Thai Journal of Mathematics
Volume 22 Number 2 (2024)
Pages 463–483

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

In memoriam Professor Charles E. Chidume (1947–2021)

Modified Popov's Extragradient-like Method for Solving a Family of Strongly Pseudomonotone Equilibrium Problems in Real Hilbert Space

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Abstract In this article, we are introducing a new proximal based extragradient method and examining its convergence analysis in order to solve equilibrium problems that incorporate strongly pseudomonotone bifunction. The main superiority of this technique, in particular that the construction of an approximation solution, proof of its convergence and also proof of its appropriateness, does not needed previous information of the modulus of strong pseudo-monotonicity and the Lipschitz-type bi-functional parameters. In addition, the method uses a decreasing and non-summable stepsize sequence. Finally, numerical experiment results are provided to illustrate the method on a test problem to equate the efficiency with previously known algorithms.

MSC: 47H10; 54H25

Keywords: equilibrium problem; strongly pseudomonotone bifunction; strong convergence theorem; Lipschitz-type conditions; variational inequality problems

Submission date: 15.08.2022 / Acceptance date: 10.06.2023

1. Introduction

We consider iterative methods for evaluating the approximate solution to the following problem of equilibrium [1]:

$$\text{Find } \xi^* \in C \text{ such that } f(\xi^*, v) \geq 0, \ \forall v \in C, \tag{1.1}$$

Published by The Mathematical Association of Thailand.

having C is a non-empty convex and closed subset of a certain Hilbert space $\mathbb{H}$ and $f : C \times C \to \mathbb{R}$ is a bifunction with $f(u, u) = 0$, for all $u \in C$.

The certain definitions of a bifunction monotonicity (see [1, 2]). A bifunction $f : C \times C \to \mathbb{R}$ is

(1) *strongly monotone* upon C if there is a $\gamma > 0$, so that

$$f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{u}) \leq -\gamma \|\widetilde{u} - \widetilde{v}\|^2, \ \forall \widetilde{u}, \widetilde{v} \in C;$$

(2) *monotone* upon C if

$$f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{u}) \leq 0, \ \forall \widetilde{u}, \widetilde{v} \in C;$$

(3) *strongly pseudomonotone* upon C if

$$f(\widetilde{u}, \widetilde{v}) \geq 0 \Longrightarrow f(\widetilde{v}, \widetilde{u}) \leq -\gamma \|\widetilde{u} - \widetilde{v}\|^2, \ \forall \widetilde{u}, \widetilde{v} \in C;$$

(4) *pseudomonotone* upon C if

$$f(\widetilde{u}, \widetilde{v}) \geq 0 \Longrightarrow f(\widetilde{v}, \widetilde{u}) \leq 0, \ \forall \widetilde{u}, \widetilde{v} \in C;$$

(5) satisfying the *Lipschitz-type condition* upon C if $L_1, L_2 \geq 0$ so that

$$f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{w}) \geq f(\widetilde{u}, \widetilde{w}) - L_1 \|\widetilde{u} - \widetilde{v}\|^2 - L_2 \|\widetilde{v} - \widetilde{w}\|^2, \ \forall \widetilde{u}, \widetilde{v}, \widetilde{w} \in C.$$

The above defined problem (1.1) arises from a number of practical applications, e.g. Nash equilibrium problems, fixed point problems, complementarity problems, optimization problems and variational inequalities problems ([1, 3]). Wide applications have prompted many researchers to study the methods of solving the (1.1) problem. Some well-known iterative methods used to solve the (1.1) problem include; proximal point methods, extragradient methods, hybrid methods, projected subgradient methods.

The proximal point method is one of the most effective methods for approximating the problem (1.1). This method is mostly utilised in approximating monotone equilibrium problems, that is, the bifunction of the problem of equilibrium must be monotone. The method was first explored by Martinet [4] for solving variational inequality problems through monotone operator. Afterwards, Rockafellar [5] extended the method to monotone operators. In 1999, Moudafi [6] also proposed a proximal point method for approximating the equilibrium problems for monotone bifunctions. As well, Konnov [7] also suggested a particular version of the proximal point approach through weaker conditions. Another prominent technique for approximating the solution of the problem (1.1) is the auxiliary problem principle. This concept for optimization problems was introduced by Cohen [8] and thus applied to the problems of variational inequalities. The technique establishes a new problem which is identical and typically simpler to carry out than the original problem. Mastroeni [9] further expanded the auxiliary problem concept to address problems including strongly monotone bifunctions. Now, in the coming, we detail some results which are set up up owing to the auxiliary problem principle. We see the extragradient method proposed by Korpelevich [10] which is one of the most celebrated two-step methods and its iterative step is as follows:

$$\begin{cases} u_0 \in C \\ v_n = P_C(u_n - \lambda F(u_n)) \\ u_{n+1} = P_C(u_n - \lambda F(v_n)) \end{cases} \tag{1.2}$$

while L is the Lipschitz constant of the operator $F : \mathbb{H} \to \mathbb{H}$ and $\lambda \in \big(0, \frac{1}{L}\big)$. The Quoc [11] and Flam [12] employ the auxiliary problem principle to build up the above two-step extragradient method in the subsequent manner:

$$\begin{cases} u_0 \in C \\ v_n = \arg\min\limits_{v \in C}\{\lambda f(u_n, v) + \frac{1}{2}\|u_n - v\|^2\} \\ u_{n+1} = \arg\min\limits_{v \in C}\{\lambda f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\} \end{cases} \tag{1.3}$$

where u_n is previously known value with $0 < \lambda < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ and L_1, L_2 are Lipschitz type constants. The weaknesses of the Korpelevich extragradient method are the computation of the values of the operator F at two separate points to be carried out at the next iteration. This fault is disposed of in the Popov′s extragradient algorithm [13, 14] written as

$$\begin{cases} u_0, v_0 \in C \\ v_n = \arg\min\limits_{v \in C}\{\lambda f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\} \\ u_{n+1} = \arg\min\limits_{v \in C}\{\lambda f(v_n, v) + \frac{1}{2}\|u_{n+1} - v\|^2\} \end{cases} \tag{1.4}$$

where u_n, v_n are known values and L is the Lipschitz constant of the operator $F : \mathbb{H} \to \mathbb{H}$, $\lambda \in \big(0, \frac{1}{2L_2+4L_1}\big)$.

On the other hand, let us discuss methods of an inertial type. Based on both the heavy ball methods of the two-order time dynamic system, Polyak [15] suggested an inertial interpretation as a speed mechanism to solve minimization problem. The inertial technique is a two-step algorithmic technique, and then the subsequent iteration is calculated by having the operate of the prior two iterations, and can be linked to as an approach to improve the iterative sequence, see [15–20]. Several inertial-like algorithms for particular cases of the problem (1.1) [18, 21–24]. Next, we can pay attention to that the functional implementation of algorithm (1.2) and algorithm (1.4) required some known information before to run the algorithm, for example, we need to have a knowledge about stepsize λ which is depended upon the values of Lipschitz constants which are not obvious to figure out. Secondly, these methods are suitable in the understanding that it can be applied for a bigger class of bifunction like pseudomonotone, but at the same time, we have weak convergence.

The main contribution of this paper is to suggest a new method to figure out an estimated solution of the problem (1.1) where the underlying mapping is strongly pseudomonotone bifunction. The algorithm combines Popov's extragradient method with inertial terms to determine a solution of an equilibrium problem. The propose algorithm can be regarded as a modification of the algorithm (1.4) for the class of equilibrium problem (1.1) involving strongly pseudomonotone bifunction. Compared to existing methods for figuring an approximate solution to the problem (1.1), the step-sizes used in our new algorithm are independent of the stongly pseudomonotone and Lipschitz-type cost-bifunction constants of the underlying mapping. This benefit comes from the use of a step-size sequence which is slowly convergent to zero. Because of it as well as the strong pseudomonotonicity of the bifunction, there is indeed a strong convergence of the methods is achieved. Though, it's not essential to know these variables, i.e., these variables are not the input variables of the method. This enables us to build estimated solution and to demonstrate the convergence of the method without needing to know the details on such variables. This is particularly noteworthy when such variables are undetermined or complicated to measure. In the meantime, we have used the variable positive step-size sequence $\{\lambda_n\}$ that is non-increased, non-summable and diminishing.

This article contains the following arrangements: Some descriptions and preliminary results are recalled in Section 2. The convergence analysis of our proposed approach is given in Section 3. Section 4 we present different numerical experiments to explain the behaviour of the stepsize sequences and also compare the performance with previously established methods.

2. Preliminaries

We consider the following important notions:

A function $g : C \to \mathbb{R}$ is a convex function and *subdifferential of* g on $u \in C$ is

$$\partial g(u) = \{w \in \mathbb{H} : g(v) - g(u) \geq \langle w, v - u\rangle, \ \forall v \in C\}.$$

The *normal cone* of C at $u \in C$ is defined by

$$N_C(u) = \{w \in \mathbb{H} : \langle w, v - u\rangle \leq 0, \ \forall v \in C\}.$$

Lemma 2.1. [25] *Let $C \subset \mathbb{H}$ be non-empty, closed and convex with $h : C \to \mathbb{R}$ be a convex, subdifferentiable with lower semicontinuous function on C. Moreover, $u \in C$ is a minimizer of a function h if and only if $0 \in \partial g(u) + N_C(u)$, where $\partial g(u)$ and $N_C(u)$ denotes the subdifferential of g at u and the normal cone of C at u respectively.*

Lemma 2.2. [26] *For any $u, v \in \mathbb{H}$ and $\beta \in \mathbb{R}$, then the following expression is true:*

$$\|\beta u + (1-\beta)v\|^2 = \beta\|u\|^2 + (1-\beta)\|v\|^2 - \beta(1-\beta)\|u - v\|^2.$$

Lemma 2.3. [27] *Let $\{p_n\}, \{q_n\} \subset [0, \infty)$ be two sequences and $\sum_{n=1}^{\infty} p_n = \infty$ with $\sum_{n=1}^{\infty} p_n q_n < \infty$, then $\liminf_{n\to\infty} q_n = 0$.*

Lemma 2.4. [28] *Let $\{p_n\}$ and $\{q_n\}$ be two sequences of non-negative real numbers with $p_{n+1} \leq p_n + q_n$ for all $n \in \mathbb{N}$. If $\sum q_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.*

Lemma 2.5. [29] *Let p_n, q_n and r_n be sequences in $[0, +\infty)$ such that*

$$p_{n+1} \leq p_n + q_n(p_n - p_{n-1}) + r_n, \ \forall n \geq 1, \quad \textit{with} \quad \sum_{n=1}^{+\infty} r_n < +\infty,$$

and also with $q > 0$ such that $0 \leq q_n \leq q < 1$ for all $n \in \mathbb{N}$. Thus, the subsequent items are true.

(*i*) $\sum_{n=1}^{+\infty} [p_n - p_{n-1}]_+ < \infty$, *with* $[t]_+ := \max\{t, 0\}$;
(*ii*) $\lim_{n\to+\infty} p_n = p^* \in [0, \infty)$.

Assumption 2.6. *Let $f : C \times C \to \mathbb{R}$ is*

(P1) *$f(u, u) = 0$, for all $u \in C$ and $EP(f, C)$ is non-empty set;*
(P2) *f is strongly pseudomonotone on C;*
(P3) *f satisfies the Lipschitz-type condition on C;*
(P4) *$f(u, \cdot)$ is convex and sub-differentiable on $\mathbb{H}$ for every fixed $u \in \mathbb{H}$.*

3. Main Results

The proposed algorithm is described as follows:

Algorithm 1 (Modified Extragradient Algorithm for Strongly Pseudomonotone EP)

Initialization: Pick $u_{-1}, u_0, v_0 \in C$, $\{\alpha_n\} \subset [0, \frac{1}{6})$ and a non-increasing sequence $\{\lambda_n\} \subset (0, +\infty)$ satisfying the following hypotheses:

$$(\text{K1}): \lim_{n\to\infty} \lambda_n = 0, \ (\text{K2}): \sum_{n=0}^{\infty} \lambda_n = +\infty.$$

Iterative Steps: Given $u_{n-1}, u_n, v_n \in C$ for $n \geq 0$. Calculate u_{n+1} and v_{n+1} the following manner:

Step 1. Evaluate

$$u_{n+1} = \arg\min_{y\in C}\{\lambda_n f(v_n, y) + \frac{1}{2}\|\xi_n - y\|^2\}.$$

where $\xi_n = u_n + \alpha_n(u_n - u_{n-1})$. If $u_{n+1} = v_n = \xi_n$, STOP. If not, take the next step.

Step 2. Evaluate

$$v_{n+1} = \arg\min_{y\in C}\{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2\}.$$

Take $n := n + 1$ and return to **Step 1.**

This section begins with the related lemmas which are needed to prove the main result.

Lemma 3.1. *The Algorithm 1 has the following relevant inequality.*

$$\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle \xi_n - u_{n+1}, y - u_{n+1}\rangle, \forall y \in C.$$

Proof. By u_{n+1} with Lemma 2.1 implies that

$$0 \in \partial_2\Big\{\lambda_n f(v_n, y) + \frac{1}{2}\|\xi_n - y\|^2\Big\}(u_{n+1}) + N_C(u_{n+1}).$$

Thus, for $\eta \in \partial_2 f(v_n, u_{n+1})$ there exists $\overline{\eta} \in N_{C(}u_{n+1})$ such that

$$\lambda_n \eta + u_{n+1} - \xi_n + \overline{\eta} = 0$$

which continues to follows that

$$\langle \xi_n - u_{n+1}, y - u_{n+1}\rangle = \lambda_n\langle \eta, y - u_{n+1}\rangle + \langle \overline{\eta}, y - u_{n+1}\rangle, \ \forall y \in C.$$

Since $\overline{\eta} \in N_C(u_{n+1})$ then $\langle \overline{\eta}, y - u_{n+1}\rangle \leq 0$, $\forall\ y \in C$. It is implies that

$$\langle \xi_n - u_{n+1}, y - u_{n+1}\rangle \leq \lambda_n\langle \eta, y - u_{n+1}\rangle, \ \forall y \in C. \tag{3.1}$$

From $\eta \in \partial f(v_n, u_{n+1})$, we've got

$$f(v_n, y) - f(v_n, u_{n+1}) \geq \langle \eta, y - u_{n+1}\rangle, \ \forall y \in \mathbb{H}. \tag{3.2}$$

From (3.1) and (3.2) we obtain

$$\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle \xi_n - u_{n+1}, y - u_{n+1}\rangle, \ \forall y \in C.$$

■

Lemma 3.2. *The Algorithm 1 also has the following relevant inequality.*

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \ \forall y \in C.$$

Proof. By v_{n+1} with Lemma 2.1, implies that

$$0 \in \partial_2 \Big\{ \lambda_{n+1} f(v_n, y) + \frac{1}{2} \|u_{n+1} - y\|^2 \Big\} (v_{n+1}) + N_C(v_{n+1}).$$

Thus, for $\eta \in \partial_2 f(v_n, v_{n+1})$ there exists $\overline{\eta} \in N_C(v_{n+1})$ such that

$$\lambda_n \eta + v_{n+1} - u_{n+1} + \overline{\eta} = 0$$

which gives that

$$\langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle = \lambda_{n+1} \langle \eta, y - v_{n+1} \rangle + \langle \overline{\eta}, y - v_{n+1} \rangle, \ \forall y \in C.$$

Since $\overline{\eta} \in N_C(v_{n+1})$ then $\langle \overline{\eta}, y - v_{n+1} \rangle \leq 0$, $\forall y \in C$. Thus, we have

$$\langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle \leq \lambda_{n+1} \langle \eta, y - v_{n+1} \rangle, \ \forall y \in C. \tag{3.3}$$

By $\eta \in \partial f(v_n, v_{n+1})$, implies

$$f(v_n, y) - f(v_n, v_{n+1}) \geq \langle \eta, y - v_{n+1} \rangle, \ \forall y \in \mathbb{H}. \tag{3.4}$$

From relation (3.3) and (3.4) we obtain

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \ \forall y \in C.$$

■

Lemma 3.3. *If $u_{n+1} = v_n = \xi_n$ in Algorithm 1, then $v_n \in EP(f, C)$.*

Proof. By Lemma 3.1, we have

$$\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle \xi_n - u_{n+1}, y - u_{n+1} \rangle, \ \forall y \in C. \tag{3.5}$$

By taking $u_{n+1} = v_n = \xi_n$, in the above expression gives that

$$\lambda_n f(v_n, y) \geq 0, \ \forall y \in C. \tag{3.6}$$

Thus, expression (3.6) and $\lambda_n \geq \lambda > 0$, gives $f(v_n, y) \geq 0$ for each $y \in C$. That's evidence $v_n \in EP(f, C)$. ■

Lemma 3.4. *If $u_{n+1} = v_{n+1} = v_n$ in Algorithm 1, then $v_n \in EP(f, C)$.*

Proof. By Lemma 3.2, we have

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \ \forall y \in C. \tag{3.7}$$

By taking $u_{n+1} = v_{n+1} = v_n$ in the above expression gives that

$$\lambda_n f(v_n, y) \geq 0, \ \forall y \in C. \tag{3.8}$$

Thus, expression (3.8) and $\lambda_n \geq \lambda > 0$, gives $f(v_n, y) \geq 0$ for all $y \in C$. This indicates that $v_n \in EP(f, C)$. ■

Lemma 3.5. *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfying the condition* (P1)-(P4)*. Thus, for each $\xi^* \in EP(f, C) \neq \emptyset$, we obtain*

$$\begin{aligned}\|u_{n+1} - \xi^*\|^2 \leq \|\xi_n - \xi^*\|^2 - (1 - 4L_1\lambda_n)\|u_n - v_n\|^2 - (1 - 2L_2\lambda_n)\|u_{n+1} - v_n\|^2 \\ + 4L_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\ - \|u_{n+1} - \xi_n\|^2 + \|u_{n+1} - u_n\|^2.\end{aligned}$$

Proof. Follows from Lemma 3.1 with $y = \xi^*$, we have

$$\lambda_n f(v_n, \xi^*) - \lambda_n f(v_n, u_{n+1}) \geq \langle \xi_n - u_{n+1}, \xi^* - u_{n+1}\rangle. \tag{3.9}$$

Since $f(\xi^*, v_n) \geq 0$, and from (P1) implies that $f(v_n, \xi^*) \leq -\gamma\|v_n - \xi^*\|^2$, gives that

$$\langle \xi_n - u_{n+1}, u_{n+1} - \xi^*\rangle \geq \lambda_n f(v_n, u_{n+1}) + \gamma\lambda_n\|v_n - \xi^*\|^2. \tag{3.10}$$

It proceeds from the context of the Lipschitz-type of f that continues.

$$f(v_{n-1}, v_n) + f(v_n, u_{n+1}) \geq f(v_{n-1}, u_{n+1}) - L_1\|v_{n-1} - v_n\|^2 - L_2\|u_{n+1} - v_n\|^2. \tag{3.11}$$

Both sides have been multiplied with $\lambda_n > 0$, such that

$$\begin{aligned}\lambda_n f(v_n, u_{n+1}) \geq \lambda_n f(v_{n-1}, u_{n+1}) - \lambda_n f(v_{n-1}, v_n) \\ - L_1\lambda_n\|v_{n-1} - v_n\|^2 - L_2\lambda_n\|u_{n+1} - v_n\|^2.\end{aligned} \tag{3.12}$$

Next, by expression (3.10) and (3.12) we reach the following:

$$\begin{aligned}\langle \xi_n - u_{n+1}, u_{n+1} - \xi^*\rangle \geq \lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \\ - L_1\lambda_n\|v_{n-1} - v_n\|^2 - L_2\lambda_n\|u_{n+1} - v_n\|^2 \\ + \gamma\lambda_n\|v_n - \xi^*\|^2.\end{aligned} \tag{3.13}$$

By Lemma 3.2 through $y = u_{n+1}$, we have

$$\lambda_n\big\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\big\} \geq \langle u_n - v_n, u_{n+1} - v_n\rangle. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\begin{aligned}\langle \xi_n - u_{n+1}, u_{n+1} - \xi^*\rangle \geq \langle u_n - v_n, u_{n+1} - v_n\rangle \\ - L_1\lambda_n\|v_{n-1} - v_n\|^2 - L_2\lambda_n\|u_{n+1} - v_n\|^2 \\ + \gamma\lambda_n\|v_n - \xi^*\|^2.\end{aligned} \tag{3.15}$$

The facts are available:

$$2\langle \xi_n - u_{n+1}, u_{n+1} - \xi^*\rangle = \|\xi_n - \xi^*\|^2 - \|u_{n+1} - \xi_n\|^2 - \|u_{n+1} - \xi^*\|^2,$$
$$2\langle u_n - v_n, u_{n+1} - v_n\rangle = \|u_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|u_n - u_{n+1}\|^2,$$

The expression (3.15) with above facts implies that

$$\begin{aligned}\|u_{n+1} - \xi^*\|^2 \leq \|\xi_n - \xi^*\|^2 - \|u_{n+1} - \xi_n\|^2 - \|u_n - v_n\|^2 - \|u_{n+1} - v_n\|^2 \\ + \|u_n - u_{n+1}\|^2 + 2L_1\lambda_n\|v_{n-1} - v_n\|^2 + 2L_2\lambda_n\|u_{n+1} - v_n\|^2 \\ - 2\gamma\lambda_n\|v_n - \xi^*\|^2.\end{aligned} \tag{3.16}$$

Now by using triangular inequality, we have

$$\|v_{n-1}-v_n\|^2 \leq \big(\|v_{n-1}-u_n\|+\|u_n-v_n\|\big)^2 \leq 2\|v_{n-1}-u_n\|^2+2\|u_n-v_n\|^2. \quad (3.17)$$

By combining the expression (3.16) and (3.17) implies that

$$\begin{aligned}\|u_{n+1}-\xi^*\|^2 \leq \|\xi_n-\xi^*\|^2 &- \|u_{n+1}-\xi_n\|^2 - \|u_n-v_n\|^2 - \|u_{n+1}-v_n\|^2 \\ &+ \|u_n-u_{n+1}\|^2 + 2L_1\lambda_n\Big[2\|v_{n-1}-u_n\|^2 + 2\|u_n-v_n\|^2\Big] \\ &+ 2L_2\lambda_n\|u_{n+1}-v_n\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2.\end{aligned} \quad (3.18)$$

Finally, we get the following

$$\begin{aligned}\|u_{n+1}-\xi^*\|^2 \leq \|\xi_n-\xi^*\|^2 &- (1-4L_1\lambda_n)\|u_n-v_n\|^2 - (1-2L_2\lambda_n)\|u_{n+1}-v_n\|^2 \\ &+ 4L_1\lambda_n\|u_n-v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2 - \|u_{n+1}-\xi_n\|^2 \\ &+ \|u_{n+1}-u_n\|^2.\end{aligned} \quad (3.19)$$

■

Theorem 3.6. *Let* $f : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ *satisfying* (P1)-(P4). *Let* $\{u_n\}$ *be a sequences in* $\mathbb{H}$ *developed by Algorithm 1, where* α_n *is non-decreasing with* $0 \leq \alpha_n \leq \alpha < \frac{1}{6}$. *Then,* $\{u_n\}$, $\{v_n\}$ *and* $\{\xi_n\}$ *strongly converges to an element* ξ^* *in* $EP(f, C)$.

Proof. In Lemma 3.5 adding the value $4L_1\lambda_n\|u_{n+1}-v_n\|^2$ on both sides, we obtain

$$\begin{aligned}&\|u_{n+1}-\xi^*\|^2 + 4L_1\lambda_n\|u_{n+1}-v_n\|^2 \\ &\leq \|\xi_n-\xi^*\|^2 - (1-4L_1\lambda_n)\|u_n-v_n\|^2 - (1-2L_2\lambda_n)\|u_{n+1}-v_n\|^2 \\ &\quad + 4L_1\lambda_n\|u_n-v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2 - \|u_{n+1}-\xi_n\|^2 \\ &\quad + \|u_{n+1}-u_n\|^2 + 4L_1\lambda_n\|u_{n+1}-v_n\|^2 \qquad (3.20) \\ &= \|\xi_n-\xi^*\|^2 - (1-4L_1\lambda_n)\|u_n-v_n\|^2 - (1-2L_2\lambda_n-4L_1\lambda_n)\|u_{n+1}-v_n\|^2 \\ &\quad + 4L_1\lambda_n\|u_n-v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2 - \|u_{n+1}-\xi_n\|^2 + \|u_{n+1}-u_n\|^2 \qquad (3.21) \\ &= \|\xi_n-\xi^*\|^2 - \frac{1}{2}(1-2L_2\lambda_n-4L_1\lambda_n)\big[2\|u_{n+1}-v_n\|^2 + 2\|u_n-v_n\|^2\big] \\ &\quad + 4L_1\lambda_n\|u_n-v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2 - \|u_{n+1}-\xi_n\|^2 + \|u_{n+1}-u_n\|^2 \qquad (3.22) \\ &\leq \|\xi_n-\xi^*\|^2 - \frac{1}{2}(1-2L_2\lambda_n-4L_1\lambda_n)\|u_{n+1}-u_n\|^2 \\ &\quad + 4L_1\lambda_n\|u_n-v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n-\xi^*\|^2 - \|u_{n+1}-\xi_n\|^2 + \|u_{n+1}-u_n\|^2. \qquad (3.23)\end{aligned}$$

By the definition of ξ_n in Algorithm 1, we have

$$\begin{aligned}\|\xi_n-\xi^*\|^2 &= \|u_n+\alpha_n(u_n-u_{n-1})-\xi^*\|^2 \\ &= \|(1+\alpha_n)(u_n-\xi^*)-\alpha_n(u_{n-1}-\xi^*)\|^2 \\ &= (1+\alpha_n)\|u_n-\xi^*\|^2 - \alpha_n\|u_{n-1}-\xi^*\|^2 + \alpha_n(1+\alpha_n)\|u_n-u_{n-1}\|^2.\end{aligned} \quad (3.24)$$

From the value of u_{n+1}, we achieve

$$\begin{aligned}\|u_{n+1} - \xi_n\|^2 &= \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \alpha_n^2\|u_n - u_{n-1}\|^2 - 2\alpha_n\langle u_{n+1} - u_n, u_n - u_{n-1}\rangle \quad (3.25)\\ &\geq \|u_{n+1} - u_n\|^2 + \alpha_n^2\|u_n - u_{n-1}\|^2 - 2\alpha_n\|u_{n+1} - u_n\|\|u_n - u_{n-1}\| \\ &\geq \|u_{n+1} - u_n\|^2 + \alpha_n^2\|u_n - u_{n-1}\|^2 - \alpha_n\|u_{n+1} - u_n\|^2 \\ &\quad - \alpha_n\|u_n - u_{n-1}\|^2 \\ &= (1 - \alpha_n)\|u_{n+1} - u_n\|^2 + (\alpha_n^2 - \alpha_n)\|u_n - u_{n-1}\|^2. \quad (3.26)\end{aligned}$$

Thus, the expression (3.23) with (3.24) and (3.26) implies that

$$\begin{aligned}&\|u_{n+1} - \xi^*\|^2 + 4L_1\lambda_{n+1}\|u_{n+1} - v_n\|^2 \\ &\leq (1 + \alpha_n)\|u_n - \xi^*\|^2 - \alpha_n\|u_{n-1} - \xi^*\|^2 + \alpha_n(1 + \alpha_n)\|u_n - u_{n-1}\|^2 \\ &\quad - \frac{1}{2}(1 - 2L_2\lambda_n - 4L_1\lambda_n)\|u_{n+1} - u_n\|^2 + 4L_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\ &\quad - (1 - \alpha_n)\|u_{n+1} - u_n\|^2 - (\alpha_n^2 - \alpha_n)\|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \quad (3.27)\\ &\leq (1 + \alpha_n)\|u_n - \xi^*\|^2 - \alpha_n\|u_{n-1} - \xi^*\|^2 + 4L_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\ &\quad - (\frac{1}{2} - L_2\lambda_n - 2L_1\lambda_n - \alpha_n)\|u_{n+1} - u_n\|^2 + 2\alpha_n\|u_n - u_{n-1}\|^2 \quad (3.28)\\ &\leq (1 + \alpha_{n+1})\|u_n - \xi^*\|^2 - \alpha_n\|u_{n-1} - \xi^*\|^2 + 4L_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\ &\quad - \rho_n\|u_{n+1} - u_n\|^2 + \zeta_n\|u_n - u_{n-1}\|^2, \quad (3.29)\end{aligned}$$

where

$$\rho_n = \Big(\frac{1}{2} - L_2\lambda_n - 2L_1\lambda_n - \alpha_n\Big)$$

and $\zeta_n = 2\alpha_n$. Next, we substitute

$$\Omega_n = \|u_n - \xi^*\|^2 - \alpha_n\|u_{n-1} - \xi^*\|^2 + 4L_1\lambda_n\|u_n - v_{n-1}\|^2.$$

From the above substitution the Expression (3.29) becomes

$$\begin{aligned}\Omega_{n+1} &\leq \Omega_n - 2\gamma\lambda_n\|v_n - \xi^*\|^2 - \rho_n\|u_{n+1} - u_n\|^2 + \zeta_n\|u_n - u_{n-1}\|^2 \\ &\leq \Omega_n - \rho_n\|u_{n+1} - u_n\|^2 + \zeta_n\|u_n - u_{n-1}\|^2.\end{aligned} \quad (3.30)$$

Furthermore, we assume that

$$\Pi_n = \Omega_n + \zeta_n\|u_n - u_{n-1}\|^2.$$

It follows from above expression and (3.30) such that

$$\Pi_{n+1} - \Pi_n = -(\rho_n - \zeta_{n+1})\|u_{n+1} - u_n\|^2, \quad (3.31)$$

continue to follows the above expression and compute

$$\begin{aligned}\rho_n - \zeta_{n+1} &= \frac{1}{2} - L_2\lambda_n - 2L_1\lambda_n - \alpha_n - 2\alpha_{n+1} \\ &\geq \frac{1}{2} - L_2\lambda_n - 2L_1\lambda_n - 3\alpha \\ &= \frac{1}{2} - \lambda_n(L_2 + 2L_1) - 3\alpha.\end{aligned} \quad (3.32)$$

Since $\lambda_n \to 0$, there is $N_0 \in \mathbb{N}$, gives that

$$0 < \lambda_n < \frac{\frac{1}{2} - 3\alpha}{L_2 + 2L_1}, \ \forall n \geq N_0.$$

From above fact we have

$$\rho_n - \zeta_{n+1} \geq 0, \ \text{for all } n \geq N_0. \tag{3.33}$$

Expression (3.31) and (3.33) implies that

$$\Pi_{n+1} - \Pi_n = -\delta \|u_{n+1} - u_n\|^2 \leq 0, \ n \geq N_0, \ \text{for some}\, \delta \geq 0. \tag{3.34}$$

The above implies that the sequence $\{\Pi_n\}$ is non-increasing for $n \geq N_0$. By the value of Π_n, we achieve

$$\begin{aligned} \|u_n - \xi^*\|^2 &\leq \Pi_n + \alpha_n \|u_{n-1} - \xi^*\|^2 \\ &\leq \Pi_{N_0} + \alpha \|u_{n-1} - \xi^*\|^2 \\ &\leq \cdots \leq \Pi_{N_0}(\alpha^{n-N_0} + \cdots + 1) + \alpha^{n-N_0}\|u_{N_0} - \xi^*\|^2 \\ &\leq \frac{\Pi_{N_0}}{1-\alpha} + \alpha^{n-N_0}\|u_{N_0} - \xi^*\|^2. \end{aligned} \tag{3.35}$$

Similarly, the value of Π_{n+1} and by above expression, we have

$$\begin{aligned} -\Pi_{n+1} &\leq \alpha_{n+1}\|u_n - \xi^*\|^2 \\ &\leq \alpha \|u_n - \xi^*\|^2 \\ &\leq \alpha \frac{\Pi_{N_0}}{1-\alpha} + \alpha^{n-N_0+1}\|u_{N_0} - \xi^*\|^2 \\ &\leq \alpha \frac{\Pi_{N_0}}{1-\alpha} + \|u_{N_0} - \xi^*\|^2. \end{aligned} \tag{3.36}$$

It follows from (3.34) and (3.36) that

$$\begin{aligned} \delta \sum_{n=N_0}^{k} \|u_{n+1} - u_n\|^2 \leq \Pi_{N_0} - \Pi_{k+1} \\ &\leq \Pi_{N_0} + \alpha \frac{\Pi_{N_0}}{1-\alpha} + \|u_{N_0} - \xi^*\|^2 \\ &\leq \frac{\Pi_{N_0}}{1-\alpha} + \|u_{N_0} - \xi^*\|^2. \end{aligned} \tag{3.37}$$

Sending $k \to \infty$ in (3.37) implies that

$$\sum \|u_{n+1} - u_n\|^2 < +\infty \quad \text{implies} \quad \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.38}$$

From (3.25) and (3.38) we get

$$\|u_{n+1} - \xi_n\| \to 0, \ \text{as } n \to \infty. \tag{3.39}$$

The expression (3.36) also implies that

$$-\Omega_{n+1} \leq \alpha \frac{\Pi_{N_0}}{1-\alpha} + \|u_{N_0} - \xi^*\|^2 + \zeta_{n+1}\|u_{n+1} - u_n\|^2. \tag{3.40}$$

Furthermore, the expression (3.22) and (3.24) for $n \geq N_0$, we have

$$\begin{aligned}&\big(1-2L_2\lambda_n-4L_1\lambda_n\big)\Big[\|u_{n+1}-v_n\|^2+\|u_n-v_n\|^2\Big]\\&\leq \Omega_n-\Omega_{n+1}+\alpha(1+\alpha)\|u_n-u_{n-1}\|^2+\|u_{n+1}-u_n\|^2.\end{aligned} \tag{3.41}$$

Now, we fix a number a natural number $k \geq N_0$, and consider the above inequality for all number $N_0, N_0+1, \ldots, k$. Summarizing using (3.40), we obtain

$$\begin{aligned}&\big(1-2L_2\lambda_n-4L_1\lambda_n\big)\sum_{n=N_0}^{k}\Big[\|u_{n+1}-v_n\|^2+\|u_n-v_n\|^2\Big]\\&\leq \Omega_{N_0}-\Omega_{k+1}+\alpha(1+\alpha)\sum_{n=N_0}^{k}\|u_n-u_{n-1}\|^2+\sum_{n=N_0}^{k}\|u_{n+1}-u_n\|^2\\&\leq \Omega_{N_0}+\alpha\frac{\Pi_{N_0}}{1-\alpha}+\|u_{N_0}-\xi^*\|^2+\zeta_{k+1}\|u_{k+1}-u_k\|^2\\&\quad+\alpha(1+\alpha)\sum_{n=N_0}^{k}\|u_n-u_{n-1}\|^2+\sum_{n=N_0}^{k}\|u_{n+1}-u_n\|^2\end{aligned} \tag{3.42}$$

sending $k \to \infty$, gives that

$$\sum_n \|u_{n+1}-v_n\|^2 < +\infty, \text{ and } \sum_n \|u_n-v_n\|^2 < +\infty, \tag{3.43}$$

and

$$\lim_{n\to\infty}\|u_{n+1}-v_n\| = \lim_{n\to\infty}\|u_n-v_n\| = 0. \tag{3.44}$$

By using above definitions we can easily infer the following.

$$\lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|u_n-\xi_n\| = \lim_{n\to\infty}\|v_{n-1}-v_n\| = 0. \tag{3.45}$$

Furthermore, the expression (3.27) with (3.38), (3.43) and Lemma 2.5, implies that

$$\lim_{n\to\infty}\|u_n-\xi^*\| = l. \tag{3.46}$$

The expression (3.45) gives that

$$\lim_{n\to\infty}\|\xi_n-\xi^*\| = \lim_{n\to\infty}\|v_n-\xi^*\| = l. \tag{3.47}$$

Now, we are showing that sequence $\{u_n\}$ converges strongly to ξ^*. The condition on λ_n for each $n \geq N_0$, and the following is still holds (even when $\alpha_n = 0$)

$$0 < \lambda_n < \frac{1}{2L_2+4L_1}, \ \forall n \geq N_0.$$

It follows from above expression and Lemma 3.5, we obtain

$$\begin{aligned}2\gamma\lambda_n\|v_n-\xi^*\|^2 &\leq \|\xi_n-\xi^*\|^2-\|u_{n+1}-\xi^*\|^2+4L_1\lambda_n\|u_n-v_{n-1}\|^2\\&\quad+\|u_{n+1}-u_n\|^2, \ \forall n \geq N_0.\end{aligned}$$

From the expression (3.24) and (3.47) implies that

$$\begin{aligned}2\gamma\lambda_n\|v_n-\xi^*\|^2 \leq &-\|u_{n+1}-\xi^*\|^2+(1+\alpha_n)\|u_n-\xi^*\|^2-\alpha_n\|u_{n-1}-\xi^*\|^2\\&+\alpha_n(1+\alpha_n)\|u_n-u_{n-1}\|^2+4L_1\lambda_n\|u_n-v_{n-1}\|^2+\|u_{n+1}-u_n\|^2\end{aligned}$$

$$\le (\|u_n - \xi^*\|^2 - \|u_{n+1} - \xi^*\|^2) + 2\alpha\|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 + (\alpha_n\|u_n - \xi^*\|^2 - \alpha_{n-1}\|u_{n-1} - \xi^*\|^2) + 4L_1\lambda_n\|u_n - v_{n-1}\|^2. \tag{3.48}$$

From the above expression with (3.38) and (3.43) implies that

$$\begin{aligned}
&\sum_{n=N_0}^{k} 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\
&\le (\|u_{N_0} - \xi^*\|^2 - \|u_{k+1} - \xi^*\|^2) + 2\alpha\sum_{n=N_0}^{k}\|u_n - u_{n-1}\|^2 + \sum_{n=N_0}^{k}\|u_{n+1} - u_n\|^2 \\
&\quad + (\alpha_k\|u_k - \xi^*\|^2 - \alpha_{N_0-1}\|u_{N_0-1} - \xi^*\|^2) + \frac{4L_1}{2L_2 + 4L_1}\sum_{n=N_0}^{k}\|u_n - v_{n-1}\|^2 \\
&\le \|u_{N_0} - \xi^*\|^2 + \alpha\|u_k - \xi^*\|^2 + 2\alpha\sum_{n=N_0}^{k}\|u_n - u_{n-1}\|^2 \\
&\quad + \frac{4L_1}{2L_2 + 4L_1}\sum_{n=N_0}^{k}\|u_n - v_{n-1}\|^2 + \sum_{n=N_0}^{k}\|u_{n+1} - u_n\|^2 \\
&\le M,
\end{aligned}$$

for $M \ge 0$. This means that

$$\sum_{n=1}^{\infty} 2\gamma\lambda_n\|v_n - \xi^*\|^2 < +\infty. \tag{3.49}$$

By Lemma 2.3 and (3.49) such that

$$\liminf \|v_n - \xi^*\| = 0. \tag{3.50}$$

Finally, by (3.46) and (3.50) implies that $\lim_{n\to\infty}\|u_n - \xi^*\| = 0$. This completes the proof. ∎

4. Numerical Experiments

We suppose that in a Nash-Cournot oligopolistic model of equilibrium there are n companies [3, 11]. Let u be a vector whose entry u_i denotes the volume of the products produced by the company i. Next, we consider that price $p_i(s)$ is a decreasing function of $s = \sum_{i=1}^{n} u_i$ such that $p_i(s) = \alpha_i - \beta_i s$, while $\alpha_i, \beta_i > 0$. Thus, the profits contributed by the company i is given by $F_i(u) = p_i(s)u_i - c_i(u_i)$ while $c_i(u_i)$ is the tax and fee for contributing u_i. Let $C_i = [u_{i,min}, u_{i,max}]$ is the policy set of firm i. Therefore the action set of the model is $C = C_1 \times \cdots \times C_n$. In fact, every company aims to achieve maximum its earnings. The standard approach towards this model is premised on the renowned Nash equilibrium idea. We're recalling that point $\xi^* \in C = C_1 \times \cdots \times C_n$ is an equilibrium point of the model if

$$F_i(\xi^*) \ge F_i(\xi^*[x_i]), \forall x_i \in C_i,\ \forall i = 1, ..., n,$$

while $\xi^*[u_i]$. It continues to stand for the vector acquired from ξ^* by replacing ξ_i^* with u_i. By taking $F(u,v) = \Psi(u,v) - \Psi(u,u)$ with $\Psi(u,v) = -\sum_{i=1}^{n} F_i(u[v_i])$. The challenge of seeking a Nash equilibrium point of the model could be developed as continues to follow:

$$\text{Determine } \xi^* \in C \text{ such that } \forall\, v \in C, F(\xi^*, v) \geq 0.$$

Herein, the bifunction F convert into the following way:

$$F(u,v) = \langle Pu + Qv + q, v - u\rangle,$$

while $q \in R^n$ and P, Q are matrices of order n such that Q is symmetric positive semi-definite and $Q - P$ is symmetric negative semi-definite. However, unlike [30–32]. Due to $Q - P$, if $F(u,v) \geq 0$, we have

$$\begin{aligned} F(v,u) &\leq F(v,u) + F(u,v) \\ &= \langle Pv + Qu + q, u - v_i\rangle + \langle Pu + Qv + q, v - u_i\rangle \\ &= \langle (P-Q)v + (Q-P)u, u - v_i\rangle \\ &= (u-v)^T(Q-P)(u-v) \\ &\leq -\gamma\|u-v\|^2 \end{aligned}$$

where $\gamma > 0$. This proves that f is strongly pseudomonotone.

We write our algorithm using Matlab programs (Matlab R2018 b) and computed on a PC Intel(R) Core(TM) i7 @ 1.80 GHz, Ram 8.00 GB. The feasible set $C = \{u \in R^5 : -5 \leq u_i \leq 5\}$, is a box inside R^5, the vector q is produced randomly and taking values in the closed interval $[-m, m]$ and P, Q randomly generated. Thus, for our experiment Algorithm 1 used and it uses four sets of stepsize sequences $\{\lambda_n\}$ which are define below:

(I) $\lambda_n = \frac{1}{(n+2)^p}, \quad p \in \{0.3, 0.5, 0.8, 1.0\};$

(II) $\lambda_n = \frac{1}{\log^p(n+3)}, \quad p \in \{0.6, 1.0, 3.0, 5.0\};$

(III) $\lambda_n = \frac{1}{(n+1)\log^p(n+3)}, \quad p \in \{0.6, 1.0, 3.0, 5.0\};$

(IV) $\lambda_n = \frac{\log^p(n+1)}{(n+1)}, \quad p \in \{0.1, 0.5, 1.0, 4.0\}.$

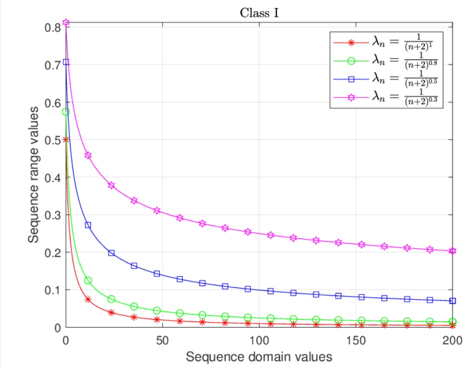
All sequence $\{\lambda_n\}$ are to meet the conditions $(K1)$ and $(K2)$. The behaviour of these four classes of stepsize λ_n defined above for the first 200 iterations you can observe in Figures 1. Next, Figures 2-9 shows the numerical results shows the relation of error term with number of iteration and elapsed time in seconds for Algorithm 1, for the 200 first iterations, using the stepsize sequences $\{\lambda_n\}$ in class (I), (II), (III) and (IV), respectively. Actually, we want to see that what is the impact of stepsize sequence $\{\lambda_n\}$ on the convergence of the iterative sequence. We give the following observation regarding our concern.

(i) We can see that the convergence of rate iterative sequence $\{u_n\}$ developed by Algorithm 1, strictly depends on the convergence rate of the stepsize sequence.

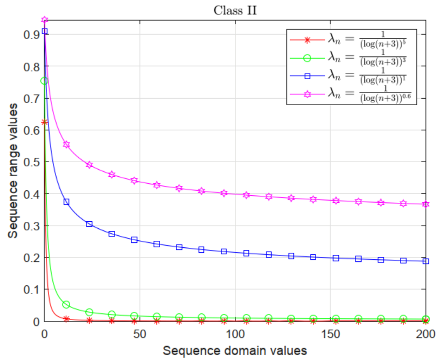
(ii) The stepsize sequence $\{\lambda_n\}$ which are slowly convergent has a great effect on D_n, for early iteration D_n going to decrease quickly, but after that, it's going to unstable.

(iii) The stepsize sequence $\{\lambda_n\}$ which are fast convergent to zero, the error term looks more stable for all iterations.

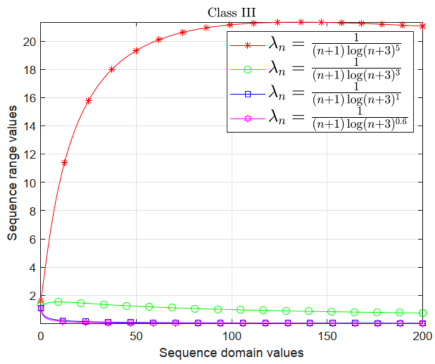
Furthermore, Figures 10-17 shows the comparison between the Algorithm 1, Algorithm 2 [33] and Algorithm 3 [33] in the term of number of iterations and execution time with respect to error term using the stepsize sequence $\{\lambda_n\}$.



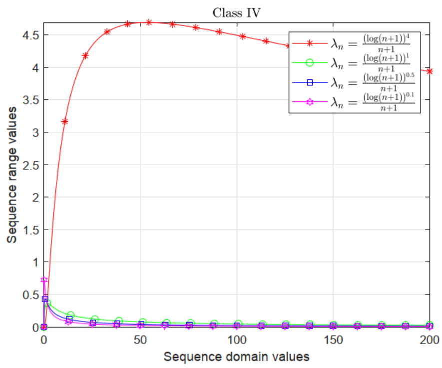
(A) $\lambda_n = \frac{1}{(n+2)^p}$



(B) $\lambda_n = \frac{1}{\log^p(n+3)}$



(C) $\lambda_n = \frac{1}{(n+1)\log^p(n+3)}$



(D) $\lambda_n = \frac{\log^p(n+1)}{(n+1)}$

FIGURE 1. The behaviour of stepsize sequence λ_n for the first 200 iterations.

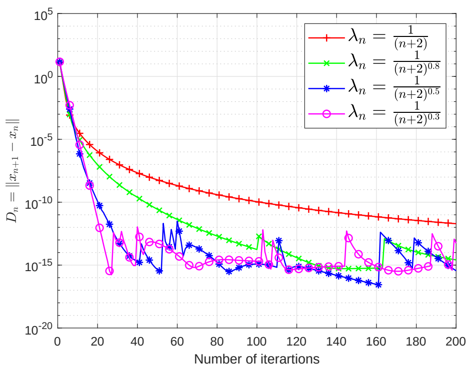


FIGURE 2. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{(n+2)^p}$.

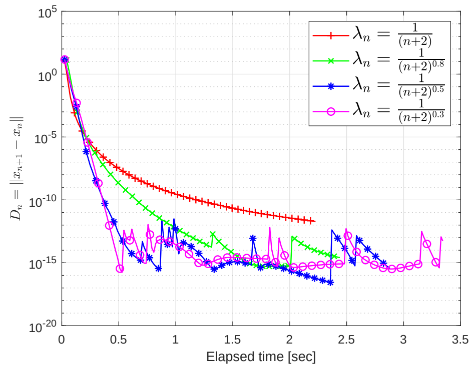


FIGURE 3. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{(n+2)^p}$.

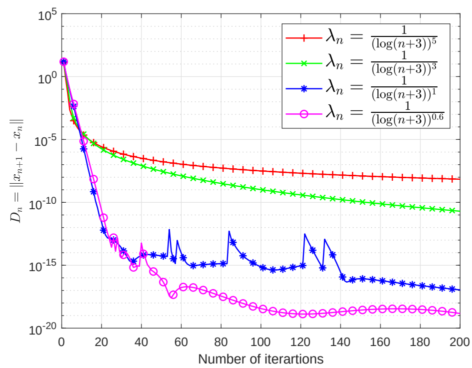


FIGURE 4. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{\log^p(n+3)}$.

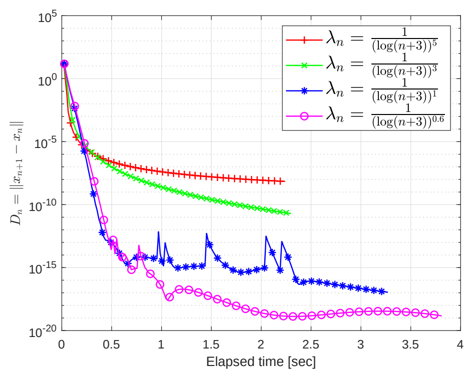


FIGURE 5. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{\log^p(n+3)}$.

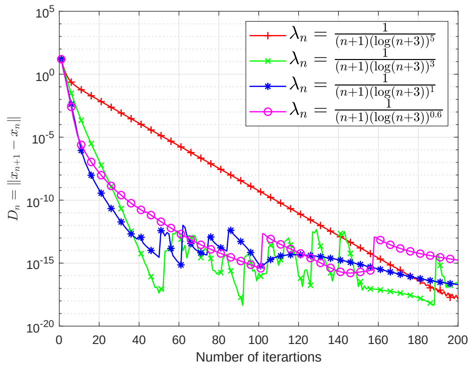


FIGURE 6. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{(n+1)\log^p(n+3)}$.

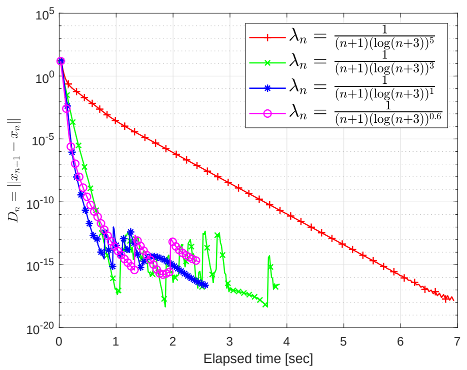


FIGURE 7. The behaviour of Algorithm 1 using $\lambda_n = \frac{1}{\log^p(n+3)}$.

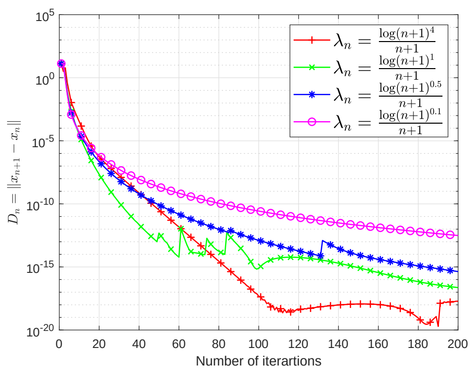


FIGURE 8. The behaviour of Algorithm 1 using $\lambda_n = \frac{\log^p(n+1)}{(n+1)}$.

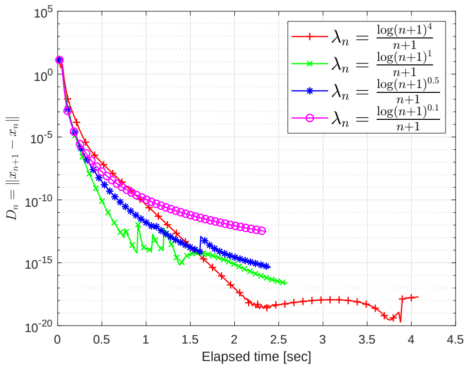


FIGURE 9. The behaviour of Algorithm 1 using $\lambda_n = \frac{\log^p(n+1)}{(n+1)}$.

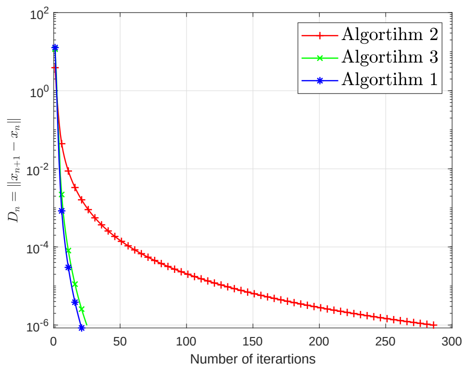


FIGURE 10. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{n+2}$.

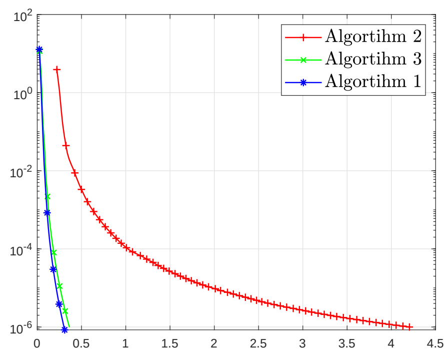


FIGURE 11. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{n+2}$.

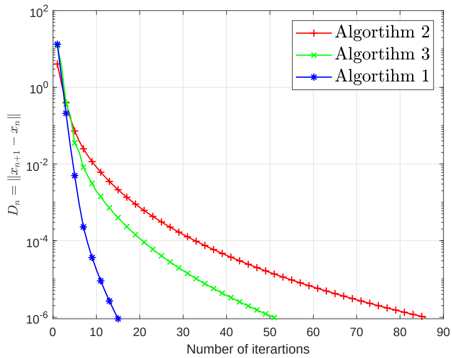


FIGURE 12. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.8}}$.

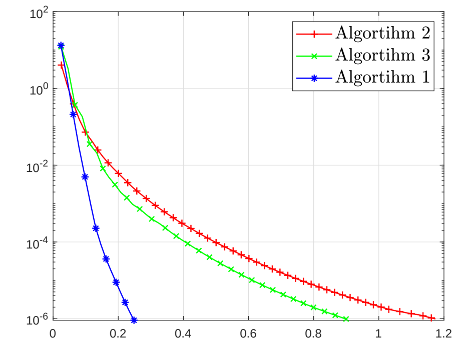


FIGURE 13. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.8}}$.

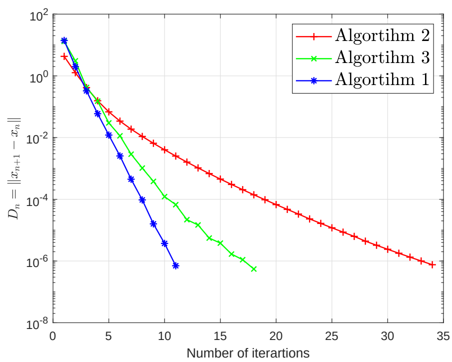


FIGURE 14. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.5}}$.

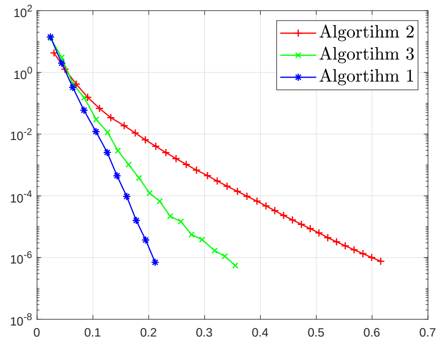


FIGURE 15. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.5}}$.

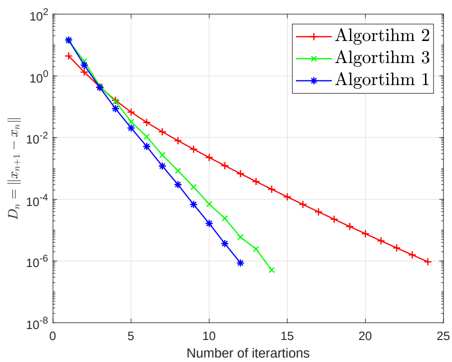


FIGURE 16. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.3}}$.

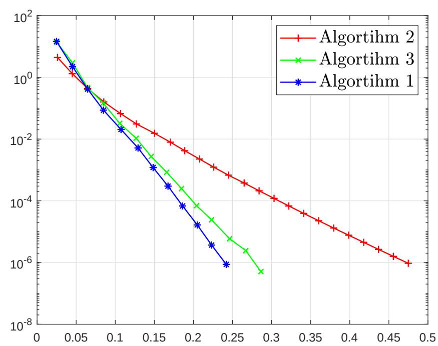


FIGURE 17. The comparison of algorithms using the stepsize sequence $\lambda_n = \dfrac{1}{(n+2)^{0.3}}$.

Acknowledgments

The first author was supported by Rajabhat Rajanagarindra University (RRU).

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