \frac{ud + d}{d} \times q = (u + 1)q = (\lfloor \frac{p}{d} \rfloor + 1)q,$$
we have
$$\lfloor \frac{pq}{d} \rfloor < (\lfloor \frac{p}{d} \rfloor + 1)q.$$
■

Next, we study *move*$(\{x, y\})$ for this chocolate bar game, and *move* represents the set of positions that can be reached by making one move from the position $\{x, y\}$.

Lemma 3.3. *For the $\{x, y\}$-chocolate bar in the game of Definition 3.1, move$(\{x, y\}) = \{\{x - t, y\} : t \leq f(x)\} \cup \{\{x, y - t\} : t \leq f(y)\}$.*

Proof. By (iii) of Definition 3.1, the chocolate can be consumed by breaking the chocolate bar along any one of the horizontal or vertical lines into two pieces and by eating one of the pieces. Therefore, players have only two options: reduce the x-coordinate and reduce the y-coordinate. Each player can eat less than or equal to $\lfloor \frac{s}{d} \rfloor$ when the size of the chocolate bar is s. The size of the chocolate bar with position $\{x, y\}$ is $(x + 1)(y + 1)$; hence, the player can eat less than or equal to
$$\lfloor \frac{(x + 1)(y + 1)}{d} \rfloor. \tag{3.1}$$

By Lemma 3.2,

$$\lfloor \frac{x+1}{d} \rfloor (y+1) \leq \lfloor \frac{(x+1)(y+1)}{d} \rfloor < (\lfloor \frac{x+1}{d} \rfloor + 1)(y+1).$$

Hence, by (3.1), the size of the biggest chocolate a player can eat by reducing the $x-$coordinate is $\lfloor \frac{x+1}{d} \rfloor (y+1)$. Therefore, a player obtains the chocolate bar of $\{x-t, y\}$ with $t \leq f(x) = \lfloor \frac{x+1}{d} \rfloor$ when he or she reduces the x-coordinate. Similarly, the player obtains the chocolate bar of $\{x, y-t\}$ with $t \leq f(y) = \lfloor \frac{y+1}{d} \rfloor$ when he or she reduces the y-coordinate. ∎

Example 3.4. When $f(m) = \lfloor \frac{m+1}{2} \rfloor$, the condition for the part to be eaten is equal to or less than half of the original size. Then, this problem is the same as the "Rectangle Game" presented at the International Olympiad in Informatics [10] and treated at [11]. Therefore, the chocolate bar game in Definition 3.1 is a generalization of the problem presented at the International Olympiad in Informatics.

Example 3.5. Let $d = 1.5$. Then, $f(4) = \lfloor \frac{5}{1.5} \rfloor = 3$ and $f(6) = \lfloor \frac{7}{1.5} \rfloor = 4$. $move(\{4,6\}) =$
$= \{\{4-t, 6\} : t \leq f(4) = 3\} \cup \{\{4, 6-t\} : t \leq f(6) = 4\}$
$= \{\{3,6\}, \{2,6\}, \{1,6\}, \{4,5\}, \{4,4\}, \{4,3\}, and \{4,2\}\}$; if you start with the chocolate bar in Figure 7, you get one of the chocolate bars in Figures 8, 9, 10, 11, 12, 13, and 14.

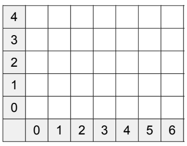

FIGURE 7. $\{4,6\}$

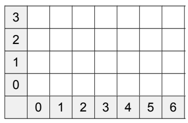

FIGURE 8. $\{3,6\}$

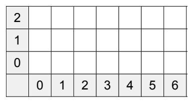

FIGURE 9. $\{2,6\}$

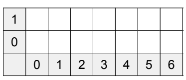

FIGURE 10. $\{1,6\}$

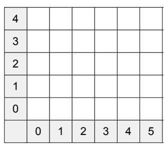

FIGURE 11. $\{4,5\}$

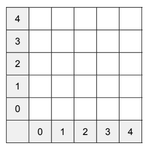

FIGURE 12. $\{4,4\}$

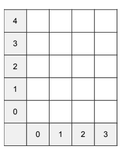

FIGURE 13. $\{4,3\}$

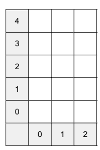

FIGURE 14. $\{4,2\}$

Theorem 3.6. *Let $h(x) = (t+1)(x+1)$. Then, the set of $\mathcal{P}$-positions of the game in Definition 3.1 is*

$$\{\{h^p(k(t+1)+u), h^q(k(t+1)+u)\} : p, q, k \in Z_{\geq 0} \text{ and } u \in N \text{ with } u \leq t\}$$
$$\cup \{\{h^p(0), h^q(0)\} : p, q \in Z_{\geq 0}\}.$$

Proof. From Lemma 3.3, the chocolate bar game in Definition 3.1 is a sum of two maximum Nim values with the rule function $f(m) = \lfloor \frac{m+1}{d} \rfloor$. Let $\mathcal{G}$ be the Grundy number of the Maximum Nim. Then, the Grundy number of the position $\{x, y\}$ of this chocolate bar game is $\mathcal{G}(x) \oplus \mathcal{G}(y)$. By Theorem 2.1, $\{x, y\}$ represents a $\mathcal{P}$-position if and only if $\mathcal{G}(x) = \mathcal{G}(y)$. Therefore, by Lemma 2.14, the set of $\mathcal{P}$-positions is

$$\begin{aligned}
&\{\{x, y\} : \mathcal{G}(x) = \mathcal{G}(y) = kt + u, k \in Z_{\geq 0} \text{ and } u \in N \text{ with } u \leq t\} \\
&\cup \{\{x, y\} : \mathcal{G}(x) = \mathcal{G}(y) = 0\} \\
=&\{\{h^p(k(t+1) + u), h^q(k(t+1) + u)\} : p, q, k \in Z_{\geq 0} \text{ and } u \in N \text{ with } u \leq t\} \\
&\cup \{\{h^p(0), h^q(0)\} : p, q \in Z_{\geq 0}\}.
\end{aligned}$$

■

Remark 3.7. In this section, we use only the result in Subsection 2.4. Hence, using the results of subsections 2.1, 2.2, and 2.3 for chocolate bar games is left as an open problem.

4. Chocolate Bars with Missing Pieces

We study the chocolate bar under the same rule as in Definition 3.1, except that the chocolate bar used in this game has one missing square, and d represents any real number such that $d > 1$. See Figure 15. When some parts of the chocolate bar are missing, the chocolate bar game becomes complicated but interesting.

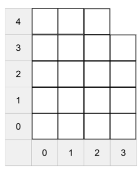

Figure 15. Chocolate bar with one missing square

Definition 4.1. (i) A chocolate bar is a rectangular array of squares with one missing square.
(ii) Two players take turns and break the chocolate bar along any one of the horizontal or vertical lines into two pieces and eat one of the pieces. When the size of the chocolate bar is s, each player can eat less than or equal to $\lfloor \frac{s}{d} \rfloor$.
(iii) We use the coordinate $\{x, y, z\}$, where $z = 1$ if there is one missing square and $z = 0$ if there is no missing square.

For the coordinates of chocolates in this subsection, see figures 16, 17, 18, 19, and 20.

Example 4.2. Let $d = 2$. Then, if you start with the chocolate in Figure 16, the size of the chocolate is $s = 5$. Next, you can eat less than or equal to $\lfloor \frac{5}{2} \rfloor = 2$. You can then obtain chocolates in Figures 17, 18, and 19, but not the chocolate in Figure 20 because you need to eat a piece of chocolate whose size is 3.

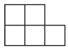

FIGURE 16. $\{1,2,1\}$

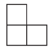

FIGURE 17. $\{1,1,1\}$

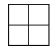

FIGURE 18. $\{1,1,0\}$

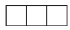

FIGURE 19. $\{0,2,0\}$

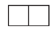

FIGURE 20. $\{0,1,0\}$

The following conjecture is based on the calculation by the computer algebra system Mathematica.

Conjecture 1. *Let $\mathcal{G}_d(x,y,z)$ be the Grundy number of chocolates in this subsection, where x,y represent the coordinates of the chocolates and $z=1$ if there is one missing square; $z=0$ if there is no missing square.*
Then, $\mathcal{G}_d(x,y,1)=\mathcal{G}_d(x,y,0)$ for any $x,y \in Z_{\geq 0}$ when $d=2$, and $\mathcal{G}_d(x,y,1) \neq \mathcal{G}_d(x,y,0)$ for some $x,y \in Z_{\geq 0}$ when $d \neq 2$.

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