Discrete and Computational Geometry, Graphs, and Games

Previous Player's Positions of Impartial Three-Dimensional Chocolate-Bar Games

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Abstract In this study, we investigate three-dimensional chocolate bar games, which are variants of the game of Chomp. A three-dimensional chocolate bar is a three-dimensional array of cubes in which a bitter cubic box is present in some part of the bar. Two players take turns and cut the bar horizontally or vertically along the grooves. The player who manages to leave the opponent with a single bitter block is the winner. We consider the $\mathcal{P}$-positions of this game, where the $\mathcal{P}$-positions are positions of the game from which the previous player (the player who will play after the next player) can force a win, as long as they play correctly at every stage. We present sufficient conditions for the case when the position (p, q, r) is a $\mathcal{P}$-position if and only if $(p-1) \oplus (q-1) \oplus (r-1)$, where p, q, and r are the length, height, and width of the chocolate bar, respectively.

1. Introduction

Chocolate bar games are variants of the Chomp game presented in [1]. A two-dimensional chocolate bar is a two-dimensional array of squares in which a bitter square printed in black is present in some part of the bar. See the chocolate bars in Figure 1.

A three-dimensional chocolate bar is a three-dimensional array of cubes in which a bitter cubic box printed in black is present in some parts of the bar. Figure 2 displays examples of three-dimensional chocolate bars. Games involving these chocolate bars may be defined as follows.

Definition 1.1. (i) Two-dimensional chocolate bar game: Each player in turn breaks the bar in a straight line along the grooves and eats the broken piece. The player who manages to leave the opponent with a single bitter block (black block) is the winner.

(ii) Three-dimensional chocolate game: The rules are the same as in (i), except that the chocolate is cut horizontally or vertically along the grooves. Examples of cutting three-dimensional chocolate bars are shown in Figure 3.
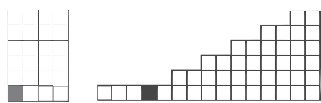

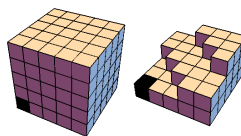FIGURE 1. Two-dimensional chocolate bars.

FIGURE 2. Three-dimensional chocolate bars.

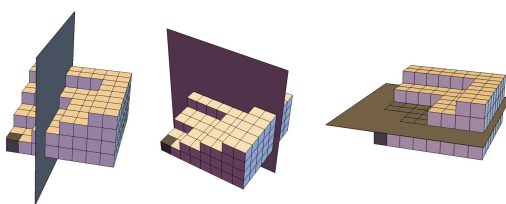**Example 1.2.** Three methods of cutting a three-dimensional chocolate bar.

FIGURE 3. (*i*) Vertical cut. (*ii*) Vertical cut. (*iii*) Horizontal cut.

For completeness, we briefly review some of the necessary concepts of combinatorial game theory; refer to [2] for greater detail. Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ be sets of non-negative integers and natural numbers, respectively.

Definition 1.3. Let x and y be non-negative integers. Then, there is $n \in \mathbb{Z}_{\geq 0}$ and $x_i, y_i \in \{0, 1\}$ for $i = 0, 1, \cdots, n$ such that $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$. We define nim-sum $x \oplus y$ as:

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i. \tag{1.1}$$

where $w_i = x_i + y_i \pmod 2$.

As chocolate bar games are impartial games without draws, only two outcome classes are possible.

Definition 1.4. (*a*) A position is called a $\mathcal{P}$-*position*, if it is a winning position for the previous player, as long as he/she plays correctly at every stage.
(*b*) A position is called an $\mathcal{N}$-*position*, if it is a winning position for the next player, as long as he/she plays correctly at every stage.

Definition 1.5. (i) For any position $\mathbf{p}$, there exists a set of positions that can be reached by making precisely one move from $\mathbf{p}$, which we will denote by $move(\mathbf{p})$.

(ii) The *minimum excluded value* (*mex*) of a set S of non-negative integers is the least non-negative integer that is not in S.

(iii) Each position $\mathbf{p}$ of an impartial game has an associated Grundy number, and we denote this as $\mathrm{G}(\mathbf{p})$.

The Grundy number is recursively defined by $\mathrm{G}(\mathbf{p}) = mex\{\mathrm{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

Theorem 1.6. *For any position* $\mathbf{p}$ *of the game,* $\mathrm{G}(\mathbf{p}) = 0$ *if and only if* $\mathbf{p}$ *is a* $\mathcal{P}$-*position.*

This is a well-known theorem reported by [3] and [4].

The original two-dimensional chocolate bar introduced by Robin [5] is the chocolate shown on the left-hand side in Figure 1. Because the horizontal and vertical grooves are independent, an $m \times n$ rectangular chocolate bar is equivalent to the game of *nim*, which includes heaps of $m - 1$ and $n - 1$ stones, respectively. Therefore, the chocolate 6×4 bar game shown on the left-hand side of Figure 1 is mathematically the same as *nim*, which includes heaps of 5 and 3 stones, respectively. It is well known that the Grundy number of *nim* with heaps of $m - 1$ stones and $n - 1$ stones is $(m - 1) \oplus (n - 1)$; therefore, the Grundy number of the $m \times n$ rectangular bar is $(m - 1) \oplus (n - 1)$. Robin [5] also presented a cubic chocolate bar, as shown on the left-hand side of Figure 2. It can be easily determined that this $5 \times 5 \times 5$ three-dimensional chocolate bar is mathematically the same as *nim* with heaps of 4, 4, and 4 stones, and the Grundy number of this cuboid bar is $4 \oplus 4 \oplus 4$. It is then natural to ask the following question.

Question 1. *What is the necessary and sufficient condition for a three-dimensional chocolate bar to fulfil the following condition* (a):

(a) *The Grundy number of the chocolate bar with the length p, height q, and width z is* $(p - 1) \oplus (q - 1) \oplus (r - 1)$.

Although the authors answered Question 1 for two-dimensional chocolate bars in [6] and for the three-dimensional case in [7], the results of these research are omitted here.

When the Grundy number of a chocolate bar with p, q, and z as the length, height, and width, is $(p - 1) \oplus (q - 1) \oplus (r - 1)$, according to Theorem 1.6 a position is a $\mathcal{P}$-position if and only if $(p - 1) \oplus (q - 1) \oplus (r - 1) = 0$ in this chocolate bar.

Therefore, it is natural to ask the following question.

Question 2. *Under what condition does a three-dimensional chocolate bar satisfy the following condition* (b):

(b) *The chocolate bar with the length p, height q, and width r is a $\mathcal{P}$-position if and only if* $(p - 1) \oplus (q - 1) \oplus (r - 1) = 0$?

Because we get Condition (b) from Condition (a) according to Theorem 1.6, Question 2 deals with the condition of the chocolate bar that is more general than the condition in Question 1. In the remainder of this paper, we present a sufficient condition for which Question 2 may be answered. Therefore, this study deals with the problem that is more general than the problem proved in [7].

Determining the necessary and sufficient conditions for this question is a very difficult unsolved problem considered by the authors. We suppose that the difficulty of presenting
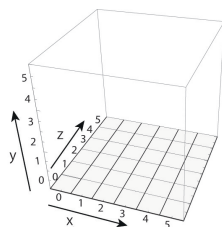
FIGURE 4. coordinate

the necessary and sufficient conditions arises from the fact that there are many kinds of sufficient conditions. For more information, see Theorems 2.11 in Section 2 and Conjecture 1 in Section 3.

We now define a three-dimensional chocolate bar.

Definition 1.7. Suppose that $f(u, v) \in \mathbb{Z}_{\geq 0}$ for $u, v \in \mathbb{Z}_{\geq 0}$. f is said to monotonically increase if $f(u, v) \leq f(x, z)$ for $x, z, u, v \in \mathbb{Z}_{\geq 0}$ with $u \leq x$ and $v \leq z$.

Definition 1.8. Let f be the monotonically increasing function in Definition 1.7.
Let $x, y, z \in \mathbb{Z}_{\geq 0}$ such that $y \leq f(x, z)$. The three-dimensional chocolate bar comprises a set of $1 \times 1 \times 1$ boxes. For $u, w \in \mathbb{Z}_{\geq 0}$ such that $u \leq x$ and $w \leq z$, the height of the column at position (u, w) is $\min(f(u, w), y) + 1$. There is a bitter box at position $(0, 0)$. We denote this chocolate bar as $CB(f, x, y, z)$. Note that $x + 1, y + 1$, and $z + 1$ are the length, height, and width of the bar, respectively.

Example 1.9. Here, we let $f(x, z) = \lfloor \frac{x+z}{3} \rfloor$, where $\lfloor \ \rfloor$ is the floor function, and we present several examples of $CB(f, x, y, z)$.
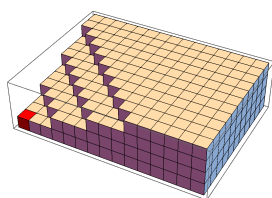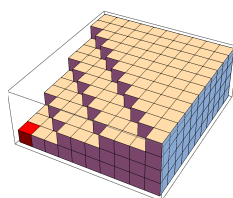

FIGURE 5. $CB(f, 14, 3, 10)$
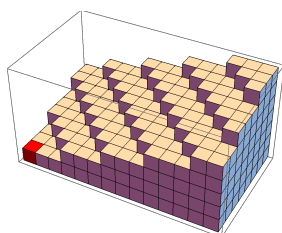

FIGURE 6. $CB(f, 9, 3, 10)$
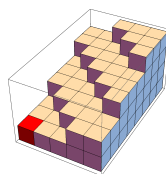

FIGURE 7. $CB(f, 13, 6, 7)$

FIGURE 8. $CB(f, 4, 3, 7)$

Next, we define $move_f((x, y, z))$ in Definition 1.10. $move_f((x, y, z))$ is a set that contains all of the positions that can be reached from position (x, y, z) in one step (directly).

Definition 1.10. For $x, y, z \in \mathbb{Z}_{\geq 0}$, we define

$$move_f((x, y, z)) = \{(u, \min(f(u, z), y), z) : u < x\} \cup \{(x, v, z) : v < y\}$$
$$\cup \{(x, \min(y, f(x, w)), w) : w < z\}, \text{ where } u, v, w \in \mathbb{Z}_{\geq 0}.$$

Remark 1.11. Definition 1.10 shows how to reduce the coordinates of the chocolate bar by cutting, and in Example 2.1, we provide concrete examples of reducing the coordinates.

2. When $f(x, z) = \lfloor \frac{x+z}{k} \rfloor$ for $k = 4m + 3$

Let $k = 4m+3$ for some $m \in \mathbb{Z}_{\geq 0}$. Let $x = \sum_{i=0}^{n} x_i 2^i$, $y = \sum_{i=0}^{n} y_i 2^i$ and $z = \sum_{i=0}^{n} z_i 2^i$ for some $n \in \mathbb{Z}_{\geq 0}$ and $x_i, y_i, z_i \in \{0, 1\}$. Throughout this section, we assume that

$$f(x, z) = \lfloor \frac{x + z}{k} \rfloor. \tag{2.1}$$

Before we prove several lemmas, we first consider the procedures provided in Example 2.1. Although this example is lengthy, the proofs of the lemmas are difficult to understand without first considering this example.

Example 2.1. Let $f(x, z) = \lfloor \frac{x+z}{3} \rfloor$.
(i) We begin with the chocolate bar shown in Figure 5. The coordinates of this chocolate bar are expressed in base 2 in Table 1. If the first coordinate $x = 14$ is reduced to $u = 9$ by cutting the chocolate bar $CB(f, 14, 3, 10)$ shown in Figure 5, by Definition 1.10 the second coordinate will be $\min(f(u, z), y) = \min(f(9, 10), 3) = \min(\lfloor \frac{19}{3} \rfloor, 3) = \min(6, 3) = 3$. Therefore, we can reduce $x = 14$ to $u = 9$ without affecting the second coordinate 3, which is the height of the chocolate bar, and we obtain the chocolate bar $CB(f, 9, 3, 10)$ shown in Figure 6 (i.e., $(9, 3, 10) \in move_f((14, 3, 10))$). The coordinates of this chocolate bar $CB(f, 9, 3, 10)$ are expressed in base 2 in Table 2.

	$x = 14$	$y = 3$	$z = 10$
$2^3 = 8$	$x_3 = 1$	$y_3 = 0$	$z_3 = 1$
$2^2 = 4$	$x_2 = 1$	$y_2 = 0$	$z_2 = 0$
$2^1 = 2$	$x_1 = 1$	$y_1 = 1$	$z_1 = 1$
$2^0 = 1$	$x_0 = 0$	$y_0 = 1$	$z_0 = 0$

TABLE 1. $CB(f, 14, 3, 10)$

	$u = 9$	$y = 3$	$z = 10$
$2^3 = 8$	$u_3 = 1$	$y_3 = 0$	$z_3 = 1$
$2^2 = 4$	$u_2 = 0$	$y_2 = 0$	$z_2 = 0$
$2^1 = 2$	$u_1 = 0$	$y_1 = 1$	$z_1 = 1$
$2^0 = 1$	$u_0 = 1$	$y_0 = 1$	$z_0 = 0$

TABLE 2. $CB(9, 3, 10)$

(ii) We begin with the chocolate bar in Figure 7. The coordinates of this chocolate bar are expressed in base 2 in Table 3. If the first coordinate $x = 13$ is reduced to $u = 4$ by cutting the chocolate bar $CB(f, 13, 6, 7)$ in Figure 7 by Definition 1.10, the second coordinate will be $\min(f(u, z), y) = \min(f(4, 7), 6) = \min(\lfloor \frac{11}{3} \rfloor, 6) = \min(3, 6) = 3$. Therefore, the second coordinate, 6, which is the height of the chocolate bar, will be reduced to 3. Then, we obtain the chocolate bar shown in Figure 8 (i.e., $(4, 3, 7) \in move_f((13, 6, 7))$). The coordinates of this chocolate bar $CB(f, 4, 3, 7)$ are expressed in base 2 in Table 4.

	$x = 13$	$y = 6$	$z = 7$
$2^3 = 8$	$x_3 = 1$	$y_3 = 0$	$z_3 = 0$
$2^2 = 4$	$x_2 = 1$	$y_2 = 1$	$z_2 = 1$
$2^1 = 2$	$x_1 = 0$	$y_1 = 1$	$z_1 = 1$
$2^0 = 1$	$x_0 = 1$	$y_0 = 0$	$z_0 = 1$

TABLE 3. $CB(f, 13, 6, 7)$

	$u = 4$	$v = 3$	$z = 7$
$2^3 = 8$	$u_3 = 0$	$v_3 = 0$	$z_3 = 0$
$2^2 = 4$	$u_2 = 1$	$v_2 = 0$	$z_2 = 1$
$2^1 = 2$	$u_1 = 0$	$v_1 = 1$	$z_1 = 1$
$2^0 = 1$	$u_0 = 0$	$v_0 = 1$	$z_0 = 1$

TABLE 4. $CB(f, 4, 3, 7)$

(iii) The procedures presented in (i) and (ii) are good examples of moving to a position whose nim-sum is 0 from a position whose nim-sum is not 0.

In (i), $14 \oplus 3 \oplus 10 = 7$, and suppose that the player wants to move to a position whose nim-sum is 0. First, let $u_3 = x_3 = 1$. Next, reduce $x_2 = 1$ to $u_2 = 0$. Note that

$$x = \sum_{i=0}^{3} x_i 2^i = 2^3 + 2^2 + 2 > 2^3 + 0 \times 2^2 + u_1 \times 2 + u_0 = \sum_{i=0}^{3} u_i 2^i = u \qquad (2.2)$$

regardless of the values of u_1, u_0. Then, reduce x_1 to $u_1 = 0$ and increase $x_0 = 0$ to $u_0 = 1$. Note that by considering (2.2), one can choose any value for u_1, u_0. Then, we obtain the position $(9, 3, 10)$ such that $9 \oplus 3 \oplus 10 = 0$.

In (ii), $13 \oplus 6 \oplus 7 = 12$, and suppose that it is desired to move to a position whose nim-sum is 0. First, we have to reduce $x_3 = 1$ to $u_3 = 0$. Because $(x_2, y_2, z_2) = (1, 1, 1)$ and $1 \oplus 1 \oplus 1 \neq 0$, we may let $(u_2, y_2, z_2) = (0, 1, 1)$ or $(u_2, v_2, z_2) = (1, 0, 1)$ by reducing y to v. Note that once we reduce x, we cannot reduce z.

If

$$(u_2, y_2, z_2) = (0, 1, 1), \qquad (2.3)$$

we have

$$f(u, z) \qquad (2.4)$$
$$= f(\sum_{i=0}^{3} u_i 2^i, \sum_{i=0}^{3} z_i 2^i)$$
$$= f(0 \times 2^3 + 0 \times 2^2 + u_1 2^1 + u_0 2^0, 7)$$
$$= \lfloor \frac{7 + u_1 2^1 + u_0 2^0}{3} \rfloor \leq \lfloor \frac{10}{3} \rfloor = 3,$$

regardless of the values of u_1, u_0. We then have $f(u, z) < 4 = y_2 2^2 \leq y$. Therefore, by Definition 1.10 (2.3) leads to a contradiction.

We should then let

$$(u_2, v_2, z_2) = (1, 0, 1), \qquad (2.5)$$

by simultaneously reducing x and y.

Next, we let $(u_1, v_1, z_1) = (0, 1, 1)$ or $(u_1, v_1, z_1) = (1, 0, 1)$.

If $(u_1, v_1, z_1) = (1, 0, 1)$, by (2.5)

$$f(u, z) \qquad (2.6)$$

$$= f(\sum_{i=0}^{3} u_i 2^i, \sum_{i=0}^{3} z_i 2^i)$$

$$= f(0 \times 2^3 + 2^2 + 2^1 + u_0 2^0, 7)$$

$$= \lfloor \frac{13 + u_0 2^0}{3} \rfloor \geq \lfloor \frac{13}{3} \rfloor$$

$$= 4 > 1 \geq \sum_{i=0}^{3} v_i 2^i = 0 \times 2^3 + 0 \times 2^2 + 0 \times 2 + v_0 = v,$$

and we obtain

$$f(u, z) > v. \qquad (2.7)$$

When we reduce x to u and y to v, by definition 1.10 we have

$$v = \min(f(u, z), y),$$

and this contradicts (2.7).

Therefore, let $(u_1, v_1, z_1) = (0, 1, 1)$. Using similar reasoning, we let $(u_0, v_0, z_0) = (0, 1, 1)$.

We then obtain the position $(4, 3, 7)$ such that $4 \oplus 3 \oplus 7 = 0$ and $v = 3 = \lfloor \frac{4+7}{3} \rfloor = f(u, z)$.

We define

$$S_t = \sum_{i=n-t}^{n} (x_i + z_i - ky_i)2^i, \qquad (2.8)$$

for $t = 0, 1, \cdots, n$.

Lemma 2.2. *We have the following relationships between $f(x, z)$ and S_n.*
(a)

$$y = f(x, z)$$

if and only if $0 \leq S_n < k$.
(b)

$$y > f(x, z)$$

if and only if $S_n < 0$.
(c)

$$y < f(x, z)$$

if and only if $S_n \geq k$.

Proof. First, note that $S_n = \sum_{i=0}^{n} (x_i + z_i - ky_i)2^i = x + z - ky$.
(a)

$$y = f(x, z) = \lfloor \frac{x + z}{k} \rfloor$$

if and only if $y \leq \frac{x+z}{k} < y+1$ if and only if $0 \leq S_n = x + z - ky < k$.

(b)

$$y > f(x,z) = \lfloor \frac{x+z}{k} \rfloor$$

if and only if $\frac{x+z}{k} < y$, which occurs if and only if $S_n = x + z - ky < 0$.

We then obtain (c) via (a) and (b). ∎

Lemma 2.3. *Let* $t \in \mathbb{Z}_{\geq 0}$. *Suppose that for* $i = n, n-1, \cdots, n-t$

$$x_i \oplus y_i \oplus z_i = 0. \tag{2.9}$$

There then exists an even number a *such that*

$$S_t = a2^{n-t}. \tag{2.10}$$

Proof. Because k is odd, by (2.9) $x_i + z_i - ky_i$ is even for $i = n, n-1, \cdots, n-t$, and therefore we have (2.10). ∎

Lemma 2.4. *Let* $t \in \mathbb{Z}_{\geq 0}$. *Suppose that for* $i = n, n-1, \cdots, n-t$

$$x_i \oplus y_i \oplus z_i = 0 \tag{2.11}$$

and

$$S_t < 0. \tag{2.12}$$

Then, for any natural number j *such that* $t < j \leq n$,

$$S_j < 0.$$

Proof. By Lemma 2.3, (2.11), and (2.12),

$$S_t = a2^{n-t} \tag{2.13}$$

for some even number a such that $a \leq -2$. Then, by (2.13), for any natural number j such that $j > t$ the following holds:

$$\begin{aligned}
S_j =& S_t + \sum_{i=n-j}^{n-t-1} (x_i + z_i - ky_i)2^i \\
\leq& S_t + 2 \times \sum_{i=0}^{n-t-1} 2^i \\
\leq& (-2)2^{n-t} + 2 \times (2^{n-t} - 1) = -2 < 0.
\end{aligned}$$

∎

Lemma 2.5. *Let* $t \in \mathbb{Z}_{\geq 0}$. *Suppose that for* $i = n, n-1, \cdots, n-t$

$$x_i \oplus y_i \oplus z_i = 0 \tag{2.14}$$

and

$$y \leq f(x,z). \tag{2.15}$$

Then,

$$S_t \geq 0. \tag{2.16}$$

Proof. If
$$S_t < 0, \tag{2.17}$$
by (2.14) and Lemma 2.4, we have
$$S_n < 0.$$
Then by (*b*) of Lemma 2.2, we have
$$y > f(x, z),$$
and this contradicts (2.15). Therefore, (2.17) is not true, and we have (2.16). ∎

Lemma 2.6. *Let $t \in \mathbb{Z}_{\geq 0}$. If*
$$S_t \geq k2^{n-t}, \tag{2.18}$$
then, for any natural number j such that $t < j \leq n$,
$$S_j \geq k2^{n-j}. \tag{2.19}$$

Proof. S_j will be smallest when $(x_i, y_i, z_i) = (0, 1, 0)$ for $i = n - t - 1, n - t - 2, \cdots, n - j$. Therefore, it is sufficient to prove (2.19) for this case. By (2.18), for any natural number j such that $j > t$,

$$
\begin{aligned}
S_j &= S_t + \sum_{i=n-j}^{n-t-1} (x_i + z_i - ky_i)2^i \\
&\geq S_t - k(2^{n-t-1} + 2^{n-t-2} + \cdots + 2^{n-j}) \\
&\geq k2^{n-t} - k(2^{n-t} - 2^{n-j}) = k2^{n-j}.
\end{aligned}
$$

∎

Lemma 2.7. *Let $t \in \mathbb{Z}_{\geq 0}$. Suppose that*
$$0 \leq S_t \leq 2m \times 2^{n-t}.$$
  Then, we have the following cases (a) and (b).
(a) *If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 1, 0)$ or $(0, 1, 1)$, then*
$$S_{t+1} < 0.$$
(b) *If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1)$ or $(0, 0, 0)$, then*
$$0 \leq S_{t+1} < k \times 2^{n-t-1}.$$

Proof. (a) If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 1, 0)$ or $(0, 1, 1)$,

$$
\begin{aligned}
S_{t+1} &= S_t + 2^{n-t-1} - k \times 2^{n-t-1} \\
&\leq 2m \times 2^{n-t} + 2^{n-t-1} - (4m+3)2^{n-t-1} \\
&= -2 \times 2^{n-t-1} < 0.
\end{aligned}
$$

  (b) If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (0, 0, 0)$,

$$
\begin{aligned}
0 &\leq S_{t+1} \\
&= S_t \leq 4m \times 2^{n-t-1} \\
&< k \times 2^{n-t-1}.
\end{aligned}
$$

If

$$(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1), \tag{2.20}$$

$$
\begin{aligned}
0 \le S_{t+1} \\
= S_t + 2 \times 2^{n-t-1} \\
\le (4m+2)2^{n-t-1} \\
< k \times 2^{n-t-1}.
\end{aligned}
$$

∎

Lemma 2.8. *Let* $t \in \mathbb{Z}_{\ge 0}$. *Suppose that*

$$(2m+2) \times 2^{n-t} \le S_t < k \times 2^{n-t} \tag{2.21}$$

and

$$x_i \oplus y_i \oplus z_i = 0 \tag{2.22}$$

for $i = n, n-1, \cdots, n-t$. *Then, we have the following cases* (*a*) *and* (*b*).
(*a*) *If* $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 1, 0)$ *or* $(0, 1, 1)$, *then*,

$$0 \le S_{t+1} < k \times 2^{n-t-1}.$$

(*b*) *If* $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1)$ *or* $(0, 0, 0)$, *then*,

$$S_{t+1} \ge k \times 2^{n-t-1}.$$

Proof. (*a*) If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 1, 0)$ or $(0, 1, 1)$, then

$$
\begin{aligned}
S_{t+1} &= S_t + 2^{n-t-1} - k2^{n-t-1} \\
&= S_t - (4m+2)2^{n-t-1}.
\end{aligned}
$$

By (2.21)

$$
\begin{aligned}
(2m+2) &\times 2^{n-t} - (4m+2)2^{n-t-1} \\
&\le S_{t+1} = S_t - (4m+2)2^{n-t-1} \\
&< (4m+3)2^{n-t} - (4m+2)2^{n-t-1}.
\end{aligned}
$$

Therefore,

$$2 \times 2^{n-t-1} \le S_{t+1} < (4m+4)2^{n-t-1}. \tag{2.23}$$

By (2.22) and Lemma 2.3, $S_{t+1} = a2^{n-t-1}$ for some even integer a, and therefore by (2.23) we have

$$0 < S_{t+1} \le (4m+2)2^{n-t-1} < k \times 2^{n-t-1}.$$

(*b*) If $(x_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1)$ or $(0, 0, 0)$, then by (2.21)

$$k \times 2^{n-t-1} < (4m+4)2^{n-t-1} \le S_t \le S_{t+1}.$$

∎

Lemma 2.9. *We assume that*

$$x \oplus y \oplus z \neq 0 \tag{2.24}$$

and

$$y \leq f(x, z). \tag{2.25}$$

Then, at least one of the following statements is true:
(1) $u \oplus y \oplus z = 0$ for some $u \in \mathbb{Z}_{\geq 0}$ such that $u < x$ and $y \leq f(u, z)$;
(2) $u \oplus v \oplus z = 0$ for some $u, v \in \mathbb{Z}_{\geq 0}$ such that $u < x, v < y$ and $v = f(u, z)$;
(3) $x \oplus v \oplus z = 0$ for some $v \in \mathbb{Z}_{\geq 0}$ such that $v < y$;
(4) $x \oplus y \oplus w = 0$ for some $w \in \mathbb{Z}_{\geq 0}$ such that $w < z$ and $y \leq f(x, w)$;
(5) $x \oplus v \oplus w = 0$ for some $v, w \in \mathbb{Z}_{\geq 0}$ such that $v < y, w < z$ and $v = f(x, w)$.

Proof. Let $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$, and $z = \sum_{i=0}^{n} z_i 2^i$.

If $n = 0$, we have $x, z \leq 1$, and $y \leq f(x, z) = \lfloor \frac{x+z}{k} \rfloor = 0$. Then, by (2.24), we have $(x, y, z) = (1, 0, 0)$ or $(0, 0, 1)$. In this case we obtain (1) for $(u, y, z) = (0, 0, 0)$ or (4) for $(x, y, w) = (0, 0, 0)$ by reducing $x = 1$ to $u = 0$ or reducing $z = 1$ to $w = 0$.

Next, we assume that $n \geq 1$ and that there exists a non-negative integer t such that

$$x_i \oplus y_i \oplus z_i = 0 \tag{2.26}$$

for $i = n, n - 1, ..., n - t + 1$ and

$$x_{n-t} \oplus y_{n-t} \oplus z_{n-t} \neq 0. \tag{2.27}$$

Let $T_h = \sum_{i=n-h}^{n} (x_i + z_i - k y_i) 2^i$ for $h = 0, 1, \cdots, t-1$, thereafter, we may define T_h for $h = t, t+1, \cdots, n$.

By (2.26), (2.25), and Lemma 2.5, we have

$$T_{t-1} \geq 0. \tag{2.28}$$

We then have three cases.

Case (1) Suppose that $(x_{n-t}, y_{n-t}, z_{n-t}) = (1, 0, 0)$. We reduce x to u and for $i = 0, 1, \cdots, t-1$ let

$$u_{n-i} = x_{n-i},$$

and we define u_{n-i} for $i = t, t+1, \cdots, n$ using an inductive method with the following steps $[I]$ and $[II]$.
Step $[I]$ Let $u_{n-t} = 0$. Then,

$$u = \sum_{i=0}^{n} u_i 2^i < \sum_{i=0}^{n} x_i 2^i = x.$$

Because $u_{n-t} = 0$ and $y_{n-t} = z_{n-t} = 0$, by (2.28) we have

$$T_t = T_{t-1} + (u_{n-t} + z_{n-t} - k y_{n-t}) 2^{n-t} = T_{t-1} \geq 0.$$

We then consider two subcases (1.1) and (1.2) according to the value of T_t.
Subcase (1.1) We suppose that

$$0 \leq T_t \leq 2m \times 2^{n-t}. \tag{2.29}$$

We then have two subsubcases for two possible values of z_{n-t-1}.

Subsubcase (1.1.1) Suppose that $z_{n-t-1} = 0$. We have two subsubsubcases for two possible values of y_{n-t-1}.

Subsubsubcase (1.1.1.1) If $y_{n-t-1} = 0$, let

$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (0, 0, 0)$$

and

$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - ky_{n-t-1})2^{n-t-1}.$$

Then, by Lemma 2.7 and (2.29)

$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.30}$$

Then, we begin Step $[II]$ with (2.30) while knowing that y has not been reduced.

Subsubsubcase (1.1.1.2) If $y_{n-t-1} = 1$, let $v_{n-t-1} = 0 < y_{n-t-1}$. Then, we have

$$\sum_{i=0}^{n} v_i 2^i < \sum_{i=0}^{n} y_i 2^i$$

for any values of v_i for $i = 0, 1, \cdots, n-t-1$. In this subsubsubcase, we reduce y to v by reducing x to u. For a concrete example of reducing y to v by reducing x to u, see (ii) and (iii) in Example 2.1.

Let

$$(u_{n-t-1}, v_{n-t-1}, z_{n-t-1}) = (0, 0, 0)$$

and

$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - kv_{n-t-1})2^{n-t-1},$$

then by Lemma 2.7 and (2.29)

$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.31}$$

Then, we begin Step $[II]$ with (2.31) while knowing that we have reduced y to v.

Subsubcase (1.1.2) Suppose that $z_{n-t-1} = 1$. We have two subsubsubcases for two possible values of y_{n-t-1}.

Subsubsubcase (1.1.2.1) If $y_{n-t-1} = 0$, let

$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1)$$

and

$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - ky_{n-t-1})2^{n-t-1},$$

then, by Lemma 2.7 and (2.29)

$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.32}$$

Then, we begin Step $[II]$ with (2.32) while knowing that y has not been reduced.

Subsubsubcase (1.1.2.2) If $y_{n-t-1} = 1$, let $v_{n-t-1} = 0 < y_{n-t-1}$. Then, we have

$$v = \sum_{i=0}^{n} v_i 2^i < \sum_{i=0}^{n} y_i 2^i = y$$

for any values of v_i for $i = 0, 1, \cdots, n-t-1$. In this subsubsubcase, we reduce y to v by reducing x to u. For a concrete example of reducing y to v by reducing x to u, see (ii) and (iii) in Example 2.1.

Then, let
$$(u_{n-t-1}, v_{n-t-1}, z_{n-t-1}) = (1, 0, 1)$$
and
$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - kv_{n-t-1})2^{n-t-1},$$
then, by Lemma 2.7 and (2.29)
$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.33}$$
Then, we begin Step $[II]$ with (2.33) while knowing that we have reduced y to v.
Subcase (1.2) We suppose that
$$(2m + 2)2^{n-t} \leq T_t < k2^{n-t}. \tag{2.34}$$
We have two subsubcases for two possible values of z_{n-t-1}.
Subsubcase (1.2.1) Suppose that $z_{n-t-1} = 0$. We have two subsubsubcases for two possible values of y_{n-t-1}.
Subsubsubcase (1.2.1.1) If $y_{n-t-1} = 1$, let
$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 1, 0)$$
and
$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - ky_{n-t-1})2^{n-t-1},$$
then, by Lemma 2.8 and (2.34)
$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.35}$$
Then, we begin Step $[II]$ with (2.35) while knowing that y has not been reduced.
Subsubsubcase (1.2.1.2) If $y_{n-t-1} = 0$, let
$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (0, 0, 0).$$
By Lemma 2.8 and (2.34)
$$T_{t+1} \geq k2^{n-t-1}. \tag{2.36}$$
Then, we begin Step $[II]$ with (2.36) while knowing that y has not been reduced.
Subsubcase (1.2.2) Suppose that $z_{n-t-1} = 1$. We have two subsubsubcases for two possible values of y_{n-t-1}.

Subsubsubcase (1.2.2.1) If $y_{n-t-1} = 1$, let
$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (0, 1, 1).$$
and
$$T_{t+1} = T_t + (u_{n-t-1} + z_{n-t-1} - ky_{n-t-1})2^{n-t-1},$$
then, by Lemma 2.8 and (2.34)
$$0 \leq T_{t+1} < k2^{n-t-1}. \tag{2.37}$$
Then, we begin Step $[II]$ with (2.37) and the fact that y has not been reduced.
Subsubsubcase (1.2.2.2) If $y_{n-t-1} = 0$, let
$$(u_{n-t-1}, y_{n-t-1}, z_{n-t-1}) = (1, 0, 1).$$

By Lemma 2.8 and (2.34)

$$T_{t+1} \geq k2^{n-t-1}. \tag{2.38}$$

Then, we begin Step $[II]$ with (2.38) while knowing that y has not been reduced.

Case (2) We suppose that $(x_{n-t}, y_{n-t}, z_{n-t}) = (0, 0, 1)$. Then, we can use the same method used for Case (1).

Case (3) We suppose that $y_{n-t} = 1$. Let $v_{n-t} = 0 < y_{n-t}$ and $v_i = x_i + z_i \pmod 2$ for $i = n-t-1, \cdots, 0$. Then, we have $x \oplus v \oplus z = 0$ and $v < y \leq f(x, z)$, and we have (3) of this lemma. In this case, we do not need Step $[II]$

Step $[II]$. We have two cases.

Case (1) This is a sequel to Case (1) of Step $[I]$. Here, the procedure consists of three subcases.

Subcase (1.1) Suppose that $T_{t+1} \geq k2^{n-t-1}$. In this case, y is not reduced to v in the last procedure, i.e., Step $[I]$. By Lemma 2.6, we let $u_i = y_i + z_i \pmod 2$ for $i = n-t-2, \cdots, 0$ without affecting the values of y_i for $i = n-t-2, \cdots, 0$ and we obtain $T_n \geq k$. According to lemma 2.2, we obtain $y < f(u, z)$. Therefore, we have (1) for this lemma.

Subcase (1.2) Suppose that $0 \leq T_{t+1} < k2^{n-t-1}$ and y was reduced to v in Step $[I]$. Then, we choose the values of u_{n-i}, v_{n-i} for $i = t+2, t+3, \cdots, n$ such that $0 \leq T_i < k2^{n-i}$ by the following (a) or (b).

(a) For $i \geq t+1$, if $0 \leq T_i < 2m \times 2^{n-i}$, then we let $(u_{n-i-1}, v_{n-i-1}, z_{n-i-1}) = (0, 0, 0)$ or $(1, 0, 1)$ when $z_{n-i-1} = 0$ or $z_{n-i-1} = 1$, respectively. Then, by Lemma 2.7, we have $0 \leq T_{i+1} < k2^{n-i-1}$.

(b) For $i \geq t+1$, if $T_i \geq (2m+2) \times 2^{n-i}$, then we let $(u_{n-i-1}, v_{n-i-1}, z_{n-i-1}) = (1, 1, 0)$ or $(0, 1, 1)$ when $z_{n-i-1} = 0$ or $z_{n-i-1} = 1$, respectively. Then, by Lemma 2.8, we have $0 \leq T_{i+1} < k2^{n-i-1}$.

Therefore, for $i = t+2, \cdots$, we have $0 \leq T_i < k2^{n-i}$, and finally we have $0 \leq T_n < k2^{n-n} = k$. We then have $v = f(u, z)$ and $u \oplus v \oplus z = 0$; therefore, we have (2) of this lemma.

Subcase (1.3) Suppose that $0 \leq T_{t+1} < k2^{n-t-1}$ and y was not reduced to v during the last procedure. In this case, we use the same method as in step $[I]$. If the inductive step finishes in this subcase, we obtain $y \leq f(u, z)$.

Case (2) This is a sequel to Case (2) of Step $[I]$. Then, we can use the same method used for Case (1) of Step $[I]$. ∎

Lemma 2.10. *We assume that $x \oplus y \oplus z = 0$ and*

$$y \leq f(x, z). \tag{2.39}$$

Then, the following hold:

(1) $u \oplus y \oplus z \neq 0$ for any $u \in \mathbb{Z}_{\geq 0}$ such that $u < x$ and $y \leq f(u, z)$;

(2) $u \oplus v \oplus z \neq 0$ for any $u, v \in \mathbb{Z}_{\geq 0}$ such that $u < x, v < y$, and $v = f(u, z)$;

(3) $x \oplus v \oplus z \neq 0$ for any $v \in \mathbb{Z}_{\geq 0}$ such that $v < y$;

(4) $x \oplus y \oplus w \neq 0$ for any $w \in \mathbb{Z}_{\geq 0}$ such that $w < z$ and $y \leq f(x, w)$;

(5) $x \oplus v \oplus w \neq 0$ for any $v, w \in \mathbb{Z}_{\geq 0}$ such that $v < y, w < z$, and $v = f(x, w)$.

Proof. If $x \oplus y \oplus z = 0$, a positive value of the nim-sum is obtained by changing the value of one of x, y, z. Therefore, we have (1), (3), and (4).

Next, we prove (2). The only way to have $u \oplus v \oplus z = 0$ for some $u, v \in \mathbb{Z}_{\geq 0}$ such that $u < x, v < y$ and

$$v = f(u, z) \tag{2.40}$$

is to reduce $(x_{n-t}, y_{n-t}, z_{n-t}) = (1, 1, 0)$ to $(u_{n-t}, v_{n-t}, z_{n-t}) = (0, 0, 0)$ for $t \in \mathbb{Z}_{\geq 0}$, such that $0 \leq t \leq n$, $u_i = x_i$ for $i = n, n-1, \cdots, n-t+1$ and $v_i = y_i$ for $i = n, n-1, \cdots, n-t+1$.

We consider two cases.

Case (1) Suppose that $0 \leq S_{t-1} \leq 2m \times 2^{n-t+1}$. Then, for $(x_{n-t}, y_{n-t}, z_{n-t}) = (1, 1, 0)$ by Lemma 2.7, we have $S_{t+1} < 0$. Then, by Lemmas 2.4 and 2.2, we have $y > f(x, z)$. This contradicts (2.39).

Case (2) Suppose that $S_{t-1} \geq (2m + 2) \times 2^{n-t+1}$. For $(u_{n-t}, v_{n-t}, z_{n-t}) = (0, 0, 0)$ by Lemma 2.8, $S_t \geq k \times 2^{n-t}$; therefore, by Lemma 2.6, $S_n \geq k$. Using Lemma 2.2, we then have $v < f(u, z)$. This contradicts (2.40).

Similarly, we can prove (5). ∎

Theorem 2.11. *Let* $f(x, z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 4m+3$. *Then, the chocolate bar* $CB(f, x, y, z)$ *is a* $\mathcal{P}$-*position if and only if*

$$x \oplus y \oplus z = 0. \tag{2.41}$$

Proof. Let $A_k = \{(x, y, z) : x \oplus y \oplus z = 0\}$ and $B_k = \{(x, y, z) : x \oplus y \oplus z \neq 0\}$. If we begin the game with a position $(x, y, z) \in A_k$, then using Theorem 2.10, any option leads to a position $(p, q, r) \in B_k$. From this position (p, q, r) by Theorem 2.9, our opponent can choose an appropriate option that leads to a position in A_k. Note that any option reduces some of the numbers in the coordinates. In this way, our opponent can always reach a position in A_k, and finally, they win by reaching $(0, 0, 0) \in A_k$. Therefore, A_k is the set of $\mathcal{P}$-positions.

If we begin the game with a position $(x, y, z) \in B_k$, then by Theorem 2.9, we can choose an appropriate option that leads to a position (p, q, r) in A_k. From (p, q, r), any option chosen by our opponent leads to a position in B_k. In this way, we win the game by reaching $(0, 0, 0)$. Therefore, B_k is the set of $\mathcal{N}$-positions. ∎

3. Other Cases

The result shown in Section 2 depends on the assumption that $k = 4m + 3$, but it seems that a similar result can be proven for an even number k with a restriction on the size of x, z.

The authors discovered the following conjecture via calculations using the computer algebra system Mathematica, but they have not managed to prove it.

Conjecture 1. *Let* $f(x, z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 2^{a+2}m + 2^{a+1}$ *and* $x, z \leq (2^{2a+2} - 2^{a+1})m + 2^{2a+1} - 1$, *where* $a, m \in \mathbb{Z}_{\geq 0}$. *Then, the chocolate bar* $CB(f, x, y, z)$ *is a* $\mathcal{P}$-*position if and only if*

$$x \oplus y \oplus z = 0.$$

Remark 3.1. If we compare Theorem 2.11 and Conjecture 1, it seems very difficult to obtain the necessary and sufficient condition for Question 2.

The authors also have the following conjecture that also may be derived via calculations using the computer algebra system Mathematica.

Conjecture 2. *Let $f(x, z) = \lfloor \frac{x+z}{k} \rfloor$ for $k = 4m+1$. Then, the chocolate bar $CB(f, x, y, z)$ is a $\mathcal{P}$-position if and only if*

$$(x + 1) \oplus y \oplus (z + 1) = 0.$$

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REFERENCES

[1] D. Gale, A curious Nim-type game, Amer. Math. Monthly 81 (1974) 876–879.

[2] M.H. Albert, R.J. Nowakowski, D. Wolfe, Lessons in Play, Second Edition, A K Peters/CRC Press, Natick, MA., United States, 2019.

[3] R.P. Sprague, Über mathematische Kampfspiele, Tohoku Mathematical Journal 41 (1935–36) 438–444.

[4] P.M. Grundy, Mathematics and games, Eureka 2 (1939) 6–8.

[5] A.C. Robin, A poisoned chocolate problem, problem corner, The Mathematical Gazette 73 (466) (1989) 341–343.

[6] S. Nakamura, R. Miyadera, Y. Nakaya, Impartial chocolate bar games, Grundy Numbers of Impartial Chocolate Bar Games, Integers 20 (2020) #G1.

[7] R. Miyadera, Y. Nakaya, Grundy numbers of impartial three-dimensional chocolate-bar games, Integers 21B (2021) #A19.