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Linear time-varying systems in Hilbert spaces: Exact controllability implies complete stabilizability

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Abstract: This paper deals with the problem of controllability and stabilizability of linear time-varying control systems in Hilbert spaces. We prove that any globally null-controllable system is completely stabilizable and conversely, under some additional conditions the complete stabilizability implies the global null-controllability. The obtained result extends existing results in the literature to infinite-dimensional and time-varying control systems.

Key words. Controllability, stabilizability, time-varying system, Hilbert space, Riccati equation.

1 Introduction

Consider a linear time-varying control system of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) \in X$ is the state, $u(t) \in U$ is the control; X and U are real Hilbert spaces of the states and the control, respectively; $A(t) : X \to X, B(t) : U \to X-$ are given linear operator functions. The problem of controllability and stabilizability for linear control systems has received a considerable amount of interest in the past decades, see; e.g. [8, 10, 14, 16] and the references therein. This problem regarding as an extension of the classical Kalman result [5] on controllability and stability of linear control systems is to find an admissible control $u(t)$ such that the corresponding solution $x(t)$ of the system has desired properties. Depending on the properties involved one defines various qualitative problems. For example, the null-controllability problem concerns the question of finding an admissible control

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$u(t)$ which steers an arbitrary state x_0 of system (1) into 0; stabilizability problem is to find a control $u(t) = K(t)x(t)$ such that the zero solution of the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t), \quad t \geq 0$$

is asymptotically stable in the Lyapunov sense. In this case one says that the system is stabilizable by the control $u(t) = K(t)x(t)$ and it is called a stabilizing feedback control of the system. Various stabilizability concepts can be adapted to investigate the stability property of control systems [3, 9, 13]. One of the extended stability properties is the concept of the complete stabilizability, originally introduced by Wonham [13], which relates to a strong exponential stability of the system. Namely, control system (1) is completely stabilizable if for every number $\delta > 0$, there exists a feedback control $u(t) = K(t)x(t)$ such that the solution $x(t, x_0)$ of the closed-loop system satisfies the condition

$$\exists N > 0 : \quad \|x(t, x_0)\| \leq Ne^{-\delta t}\|x_0\|, \quad \forall t \geq 0.$$

This means that for every positive number $\delta > 0$, the system zero-input response of the closed-loop system decays faster than $e^{-\delta t}$. In other words, for any given in advance the decay rate $\delta > 0$, the system can be $\delta-$exponentially stabilizable. Such definition may arise because of controlling of the speed of the real models. It is well known that if a finite-dimensional time-invariant control system is globally null-controllable in finite time then it is stabilizable, but the converse is not true as shown by Kalman [5], Wonham [14]. However, if the system is completely stabilizable, then it is globally null-controllable in finite time, see; e.g. Wonham [13]. In the infinite-dimensional control theory, investigations of controllability and stabilizability are more complicated and require more sophisticated techniques. The difficulties increase to the same extent as passing from time-invariant to time-varying systems. Some extensions have been developed by Slemrod [12], Zabczyk [15] for time-invariant control systems in Hilbert spaces. For time-varying control systems in finite-dimensional spaces, using Kalman's decomposition method, Ikeda et al. [4] proved that the system is completely stabilizable if it is uniformly globally null-controllable and Phat and Ha [11] extends some results of [4] to time-varying control systems.

In this paper, we develop the result of Phat and Ha [11] on the relationship between the exact controllability and complete stabilizability for linear time-varying control systems in Hilbert spaces. We show that the system is completely stabilizable if it is globally null-controllable in finite time, and conversely under some additional growth condition on the evolution operator of the system, the system is globally null-controllable in finite time if it is completely stabilizable. The result of the paper can be considered as further extensions of Wonham [12], Ikeda et al. [4], Phat and Ha [11] to infinite-dimensional systems, of Slemrod [11], Zabczyk [14, 15] to the time-varying systems.

2 Notation and mathematical preliminaries

The following notation will be used throughout this paper. R denotes the set of all real numbers; R^+ denotes the set of all non-negative real numbers; X denotes an infinite-dimensional real Hilbert space with the inner product $\langle .,. \rangle$; X^* denotes the dual space of X; $L(X)$ (respectively, $L(X,Y)$) denotes the Banach space of all linear bounded operators mapping X into X (respectively, X into Y); $L_2([t,s],X)$ denotes the set of all strongly measurable L_2-integrable and $X-$ valued functions on $[t,s]$; $D(A), A^{-1}$ and A^* denote the domain, the inverse and the adjoint of the operator A, respectively; cl$M$ denotes the closure of a set M; I denotes the identity operator; A is self-adjoint if $A = A^*$; $\rho(A)$ and $R(\lambda, A)$ denote the resolvent set and the resolvent of A, respectively; An operator $Q \in L(X)$ is called non-negative definite ($Q \geq 0$) if $\langle Qx, x \rangle \geq 0$, for all $x \in X$; $LO([t,\infty), X^+)$ denotes the set of all linear bounded self-adjoint non-negative definite operator-valued function on $[0,\infty)$; The operator $A : D(A) \to X$ generates the C_0-semigroup $S(t)$ in X if

$$Ax = \lim_{t\to 0^+} \frac{1}{t}[S(t) - I]x, \quad \forall x \in D(A),$$

where $D(A) = \{x \in X : \text{such that } \lim_{h\to 0^+} \frac{1}{h}[S(h)x - x] \text{ exists}\}$. Let $\{A(t)\}, t \in R^+$ be a family of linear operator-valued functions.

$A(t)$ is called bounded on R^+ if

$$\exists M > 0 : \sup_{t\in R^+} \|A(t)\| \leq M;$$

Consider linear time-varying control system (1), where X, U are infinite-dimensional real Hilbert spaces; $A(t) : X \to X, t \in R^+$, is a linear unbounded operator and $B(t) \in L(U,X)$. In this paper we consider a class of admissible controls $u(t) \in L_2([0,t),U)$ for all $t \in R^+$. As in [1] we will assume the following conditions that guarantee the existence and uniqueness of the solution of linear control system (1).

(a) Operator functions $A(.)x, B(.)u$ are continuous and bounded in $t \in R^+$ for all $x \in X, u \in U$;
(b) cl$D(A(t)) = X, t \in R^+$ and $A(.)x$ is a continuous function on R^+ for every $x \in D(A(.))$,
(b) For each $t \in R^+, A(t)$ generates a C_0- semigroup on X and there is a evolution operator $U(t,s) : \{(t,s) : t \geq s \geq 0\} \to L(X)$, such that $U^*(t,s)$ is continuous in t, s and for each $x \in D(A(t)), U(t,s)x \in D(A(t))$ the following conditions hold:

(i)

$$\frac{\partial U(t,s)x}{\partial t} = A(t)U(t,s)x, \quad U(s,s) = I,$$

$$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x,$$

where $U_n(t,s)$ is the evolution operator generated by the Yosida approximation [7]

$$A_n(t) = n^2[nI - A(t)]^{-1} - nI$$

of $A(t)$.
(ii) $U(t,s) = U(t,r)U(r,s)$, for all $t \geq r \geq s \geq 0$.

In this case, we say that $A(t)$ generates a strongly continuous evolution operator $U(t,s)$, and then for every initial state $x_0 \in X$, for every admissible control $u(t)$, linear control system (1) has a mild solution given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)B(s)u(s)ds.$$

Remark 2.1. The evolution operator is a natural extension to the C_0 semigroup of time-invariant linear systems. For instance, if $A(t) = A$ is independent of t then $U(t,s)S(t)S^{-1}(\tau)$ and the two parameter family of semigroup operators reduces to the one parameter family $S(t)$, which is the standard C_0 semigroup generated by A. It is well known that if the operator $A(t) \in L(X), t \in R^+$ r, which is bounded on R^+, then the semigroup evolution operator $U(t,s)$ satisfying the above conditions always exists. However, if $A(t), t \in R^+$ is unbounded, then the evolution operator $U(t,s)$ exists provided additional assumptions, see; e.g. [1, 7] for the details.

In the sequel, sometimes for the sake of brevity, we will omit the arguments of operator-valued functions, if it does not cause any confusion.

Definition 2.1. Linear control system (1) is globally null-controllable (GNC) in finite time if for every $x_0 \in X$, there exist a number $T > 0$ and an admissible control $u(t)$ such that

$$U(T,0)x_0 + \int_0^T U(T,s)B(s)u(s)ds = 0.$$

We state the following well-known controllability criterion for infinite-dimensional control system that will be used later.

Proposition 2.1. [1, 2] *Linear control system (1) is GNC in finite time if and only if*

$$\exists T > 0, c > 0: \quad \int_0^T \|B^*(s)U^*(T,s)x^*\|^2 ds \geq c\|U^*(T,0)x^*\|^2, \quad \forall x^* \in X^*.$$

Definition 2.2. Linear control system (1) is completely stabilizable (CSz) if for every number $\delta > 0$, there exists a feedback control $u(t) = K(t)x(t)$, where $K(t) \in L(X,U)$ is bounded on R^+, such that the solution $x(t,x_0)$ of the closed-loop system $\dot{x}(t)[A(t) + B(t)K(t)]x(t), x(0) = x_0$, satisfies

$$\exists N > 0: \quad \|x(t,x_0)\| \leq Ne^{-\delta t}\|x_0\|, \quad \forall t \geq 0.$$

The solution to the stabilizability problem involves a Riccati operator equation (ROE) of the form

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)B^*(t)P(t) + Q(t) = 0, \tag{2}$$

where $Q(t) \geq 0$ is a given self-adjoint operator function. Since $A(t), t \in R^+$ is an unbounded operator, it is not clear a priori what a solution of ROE means. We will defined, as in [1], the solution of ROE (4) as follows.

Definition 2.3. The (mild) solution of ROE (2) is a linear operator function $P(t) \in L(X)$ satisfying the following two conditions:
(i) The scalar function $\langle P(\cdot)x, y\rangle$ is differentiable on $[0, \infty)$ for every $x, y \in D(A(.))$.
(ii) For all $x, y \in D(A(t)), t \in R^+$:

$$\frac{d}{dt}\langle P(t)x, y\rangle + \langle P(t)x, A(t)y\rangle + \langle P(t)A(t)x, y\rangle -$$
$$-\langle P(t)B(t)B^*(t)P(t)x, y\rangle + \langle Q(t)x, y\rangle = 0.$$

In the sequel, we state the following sufficient condition which guarantees the existence of a bounded solution $P(t)$ of ROE (2).

Definition 2.4. Let $Q(t) \in LO([0, \infty), X^+)$. The control system (1) is called $Q(t)-$stabilizable if for every initial state x_0, there is a control $u(t) \in L_2([0, \infty), U)$ such that the cost function

$$J(u) = \int_0^\infty [\|u(t)\|^2 + \langle Q(t)x(t, x_0), x(t, x_0)\rangle]dt, \qquad (3)$$

exists and is finite.

Proposition 2.2. [1] *If linear control system (1) is $Q(t)-$stabilizable, then the ROE (2) has the solution $P(t) \in LO([0, \infty), X^+)$ bounded on R^+.*

3 Main result

Consider the linear time-varying control system (1) in Hilbert spaces. As we have already mentioned in Introduction, for time-invariant control systems in finite-dimensional spaces Wonham [12] proved the equivalence of the complete stabilizability and global null-controllability and for the case of infinite-dimensional systems, assuming a compactness property on the semigroup Slemrod [10] showed that the time-invariant control system in Hilbert spaces is completely stabilizable iff it is globally null-controllable in finite time. In this section we us the following growth condition on the evolution operator $U(t, s)$:

H. $\exists M > 0, \alpha > 0 : \quad \|U(t, s)\| \leq Me^{\alpha|t-s|}, \quad \forall t, s \geq 0.$

It's known from [7] that the growth condition H hods for the time-invariant system when $A \in L(X)$ is a linear continuous constant operator as well as for the time-varying system when $A(t)$ is a matrix function uniformly bounded in $t \in R^+$. The main result of the paper is the following.

Theorem 3.1. *Linear time-varying control system (1) is completely stabilizable (CSz) if and only if it is globally null-controllable (GNC) in finite time. Conversely, assume the condition H, the system is GNC in finite time if it is CSz*

Proof. The proof of this theorem follows similar arguments used in [11] by employing some more techniques in infinite-dimensional analysis. We start by showing that GNC implies CSz. Assume that the linear time-varying control system (1) is GNC in finite time. Let $\delta > 0$ be any given number. We take a change of the state variable $y(t)e^{\delta t}x(t)$, then the linear control system (1) is transformed to the system

$$\dot{y}(t) = \tilde{A}(t)y(t) + \tilde{B}(t)u(t), \quad y(0) = y_0 x_0, \tag{4}$$

where $\tilde{A}(t) = A(t) + \delta I, \tilde{B}(t) = e^{\delta t}B(t)$. We choose an operator function $Q \in LO([0,\infty), X^+)$ bounded on R^+ such that

$$Q(t) \geq 2\tilde{A}(t) + B(t)B^*(t), \quad t \geq 0. \tag{5}$$

We first show that the linear control system $[\tilde{A}(t), B(t)]$:

$$\dot{z}(t) = \tilde{A}(t)z(t) + B(t)u(t), \quad z(0) = z_0, \quad t \in R^+,$$

is globally null-controllable in finite time. Indeed, by the GNC of the former system $[A(t), B(t)]$, for every $z_0 \in X$ there are a time $T > 0$ and admissible control $u(t) \in L_2([0,T], U)$ such that

$$U(T,0)z_0 + \int_0^T U(T,s)B(s)u(s)ds = 0. \tag{6}$$

Multiplying both sides of (6) with $e^{\delta T}$ and observing that $U_{\tilde{A}}(t,s) = e^{\delta(t-s)}U(t,s)$ we find

$$U_{\tilde{A}}(T,0)z_0 + \int_0^T U_{\tilde{A}}(T,s)B(s)\tilde{u}(s)ds = 0,$$

where $\tilde{u}(s) = e^{\delta s}u(s)$. This implies that the system $[\tilde{A}(t), B(t)]$ is GNC in finite time. Let $u_z(t)$ be an admissible control according to the solution $z(t)$ of system $[\tilde{A}(t), B(t)]$ transferring $z_0 \in X$ into 0 in time T. For every initial state $z_0 \in X$ there is an admissible control $u_z(t) \in L_2([0,T], U)$ such that the solution $z(t)$ of the system according to the control $u_z(t)$ satisfies $z(0) = z_0, z(T) = 0$. Define the admissible control $\tilde{u}_z(t) \in L_2([0,\infty), U), t \geq 0$ by

$$\tilde{u}_z(t) = \begin{cases} u_z(t), & \text{if } t \in [0,T] \\ 0, & \text{if } t > T. \end{cases}$$

Therefore, we have

$$\begin{aligned} J(\tilde{u}_z) &= \int_0^\infty [\|\tilde{u}_z(t)\|^2 + \langle Q(t)z(t), z(t)\rangle]dt \\ &= \int_0^T [\|u_z(t)\|^2 + \langle Q(t)z(t), z(t)\rangle]dt < +\infty, \end{aligned}$$

which means that the linear control system $[\tilde{A}(t), B(t)]$ is $Q(t)-$stabilizable. Applying Proposition 2.2 to the cost function (3), we can find an operator function $P \in LO([0,\infty), X^+)$, which is a solution of the following ROE

$$\dot{P}(t) + \tilde{A}^*(t)P(t) + P(t)\tilde{A}(t) - P(t)B(t)B^*(t)P(t) + Q(t) = 0,$$

or equivalently

$$\dot{P}(t) + \tilde{A}^*(t)P(t) + P(t)\tilde{A}(t) - e^{-2\delta t}P(t)\tilde{B}(t)\tilde{B}^*(t)P(t) + Q(t) = 0, \tag{7}$$

We now consider a Lyapunov-like function

$$V(t,y) = \langle P(t)y, y\rangle + \|y\|^2,$$

and construct a feedback control of the form

$$u(t) = -\frac{e^{-2\delta t}}{2}\tilde{B}^*(t)[P(t) - I]y(t). \tag{8}$$

Taking the derivative of $V(.)$ in t along the solution of $y(t)$ of the system (4) and using the chosen feedback control and the ROE (7), we have

$$\begin{aligned}
\dot{V}(t,y(t)) &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P(t)\dot{y}(t), y(t)\rangle + 2\langle \dot{y}(t), y(t)\rangle, \\
&= \langle (-\tilde{A}^*P - P\tilde{A} + e^{-2\delta t}P\tilde{B}\tilde{B}^*P - Q)y, y\rangle \\
&\quad +2\langle P(\tilde{A}y + \tilde{B}u), y\rangle + 2\langle \tilde{A}y + \tilde{B}u, y\rangle, \\
&= e^{-2\delta t}\langle P\tilde{B}\tilde{B}^*Py, y\rangle + 2\langle P\tilde{B}u, y\rangle + 2\langle \tilde{A}y, y\rangle + 2\langle \tilde{B}u, y\rangle - \langle Qy, y\rangle \\
&= -\langle [Q(t) - 2\tilde{A}(t) - e^{-2\delta t}\tilde{B}(t)\tilde{B}^*(t)]y(t), y(t)\rangle \\
&= -\langle [Q(t) - 2\tilde{A}(t) - B(t)B^*(t)]y(t), y(t)\rangle.
\end{aligned}$$

By choosing $Q(t)$ from (5), we find $[Q(t) - 2\tilde{A}(t) - B(t)B^*(t)] \geq 0$, and hence $\dot{V}(y(t)) \leq 0, \quad \forall t \geq 0$. This inequality shows the boundedness of the solution $y(t)$ of the system (4). Indeed, by integrating both sides of the above inequality from 0 to t, we obtain that

$$V(t, y(t)) - V(0, y_0) \leq 0,$$

and hence

$$\langle P(t)y(t), y(t)\rangle - \|y(t)\|^2 \leq \langle P(0)y_0, y_0\rangle + \|y_0\|^2.$$

Since $P(t) \geq 0$, for all $t \in R^+$,

$$\|y(t)\| \leq \langle P(0)y_0, y_0\rangle + \|y_0\|^2, \quad \forall t \geq 0,$$

we find

$$\exists N > 0 : \quad \|y(t)\| \leq N\|y_0\|, \quad \forall t \geq 0.$$

Therefore, by returning to the solution $x(t)$ of system (1), we finally obtain that

$$\|x(t)\| \leq N\|x_0\|e^{-\delta t}, \quad \forall t \geq 0.$$

This means that the control system (1) is completely stabilizable by the feedback control (8) transformed in the state $x(t)$ as

$$u(t) = -\frac{e^{-2\delta t}}{2}\tilde{B}^*(t)[P(t) - I]y(t) = K(t)x(t),$$

where $K(t) = -\frac{1}{2}B^*(t)[P(t) - I]$, which is bounded on R^+. Thus GNC implies CSz.

Next we show that CSz implies GNC. Assume that linear control system (1) is strongly stabilizable. By the property of the evolution operator we have

$$\exists M > 0, \alpha > 0: \quad \|U^*(t,s)\| \leq Me^{\alpha|t-s|}, \quad \forall t, s \geq 0. \tag{9}$$

Due to the strong stabilizability of (1), for a chosen number $\delta > \alpha > 0$, there is an operator function $K \in L(X,U)$, which is bounded on R^+ such that the solution $x(t,x_0) = U_K(t,0)x_0$, where $U_K(t,s)x_0$ is the evolution operator generated by the operator $[A(t) + B(t)K(t)]$, satisfies

$$\exists N > 0: \quad \|x(t,x_0)\| = \|U_K(t,0)x_0\| \leq Ne^{-\delta t}\|x_0\|, \quad \forall t \geq 0. \tag{10}$$

On the other hand, for every $x_0 \in X$ and feedback control $u(t) = K(t)x(t)$, the solution $x(t,x_0)$ of system (1) is defined as

$$x(t,x_0) = U(t,0)x_0 + \int_0^t U(t,s)B(s)u(s)ds.$$

Therefore,

$$U(t,0)x_0 = U_K(t,0)x_0 - \int_0^t U(t,s)B(s)K(s)U_K(s,0)x_0 ds, \quad t \in R^+.$$

Since the above relation holds for every $x_0 \in X$, the following estimate holds for every $x^* \in X^*$:

$$\|U^*(t,0)x^*\| \leq \|U_K^*(t,0)x^*\| + \int_0^t \|U_K^*(s,0)K^*(s)B^*(s)U^*(t,s)x^*\|ds.$$

Taking the condition (10) into account, we have

$$\begin{aligned} \|U^*(t,0)x^*\| &\leq Ne^{-\delta t}\|x^*\| + Nk\int_0^t e^{-\delta s}\|B^*(s)U^*(t,s)\|ds, \\ &\leq Ne^{-\delta t}\|x^*\| + Nk(\int_0^t e^{-2\delta s}ds)^{1/2} \times \\ &\quad \times(\int_0^t \|B^*(s)U^*(t,s)x^*\|^2 ds)^{1/2}, \end{aligned} \tag{11}$$

where $k := \sup\{\|K(s)\| : s \in [0, \infty)\}$. Setting $\beta(t) = (\int_0^t e^{-2\delta s} ds)^{1/2}$, we have

$$\beta(t) = (\frac{1}{2\delta} - \frac{1}{2\delta} e^{-2\delta t})^{1/2}. \tag{12}$$

To prove the assertion, we assume to the contrary that system (1) is not globally null-controllable in any finite time $t > 0$. Then, by Proposition 2.1, for every $t > 0$ and for some chosen numbers $c > 0, \epsilon \in (0, 1)$, satisfying

$$c < \Big[\frac{(1-\epsilon)\sqrt{2\delta}}{Nk}\Big]^2, \tag{13}$$

there is $x_0^* \in X^*$ such that

$$\int_0^t \|B^*(s)U^*(t, s)x_0^*\| ds < c\|U^*(t, 0)x_0^*\|^2. \tag{14}$$

The strict inequality (14) shows that $x_0^* \neq 0$, we may assume, without loss of generality, that the inequality (14) holds for all $x_0 : \|x_0^*\| = 1$; otherwise we can take $x_1^* = \frac{x_0^*}{\|x_0^*\|}$. Therefore, from (11), (14) it follows that

$$\|U^*(t, 0)x_0^*\| < Ne^{-\delta t} + \sqrt{c}Nk\beta(t)\|U^*(t, 0)x_0^*\|. \tag{15}$$

On the other hand, we observe that

$$1 = \|x_0^*\| = \|U^*(0, t)U^*(t, 0)x_0^*\| \leq \|U^*(0, t)\|\|U^*(t, 0)x_0^*\|,$$

which, due to (9) and $\|U^*(t, 0)x_0^*\| \neq 0$, gives

$$\frac{1}{\|U^*(t, 0)x_0^*\|} \leq \|U^*(0, t)\| \leq Me^{\alpha t}, \quad \forall t > 0. \tag{16}$$

Therefore, by using (15) and (16), we obtain that

$$1 < \frac{Ne^{-\delta t}}{\|U^*(t, 0)x_0^*\|} + \sqrt{c}Nk\beta(t) < NMe^{-(\delta-\alpha)t} + \sqrt{c}Nk\beta(t), \quad t > 0,$$

hence

$$1 - \sqrt{c}Nk\beta(t) < NMe^{-(\delta-\alpha)t}, \quad \forall t > 0.$$

The above relation does not depend on x_0^*, we can let t go to infinity and noticing from (12) that $\beta(t) \to (1/\sqrt{2\delta})$, the right hand-side goes to 0 because of $\delta > \alpha$, we have

$$1 - \sqrt{c}N\frac{1}{\sqrt{2\delta}}k \leq 0.$$

Then, from (13) it follows the condition

$$\epsilon < 1 - \sqrt{c}N\frac{1}{\sqrt{2\delta}}k \leq 0,$$

which leads to a contradiction. The system is therefore globally null-controllable in finite time and the proof is complete.

Remark 3.1. Note that if $A(t) = A$ and generates the C_0-strong continuous semigroup $S(t)$, then we have the following result for linear time-invariant systems.

Corollary 3.1. [11, 14] *A linear time-invariant control system in Hilbert spaces is CSz if and only if it is GNC in finite time.*

For linear time-varying systems in finite-dimensional spaces, we also have the following consequence.

Corollary 3.2. [4] *Assume that* $X = R^n, U = R^m$ *and* $A(t), B(t)$ *are matrix function bounded on* R^+. *Linear time-varying control system (1) is CSz if and only if it is GNC in finite time.*

4 Conclusions

We have established the equivalence of complete stabilizability and exact controllability for linear rime-varying control systems in Hilbert spaces. The obtained result extends existing results in the literature to infinite-dimensional and time-varying control systems.

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References

[1] A. Bensoussan, G.Da Prato, M.C. Delfour and S.K. Mitter, *Representation and Control of Infinite-dimensional Systems,* vol. I, II, Birkhauser, 1992.

[2] R.F. Curtain and H. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory.* Springer-Verlag, New York, 1995.

[3] H. Jerbi, Asymptotic stabilizability of three-dimensional homogeneous polimial systems of degree three, *Appl. Math. Lett.,* **17**(2004), 357-366.

[4] M. Ikeda, H. Maeda and S. Kodama, Stabilization of linear systems. *SIAM J. on Control,* **10**(1972), 716-729.

[5] R.E. Kalman, Y.C. Ho and K.S. Narenda, Controllability of linear dynamical systems. *Differential Equations,* **1**(1963), 189-213.

[6] P. Niamsup and V.N. Phat, Stabilization of linear non-autonomous systems with norm bounded controls. *J. Optim. Theory Appl.,* **131**(2006), 135-149.

[7] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations.* Springer, New York, 1983.

[8] V.N. Phat, *Constrained Control Problems of Discrete Processes.* World Scientific, Singapore, 1996.

[9] V.N. Phat, Weak asymptotic stabilizability of discrete-time inclusions given by set-valued operators, *J. Math . Anal. Appl.* **202**(1996), 353-369.

[10] V.N. Phat, New stabilization criteria for linear time-varying systems with state delay and normed bounded uncertainties. *IEEE Trans. Aut. Control,* **12**(2002), 2095-2098.

[11] V.N. Phat and Q.P.Ha, New characterization of stabilizability via Riccati equations for LTV systems, *IMA J. Math. Contr. Inform.,* **25**(2008), 419-429.

[12] M. Slemrod, A note on complete controllability and stabilizability for linear control systems in Hilbert spaces. *SIAM J. on Control,* **12**(1974), 500-508.

[13] W.M. Wonham, On pole assignment in multi-input controllable linear systems, *IEEE Trans. Aut. Control*, **12**(1967), 660-665.

[14] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach,* Springer-Verlag, Berlin, 1979.

[15] J. Zabczyk, Complete stabilizability implies exact controllability, *Seminaru Ecuati Functionale*, **38**(1977), 1-10.

[16] J. Zabzyk, *Mathematical Control Theory: An Introduction,* Birkhauzer, Boston, 1992.

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