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An Optimal Control Technique for Epidemiological Model with Limited Vaccination Supply

Printaporn Sanguansuttigul

Department of Mathematics and Computer Science, Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
39 Moo 1, Klong hok, Khlong Luang, Pathum Thani 12110, Thailand
e-mail : printaporn_s@rmutt.ac.th

Abstract In this paper, the author proposes an optimal control model describing the containment of an outbreak used for determining an optimal vaccination schedule where the vaccination supply is limited within a given time period. The model also takes into account different age groups, which incorporates varying contact rates within and across the groups. The optimality is provided using the Pontryagin maximal principle via an adjoint system. A numerical example is then presented based on the age-specific contact behaviors and dermographic data of Bangkok to illustrate how the model can be used to obtain the optimal schedule.

MSC: 34H05; 34H15; 92B05; 92D25

Keywords: optimal control; pontryagin maximal principle; epidemiological model; limited vaccination supply

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1. Introduction

An infectious diseases, also known as a transmissible disease, is caused by organisms for example bacteria, viruses, fungi or parasites. There are different types of infectious diseases, some infectious diseases can be passed directly from one person to another while some are transmitted by insects or other vector animals. Many infectious diseases can be prevented by vaccines such as influenza, chickenpox etc.

In late 2019, the world is facing a global pandemic. The first confirmed case started in Wuhan, China, then the Chinese government announced severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) variants, known more commonly as the coronavirus disease 2019 (COVID-19) [7, 15]. COVID-19 rapidly spreading all countries around the world. The World Health Organization (WHO) informed about the COVID-19 is an infectious disease caused by a newly discovered coronavirus and could spread from person to person [6]. As of today, most countries around the world have been affected from COVID-19. In December of 2020, the Alpha variant was detected in the UK [5] and in

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middle of 2021, So many countries cannot control the COVID-19, the Delta variant being the most concerning variant so far because almost all of countries reinfect with delta variant after recovery from breakthrough infection by alpha variant.

To effectively slow down and reduce infection rate of the SARS-CoV-2 pandemic, so many countries have locked down since the end of January 2020. Other means of protection include wearing a mask, keeping social distance and increasing oneself's hygiene. However, these approaches are just temporary procedures to confront and contain the dreadful disease with enormous direct and indirect costs. As of now, one have observe the massive impacts of such harsh measures including global economical collapse, rise in poverty, human rights violations, mental health problems, big halts in social and academic activities, etc. We have also seen a huge improvement over the recent period where vaccination enters the stage and helps reducing the severity of the symptoms as well as the death rate. It was also obvious that the availability of the vaccine is never sufficient to the needs and so many people would need to wait for their turn to get vaccinated.

With the above motivation, the author propose a systematic study to determine an optimal vaccination schedule, under the supply limitation constraint in mind, using an optimal control technique attached to an epidemiological model that takes into account the age-group strucutres. This special structure is adopted as we perceive different activities and behaviors as well as the difference in how the body reacts to the disease in people from different age groups. The general analysis regarding the stability, reproduction number, and the state-adjoint dynamics were carried out using eigenvalue criteria, next-generation method, and Pontryagin maximal principle. We also simulate our model numerically the COVID-19 situation with age-specific dataset in Thailand and discuss the resulting simulations.

2. Model Description

2.1. Epidimiological Dynamics

The infectious disease model is a mathematical tool that has been widely used to study the mechanisms by which a disease spreads, to predict the future course of an outbreak and to evaluate strategies to control an epidemic [8]. We consider the infectious diseases model based on [9], as described visually by Figure 1 and analytically by (2.1). In our model, when a susceptible individual (in S) got exposed to the disease, he/she will be moved to exposed state E, and then to symptomatic (I) or asymptomatic (A) infectious states. An asymptomatic infected individual is assumed to recover (the state R) by themselves, while a symptomatic infected individual would need to go to a quarantine (Q), before recovering (going to R) or dying (going to D). Every recovered individuals would eventually lose their immunity and would again become susceptible (S). We also introduce a vaccinated state V, where a vaccinated induvidual can still get the disease, but with only a mild symptom, similar to those in state A. This also means that the affected vaccinated individual would be able to recover by oneself. In this paper, we assume that the vaccinated individuals would not lose their protections over time, which can also be corresponded to the scenario where the vaccinated ones would get their next doses iteratively. This comparmental structure is then repeated in all the age groups. For simplicity, we would adopt the following four age groups in our model: (1) 0-19 years, (2) 20-39 years, (3) 40-59 years and (4) 60+ years. The different age-group structure can also be adopted and analyzed with the same technique presented in the sequel of this paper.

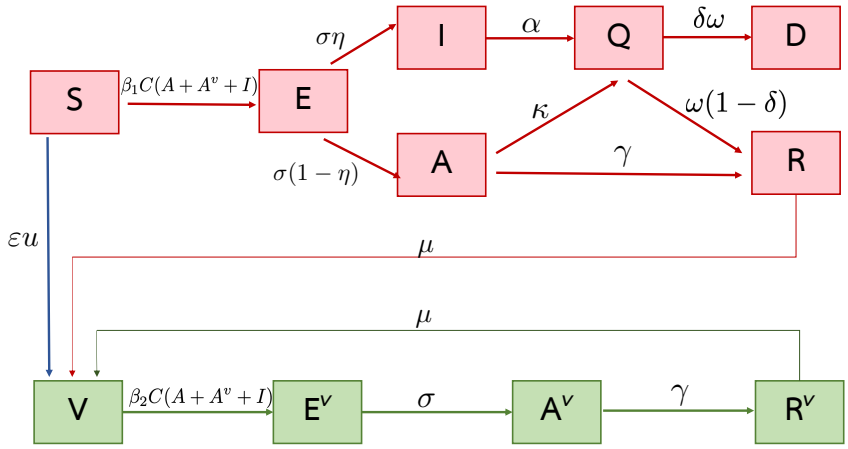

FIGURE 1. Infectious diseases model.

From Figure 1, the following dynamical system is derived:

$$\frac{dS_i}{dt} = \mu R_i - \varepsilon u S_i - \frac{\beta_1}{N}\sum_{j=1}^{K} C_{ij} S_i (A_j + A_j^v + I_j) \tag{2.1a}$$

$$\frac{dE_i}{dt} = -\sigma E_i + \frac{\beta_1}{N}\sum_{j=1}^{K} C_{ij} S_i (A_j + A_j^v + I_j) \tag{2.1b}$$

$$\frac{dI_i}{dt} = \eta_i \sigma E_i - \alpha I_i \tag{2.1c}$$

$$\frac{dA_i}{dt} = (1-\eta_i)\sigma E_i - \gamma A_i - \kappa A_i \tag{2.1d}$$

$$\frac{dQ_i}{dt} = \alpha I_i - \delta_i \omega Q_i + \kappa A_i \tag{2.1e}$$

$$\frac{dR_i}{dt} = \gamma A_i + (1-\delta_i)\omega Q_i - \mu R_i \tag{2.1f}$$

$$\frac{dV_i}{dt} = \mu R_i^v + \varepsilon u S_i - \frac{\beta_2}{N}\sum_{j=1}^{K} C_{ij} V_i (A_j + A_j^v + I_j) \tag{2.1g}$$

$$\frac{dE_i^v}{dt} = -\sigma E_i^v + \frac{\beta_2}{N}\sum_{j=1}^{K} C_{ij} V_i (A_j + A_j^v + I_j) \tag{2.1h}$$

$$\frac{dA_i^v}{dt} = \sigma E_i^v - \gamma A_i^v \tag{2.1i}$$

$$\frac{dR_i^v}{dt} = \gamma A_i^v - \mu R_i^v \tag{2.1j}$$

$$\frac{dD_i}{dt} = \delta_i \omega Q_i \tag{2.1k}$$

2.2. Limited Vaccination Supply Scenario

To introduce a supply limit constraint of avalilable vaccines, we need to introduce a virtual state W describing the total number of individuals that have been vaccinated so far. If $X \geq 0$ denotes the number of available vaccines during the period $[0, T]$, we may formulate the dynamics of W as well as their boundary conditions by

$$\frac{dW}{dt} = (S_1(t) + S_2(t) + S_3(t) + S_4(t))u(t) \tag{2.2a}$$

$$W(0) = 0 \tag{2.2b}$$

$$W(T) \leq X. \tag{2.2c}$$

This state should be added to the system (2.1) to form a correct dynamics in the limited vaccination supply scenario. Also note that we allows the leftover vaccine stocks. This is explainable as the cost of vaccination (u^2 in the integrand) may be overcame by the already small affected cases.

2.3. Optimal Control Model of Vaccination Schedule

A dynamical system, like the one we used to describe the states of infection of a disease, is a system that follows some particular rules of the dynamics given by ODEs. Hence, we cannot control the states directly to meet our preference. However, this can be done indirectly via the control variable. Optimal control was originally an engineering topic that enables the optimal selection of control among the vastly available choices in order to get a desirable states (see e.g. [2, 16–18]). The technique is mostly popular with the electrical engineers for several reasons. One may think of the effect of a control as a conditioning which will drive the systems to produce the best outcome. Several events in systems biology can be affected by the human activities and decisions [10–12]. The authors consider control the spread of an infectious disease which can be achieved through various vaccination strategies. We emphasize the optimal control method with disease model to present the vaccination schedule when vaccination supplies are limited. We used the mathematical model to explain the behaviour of infectious diseases and show the outcome of an epidemic. The modelling can also predict future growth patterns, which can help public health.

In our model (2.1), $u = u(t)$ denotes the percentage of susceptible is individuals being vaccinated per unit of time. If $u = 0$, then no vaccination is done and $u = 1$ indicates that all susceptible population is vaccinated [4]. We consider in this paper an optimal vaccination schedule, which explains the rate of vaccination over the time period $[0, T]$. The following optimal control problem is then used to derive an optimal vaccination schedule:

$$\begin{cases} \min\limits_{u} \quad J[u] := \displaystyle\int_0^T a \sum_{i=1}^{K} (I_i(t) + A_i(t) + Q_i(t)) + u^2(t)dt \\ \text{s.t.} \quad \text{(2.1a)–(2.1k) and (2.2a)–(2.2c)}, \end{cases} \tag{2.3}$$

where $a > 0$ is a weight constant for the number of sick individuals. If the weight a is close to 0, then we expect to see less vaccination. Otherwise, the vaccination rate increases with the higher value of a.

3. Model Analysis

3.1. Basic Reproduction Number

In this section, we calculate the basic reproduction number or $\mathscr{R}_0$ which is the number of secondary infections occurring in population. This number represents the number of individuals that could be infected from a single infectious individual in a population, hence determining the spreadability of the disease. In epidemiological modelling, the next-generation matrix method is used to derive $\mathscr{R}_0$ for a compartmental model of the spread of infectious diseases.

To find $\mathscr{R}_0$, we use the formula

$$\mathscr{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}), \tag{3.1}$$

where ρ denotes the spectral radius of the corresponding matrix. Here, the matrices $\mathcal{F}$ and $\mathcal{V}$ are called the transmission and transition parts, respectively. The transmission part describes the newly infected while the transition part describes changing of states. The product $\mathcal{F}\mathcal{V}^{-1}$ is then called the next-generation matrix.

For our model (2.1), we may compute the matrices $\mathcal{F}$ and $\mathcal{V}$ by

$$\mathcal{F} = \begin{bmatrix} \frac{\partial f_{E,i}}{\partial E_i} & \frac{\partial f_{E,i}}{\partial I_i} & \frac{\partial f_{E,i}}{\partial A_i} & \frac{\partial f_{E,i}}{\partial E_i^v} & \frac{\partial f_{E,i}}{\partial A_i^v} \\ \frac{\partial f_{I,i}}{\partial E_i} & \frac{\partial f_{I,i}}{\partial I_i} & \frac{\partial f_{I,i}}{\partial A_i} & \frac{\partial f_{I,i}}{\partial E_i^v} & \frac{\partial f_{I,i}}{\partial A_i^v} \\ \frac{\partial f_{A,i}}{\partial E_i} & \frac{\partial f_{A,i}}{\partial I_i} & \frac{\partial f_{A,i}}{\partial A_i} & \frac{\partial f_{A,i}}{\partial E_i^v} & \frac{\partial f_{A,i}}{\partial A_i^v} \\ \frac{\partial f_{E^v,i}}{\partial E_i} & \frac{\partial f_{E^v,i}}{\partial I_i} & \frac{\partial f_{E^v,i}}{\partial A_i} & \frac{\partial f_{E^v,i}}{\partial E_i^v} & \frac{\partial f_{E^v,i}}{\partial A_i^v} \\ \frac{\partial f_{A^v,i}}{\partial E_i} & \frac{\partial f_{A^v,i}}{\partial I_i} & \frac{\partial f_{A^v,i}}{\partial A_i} & \frac{\partial f_{A^v,i}}{\partial E_i^v} & \frac{\partial f_{A^v,i}}{\partial A_i^v} \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \frac{\partial v_{E,i}}{\partial E_i} & \frac{\partial v_{E,i}}{\partial I_i} & \frac{\partial v_{E,i}}{\partial A_i} & \frac{\partial v_{E,i}}{\partial E_i^v} & \frac{\partial v_{E,i}}{\partial A_i^v} \\ \frac{\partial v_{I,i}}{\partial E_i} & \frac{\partial v_{I,i}}{\partial I_i} & \frac{\partial v_{I,i}}{\partial A_i} & \frac{\partial v_{I,i}}{\partial E_i^v} & \frac{\partial f_{I,i}}{\partial A_i^v} \\ \frac{\partial v_{A,i}}{\partial E_i} & \frac{\partial v_{A,i}}{\partial I_i} & \frac{\partial v_{A,i}}{\partial A_i} & \frac{\partial v_{A,i}}{\partial E_i^v} & \frac{\partial f_{A,i}}{\partial A_i^v} \\ \frac{\partial v_{E^v,i}}{\partial E_i} & \frac{\partial v_{E^v,i}}{\partial I_i} & \frac{\partial v_{E^v,i}}{\partial A_i} & \frac{\partial v_{E^v,i}}{\partial E_i^v} & \frac{\partial v_{E^v,i}}{\partial A_i^v} \\ \frac{\partial v_{A^v,i}}{\partial E_i} & \frac{\partial f_{A^v,i}}{\partial I_i} & \frac{\partial v_{A^v,i}}{\partial A_i} & \frac{\partial v_{A^v,i}}{\partial E_i^v} & \frac{\partial v_{A^v,i}}{\partial A_i^v} \end{bmatrix},$$

where f and v are given by

$$f = \begin{bmatrix} f_{E,i} \\ f_{I,i} \\ f_{A,i} \\ f_{E^v,i} \\ f_{A^v,i} \end{bmatrix} = \begin{bmatrix} \frac{\beta_1}{N}\sum_{j=1}^{K} C_{ij}S_i(A_j + A_j^v + I_j) \\ 0 \\ 0 \\ \frac{\beta_2}{N}\sum_{j=1}^{K} C_{ij}V_i(A_j + A_j^v + I_j) \\ 0 \end{bmatrix}$$

and

$$v = \begin{bmatrix} v_{E,i} \\ v_{I,i} \\ v_{A,i} \\ v_{E^v,i} \\ v_{A^v,i} \end{bmatrix} = \begin{bmatrix} \sigma E_i \\ \alpha I_i - \eta_i \sigma E_i \\ \gamma A_i + \kappa A_i + (1-\eta_i)\sigma E_i) \\ \sigma E_i^v \\ \gamma A_i^v - \sigma E_i^v \end{bmatrix}.$$

The next-generation matrix is then computed via (3.1), yielding

$$\mathcal{F}\mathcal{V}^{-1} = \left[a_{ij}\right],$$

where the entries are given by

$$a_{ij} = \begin{cases} \dfrac{C_{i,j-4\left\lfloor \frac{j-1}{4} \right\rfloor} S_i \beta_1 b_j}{N} & \text{where } i \leq 4 \\ \dfrac{C_{i-12,j-4\left\lfloor \frac{j-1}{4} \right\rfloor} V_{i-12} \beta_2 b_j}{N} & \text{where } 13 \leq i \leq 16 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$b_j = \begin{cases} \left(\dfrac{\eta_j}{\alpha} - \dfrac{(\eta_j - 1)}{\gamma + \kappa}\right) & \text{where } j \leq 4 \\ \dfrac{1}{\alpha} & \text{where } 5 \leq j \leq 8 \\ \dfrac{1}{\gamma + \kappa} & \text{where } 9 \leq j \leq 12 \\ \dfrac{1}{\gamma} & \text{where } j \geq 13. \end{cases}$$

Since our system is large, the general form of $\mathscr{R}_0$ is omitted. To this end, we resort to computing the reproduction number only in our selected situation, where the parameters are given as in Table 1.

$$\begin{aligned} \mathscr{R}_0 &= \rho(\mathcal{F}\mathcal{V}^{-1}) \\ &= 4.342 \end{aligned}$$

TABLE 1. Parameter Values

Parameter	Description	Value	Reference
β_1	The effective contact rate	~ 1	[1]
β_2	The effective contact rate	~ 1	[1]
$1/\gamma$	Recovery period	21 Days	[3]
$1/\alpha$	Pre-isolation infection period	4.6 Days	[3]
$1/\omega$	Post-isolation recovery period	16.4 Days	[3]
$1/\mu$	Immunity duration	1 year	Estimated
$1/\sigma$	Latent period	5.1 Days	[13]
κ	Transmission rate state A transfer to Q	0.2 Day^{-1}	Estimated
η	Proportion of symptomatic infections	Age-specific	[21]
δ	Proportion of disease mortality	Age-specific	[14]

3.2. Optimal Control

To solve the optimal control model (2.3), we exploit the well-known Pontryagin's Maximum Principle (see e.g. [19]) where we state as follows for the convenience of the readers.

Theorem 3.1. *If $u^*(t)$ and $x^*(t)$ are optimal for problem (2.3), then there exists variable $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)),$$

at each time, for all u with values in U_1 where the Hamiltonian H is defined by

$$H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t))$$

and

$$\lambda'(t) = -\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x}$$
$$\lambda(t) = 0.$$

Optimal Control method will be used for controlling the spread of an infectious disease with vaccination strategies. This is described by the problem (2.3). Using Pontryagin's Maximum Principle (Theorem 3.1) to find the optimal vaccination schedule with limited supply, we get the Hamiltonian in the form

$$\begin{aligned} H = {} & a \sum_{i=1}^{K} (I_i(t) + A_i(t) + Q_i(t)) + u^2 \\ & + \sum_{i=1}^{K} \lambda_{S_i} \left(\mu R_i - \varepsilon u S_i - \frac{\beta_1}{N} \sum_{j=1}^{K} C_{ij} S_i (A_j + A_j^v + I_j) \right) \\ & + \sum_{i=1}^{K} \lambda_{E_i} \left(-\sigma E_i + \frac{\beta_1}{N} \sum_{j=1}^{K} C_{ij} S_i (A_j + A_j^v + I_j) \right) \\ & + \sum_{i=1}^{K} \lambda_{I_i} (\eta_i \sigma E_i - \alpha I_i) + \sum_{i=1}^{K} \lambda_{A_i} ((1 - \eta_i) \sigma E_i - \gamma A_i - \kappa A_i) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{K}\lambda_{Q_i}(\alpha I_i-\delta_i\omega Q_i+\kappa A_i)+\sum_{i=1}^{K}\lambda_{R_i}(\gamma A_i+(1-\delta_i)\omega Q_i-\mu R_i) \qquad (3.2)\\
&+\sum_{i=1}^{K}\lambda_{V_i}\left(\mu R_i^v+\varepsilon u S_i-\frac{\beta_2}{N}\sum_{j=1}^{K}C_{ij}V_i(A_j+A_j^v+I_j)\right)\\
&+\sum_{i=1}^{K}\lambda_{E_i^v}\left(-\sigma E_i^v+\frac{\beta_2}{N}\sum_{j=1}^{K}C_{ij}V_i(A_j+A_j^v+I_j)\right)\\
&+\sum_{i=1}^{K}\lambda_{A_i^v}(\sigma E_i^v-\gamma A_i^v)+\sum_{i=1}^{K}\lambda_{R_i^v}(\gamma A_i^v-\mu R_i^v)+\sum_{i=1}^{K}\lambda_{D_i}(\delta_i\omega Q_i)\\
&+\lambda_W u\sum_{i=1}^{K}S_i.
\end{aligned}
$$

The values λ_{S_i}, λ_{E_i}, λ_{I_i}, λ_{A_i}, λ_{Q_i}, λ_{D_i}, λ_{R_i}, λ_{V_i}, $\lambda_{E_i^v}$, $\lambda_{A_i^v}$, $\lambda_{R_i^v}$ and λ_W are the associated adjoints for the states S_i, E_i, I_i, A_i, Q_i, D_i, R_i, V_i, E_i^v, A_i^v, R_i^v and W, respectively. By differentiating the Hamiltonian with respect to each state variable, we find the differential equation for the associated adjoints in the following

$$
\begin{aligned}
\lambda'_{S_i}&=\lambda_{S_i}\left(\varepsilon u+\frac{\beta_1}{N}\sum_{j=1}^{K}C_{ij}(A_j+A_j^v+I_j)\right)-\lambda_{V_i}(\varepsilon u)\\
&\quad-\lambda_{E_i}\left(\frac{\beta_1}{N}\sum_{j=1}^{K}C_{ij}(A_j+A_j^v+I_j)\right) \qquad (3.3a)\\
\lambda'_{E_i}&=\lambda_{E_i}(\sigma)-\lambda_{A_i}((1-\eta_i)\sigma)-\lambda_{I_i}(\eta_i\sigma) \qquad (3.3b)\\
\lambda'_{I_i}&=-a+\lambda_{I_i}\alpha-\lambda_{Q_i}\alpha+(\lambda_{S_i}-\lambda_{E_i})\frac{\beta_1}{N}C_{ii}S_i\\
&\quad+(\lambda_{V_i}-\lambda_{E_i^v})\frac{\beta_2}{N}C_{ii}V_i \qquad (3.3c)\\
\lambda'_{A_i}&=-a+\lambda_{A_i}(\gamma+\kappa)-\lambda_{Q_i}(\kappa)-\lambda_{R_i}(\gamma) \qquad (3.3d)\\
&\quad+(\lambda_{S_i}-\lambda_{E_i})\frac{\beta_1}{N}C_{ii}S_i+(\lambda_{V_i}-\lambda_{E_i^v})\frac{\beta_2}{N}C_{ii}V_i \qquad (3.3e)\\
\lambda'_{Q_i}&=-a+\lambda_{Q_i}(\delta_i\omega)-\lambda_{R_i}(\omega(1-\delta_i))-\lambda_{D_i}(\delta_i\omega) \qquad (3.3f)\\
\lambda'_{R_i}&=\lambda_{R_i}\mu \qquad (3.3g)\\
\lambda'_{V_i}&=\lambda_{V_i}\left(\frac{\beta_2}{N}\sum_{j=1}^{K}C_{ij}(A_j+A_j^v+I_j)\right)\\
&\quad-\lambda_{E_i^v}\left(\frac{\beta_2}{N}\sum_{j=1}^{K}C_{ij}(A_j+A_j^v+I_j)\right) \qquad (3.3h)\\
\lambda'_{E_i^v}&=\lambda_{E_i^v}(\sigma)-\lambda_{A_i^v}(\sigma) \qquad (3.3i)
\end{aligned}
$$

$$\lambda'_{A_i^v} = \lambda_{A_i^v}(\gamma) - \lambda_{R_i^v}(\gamma) + (\lambda_{S_i} - \lambda_{E_i})\frac{\beta_1}{N}C_{ii}S_i - (\lambda_{V_i} - \lambda_{E_i^v})\frac{\beta_2}{N}C_{ii}V_i \tag{3.3j}$$

$$\lambda'_{R_i^v} = \lambda_{R_i^v}(\mu) \tag{3.3k}$$

$$\lambda'_{D_i} = 0 \tag{3.3l}$$

$$\lambda'_W = 0. \tag{3.3m}$$

Except λ_W, all the other adjoint variables, denoted generically by λ, are attached with the transversality condition $\lambda(T) = 0$.

4. Numerical Simulations and Discussions

In this section, we simulate our model numerically to illustrate how the model works in general. We simulate using the contact matrix between different age groups in Thailand taken from [20] and the initial population data is taken from the dermographic data provided by National Statistical Office of Thailand. We take the time period of 180 days, which is related to an accepted effective period of any COVID-19 vaccines. We assume that 1,600,000 doses of vaccines are available within this time span.

The optimal control model (2.3) is solved using the forward-backward sweeping technique, which is quite a standard approach to solve an optimal control problem. The approach solves alternatively the state and adjoint variables through the state dynamical system (2.1) and the adjoint dynamical system (3.3). The following Figure 3 shows the result of our simulation.

From our simulations, it turns out that the vaccination should take place at full capacity in the early period (in this case it is 20 percent of the susceptible individuals) and then a quick ramp down means the that we may halt the program almost immediately. This can by understood also by looking at the number of individuals in other states keeping in mind that only susceptible individuals are allowed to get the vaccine. Particularly, one may observe that the number of exposed, infected (symptomatically and asymptomatically) and vaccinated individuals are rising up fast. This results in a quick decreasing in the population eligible to the vaccination.

Another important observation from Figure 3 is the difference in state transitions of different age groups. For instance, the number of individuals in the last age group (age 60+ y.o.) is higher than of the first group (age 0–19 y.o.) but due to their behaviors, the exposure to the disease after vaccination of the young individuals (first group) are significantly higer than the old individuals (fourth group). This causes the same trends for the infected and recovered states after being vaccinated. However, the disease is more fatal to the old individuals from the fourth group.

It is not against our intuition that the maximum vaccination rate would be preferrable in order to contain the outbreak. Setting our mind in this direction, our model is helpful in determining the duration of time that vaciination program is needed. To see this more clearly, compare the main experimental result in Figure 3 to the Figure 2 below where the exposure rate β_1 is assumed to be 10 times less aggressive. As expected, a shorter and smaller vaccination program can be launched and less number of vaccination is required. As a consequence of this simulation, we can spend less money by ordering a smaller stock to contain an outbreak.

Finally, we would also like to add a more technical discussion as well. Solving the adjoint system (3.3) results in λ_W being a constant function, with some undetermined value $k > 0$. In principle, the value k is reversely proportional to the final state $W(T)$. The value of k is, in general, then adjusted by hand until $W(T)$ falls within the scope of satisfaction.

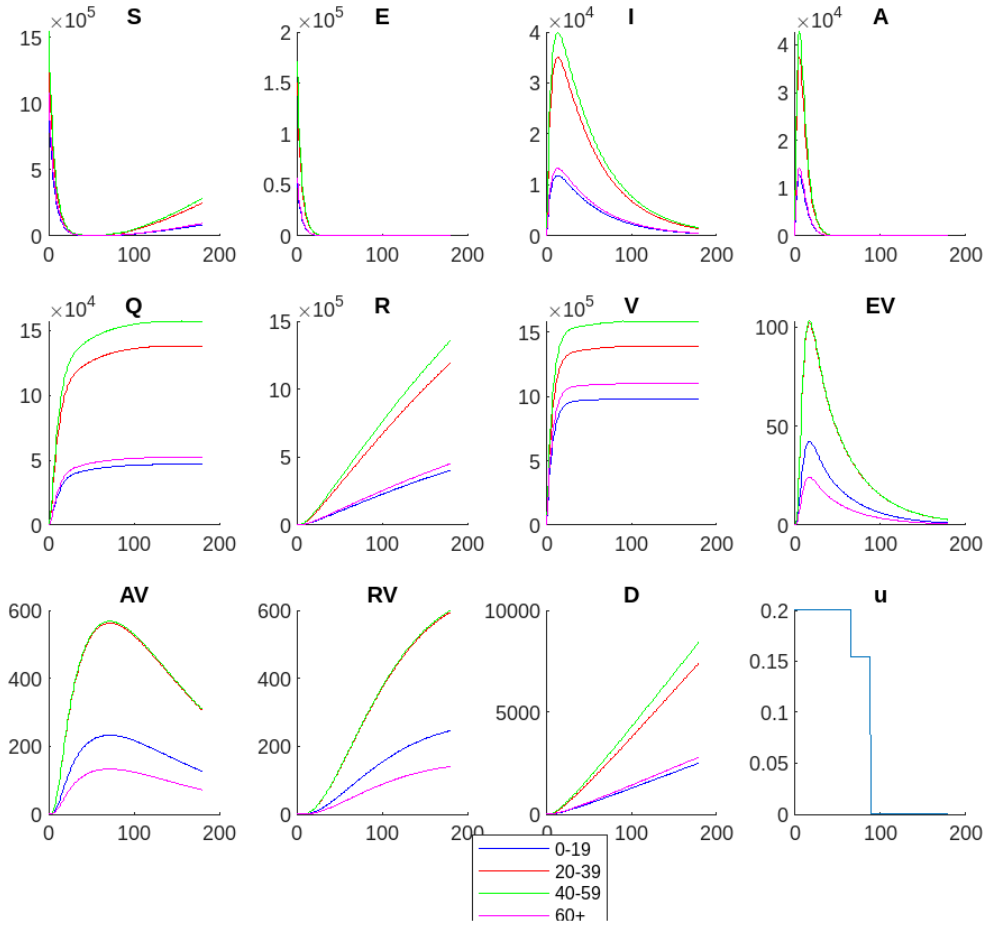

FIGURE 2. States of disease of each age groups and optimal vaccination schedule with β_1 adjusted to be 10 times less aggressive (compared to Figure 3).

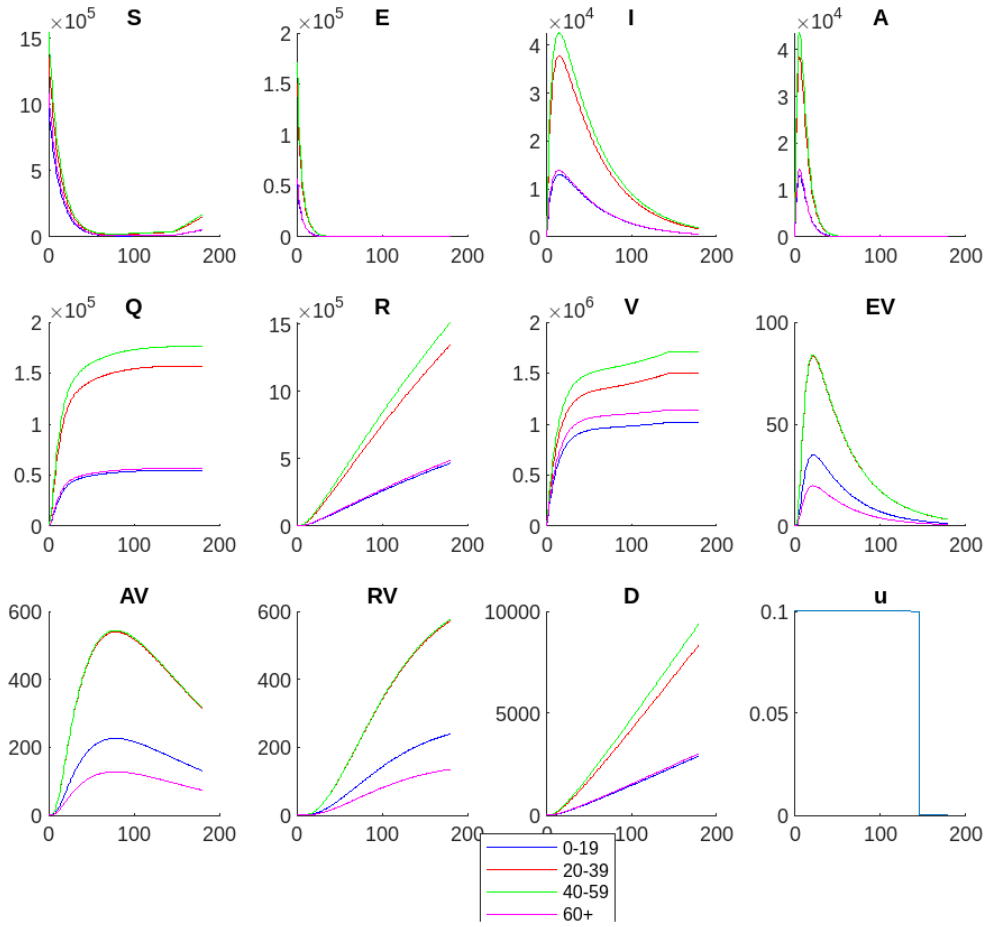

FIGURE 3. States of disease of each age groups and optimal vaccination schedule.

5. Conclusion

In this paper, we have proposed an optimal control model for optimal vaccination schedule paying attention to (1) the limited vaccination supply, (2) the different behaviors of different age groups, and (3) the post-vaccination infection. It was not unexpected to see that the vaccination rate would be maximized and get done as rapid as possible until the immuned population is sufficiently large. Our model is, however, still very helpful in determining the length of the vaccination program within a given scope of time.

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