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Sequential Hilfer-Hadamard Fractional Three-Point Boundary Value Problems

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Abstract In this paper, we study existence and uniqueness of solutions for three-point boundary value problem for Hilfer-Hadamard sequential fractional differential equations, via standard fixed point theorems. The existence is proved by Schaefer and Krasnoselskii fixed point theorems as well and Leray-Schauder nonlinear alternative, while the existence and uniqueness by Banach contraction mapping principle. Illustrative examples are also discussed.

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1. Introduction

The fractional calculus has always been an interesting research topic for many years. This is because, fractional differential equations describe many real world process related to memory and hereditary properties of various materials more accurately as compared to classical order differential equations. Fractional differential equations arise in lots of engineering and clinical disciplines which includes biology, physics, chemistry, economics, signal and image processing, control theory and so on; see the monographs as [1–8].

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Various types of fractional derivatives were introduced among which the following Riemann-Liouville and Caputo are the most widely used ones.

(1) **Riemann-Liouville definition**. For $n-1<\alpha<n$, the derivative of u is

$$^{RL}D^{\alpha}u(t) := D^{n}I^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-\alpha-1}u(s)ds.$$

(2) **Caputo definition**. For $n-1<\alpha<n$, the derivative of u is

$$^{C}D^{\alpha}u(t) := I^{n-\alpha}D^{n}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}\left(\frac{d}{ds}\right)^{n}u(s)ds.$$

Both Riemann-Liouville definition and Caputo definition are defined via fractional integral, the Riemann-Liouville fractional integral, which is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-s)^{\alpha-1}u(s)ds, \qquad n-1<\alpha<n.$$

A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [9], the known as *the Hilfer fractional derivative* of order α and a type $\beta\in[0,1]$, which interpolates between the Riemann-Liouville and Caputo derivative, since it is reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta=0$ and $\beta=1$, respectively. The Hilfer fractional derivative of order α and parameter β of a function u is defined by

$$^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}u(t),$$

where $n-1<\alpha<n,\ \ 0\le\beta\le1,\ \ t>a>0,\ \ D=\dfrac{d}{dt}$. Some properties and applications of the Hilfer derivative are given in [10], [11] and references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see for example [12–15] and references therein. Nonlocal boundary value problems for Hilfer fractional derivative were studied in [16, 17].

Existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions were studied in [18].

In this paper, we study existence and uniqueness of solutions for boundary value problems for sequential Hilfer-Hadamard fractional differential equations with three-point boundary conditions,

$$({}_{H}D_{1+}^{\alpha,\beta} + k\,{}_{H}D_{1+}^{\alpha-1,\beta})u(t) = f(t,u(t)),\ 1<\alpha\le2,\ t\in[1,e], \tag{1.1}$$

$$u(1)=0,\ u(e)=\lambda u(\theta),\ \theta\in(1,e), \tag{1.2}$$

where ${}_{H}D_{1+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha\in(1,2]$ and type $\beta\in[0,1]$, $n=2,\gamma=\alpha+\beta(2-\alpha)$, $\gamma\in[\alpha,2]$, k $\in\mathbb{R}_{+}:=[0,\infty)$, $\lambda\in\mathbb{R}^{+}\setminus\{\frac{1}{(\log\ \theta)^{\gamma-1}}\}$ and $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

Existence and uniqueness results are established by using classical fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while Schaefer and Krasnoselskii's fixed point theorem [19] as well nonlinear alternative of Leray-Schauder type [20] are applied to obtain the existence results for the problem (1.1)–(1.2).

The paper is constructed as follows: In Section 2 we recall some basic facts needed in our study. The main results are proved in Section 3. Examples illustrating the main results are presented in Section 4.

2. Preliminaries

In this section, some basic definitions, lemmas and theorems are mentioned.

Definition 2.1. (Hadamard fractional integral [2]). The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$ {}_H I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \ \ (t > a) \tag{2.1} $$

provide the integral exists, where $\log(.) = \log_e(.)$.

Definition 2.2. (Hadamard fractional derivative [2]). The Hadamard fractional derivative of order $\alpha > 0$, applied to the fuction $f : [a, \infty) \to \mathbb{R}$, is defined as follows:

$$ {}_H D_{a^+}^{\alpha} f(t) = \delta^n ({}_H I_{a^+}^{n-\alpha} f(t)), \ \ n-1 < \alpha < n, \ \ n = [\alpha] + 1 \tag{2.2} $$

where $\delta^n = (t\frac{d}{dt})^n$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. (Hilfer-Hadamard fractional derivative [11]). Let $n-1 < \alpha < n$ and $0 \le \beta \le 1$, $f \in L^1(a, b)$. The Hilfer-Hadamard fractional derivative of order α and tybe β of f is defined as

$$ \begin{aligned} ({}_H D_{a^+}^{\alpha,\beta} f)(t) &= ({}_H I_{a^+}^{\beta(n-\alpha)} \delta^n \ {}_H I_{a^+}^{(n-\alpha)(1-\beta)} f)(t) \\ &= ({}_H I_{a^+}^{\beta(n-\alpha)} \delta^n \ {}_H I_{a^+}^{(n-\gamma)} f)(t); \ \gamma = \alpha + n\beta - \alpha\beta \\ &= ({}_H I_{a^+}^{\beta(n-\alpha)} \ {}_H D_{a^+}^{\gamma} f)(t), \end{aligned} $$

where ${}_H I_{a^+}^{(.)}$ and ${}_H D_{a^+}^{(.)}$ is the Hadamard fractional integral and derivative defined by (2.1) and (2.2), respectively.

The Hilfer-Hadamard fractional derivative may be viewed as interpolating the Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative.

We recall the following known theorem by Kilbas *et al.* [2] which I will use in the following.

Theorem 2.4 ([2]). *Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n-1 < \gamma < n$, $n = [\alpha] + 1$ and $0 < a < b < \infty$. If $f \in L^1(a, b)$ and $({}_H I_{a^+}^{n-\gamma} f)(t) \in AC_\delta^n[a, b]$*

$$ \begin{aligned} {}_H I_{a^+}^{\alpha} ({}_H D_{a^+}^{\alpha,\beta} f)(t) &= {}_H I_{a^+}^{\gamma} ({}_H D_{a^+}^{\gamma} f)(t) \\ &= f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_H I_{a^+}^{n-\gamma} f))(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a} \right)^{\gamma-j-1}. \end{aligned} $$

Finally, we will use the following well known fixed point theorems on Banach space for proving the existence and uniqueness of the solutions to Hilfer-Hadamard fractional boundary value problem (1.1)-(1.2):

Theorem 2.5. (Banach's contraction principle [21]). *Let X be a Banach space, $D \subset X$ be closed and $\mathcal{F} : D \to D$ be a contraction (i.e., there exists a constant $L \in (0, 1)$ such that for any $x, y \in X$, $\|\mathcal{F}x - \mathcal{F}y\| \le L\|x - y\|$). Then T has a unique fixed point on X.*

Theorem 2.6. (Schaefer fixed point theorem [22]). *Let $\mathcal{F} : E \to E$ be a completely continuous operator (i.e., a continuous map $\mathcal{F}$ restricted to any bounded set in E is compact). Let $\varepsilon(\mathcal{F}) = \{x \in E : x = \lambda\mathcal{F}(x), \ 0 \le \lambda \le 1\}$. Then either the set $\varepsilon(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fixed point.*

Theorem 2.7. (Krasnoselskii's fixed point theorem [19]). *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the operators satisfying the conditions: (i) $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) F_1 is compact and continuous; (iii) $\mathcal{F}_2$ is a contraction mapping. Then there exists $y \in Y$ such that $y = \mathcal{F}_1 y + \mathcal{F}_2 y$.*

Theorem 2.8. (Leray-Schauder nonlinear alternative [20]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{F} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{F}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{F}$ has a fixed point in $\bar{U}$, or
(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{F}(x)$.

3. Main Results

We start by proving a basic lemma concerning a linear variant of the boundary value problem (1.1)-(1.2), which be used to transform the boundary value problem (1.1)-(1.2) into an equivalent integral equation.

In this section, we prove existence and uniqueness of solutions for Hilfer-Hadamard sequential fractional boundary value problem (1.1)-(1.2).

Lemma 3.1. *Let $h \in C([1,e],\mathbb{R})$. Then u is a solution of the following Hilfer-Hadamard sequential fractional differential equation*

$$({}_HD^{\alpha,\beta}_{1^+} + k\,{}_HD^{\alpha-1,\beta}_{1^+})u(t) = h(t), \quad 1 < \alpha \le 2, \quad t \in [1,e] \tag{3.1}$$

supplemented with the boundary conditions (1.2), if and only if

$$\begin{aligned} u(t) \;=\; & \frac{(\log t)^{\gamma-1}}{1-\lambda(\log\ \theta)^{\gamma-1}}\left\{k\left[\int_1^e \frac{u(s)}{s}ds - \lambda\int_1^\theta \frac{u(s)}{s}ds\right]\right. \\ & \left. +\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta \left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds - \int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\right]\right\} \\ & -k\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds,\ t \in [1,e]. \end{aligned} \tag{3.2}$$

Proof. Taking the Hadamard fractional integral of order α to both sides of (3.1), we get

$${}_HI^{\alpha}_{1^+}({}_HD^{\alpha,\beta}_{1^+})u(t) + k\,{}_HI^{\alpha}_{1^+}({}_HD^{\alpha-1,\beta}_{1^+})u(t) = {}_H\,I^{\alpha}_{1^+}h(t).$$

By Theorem 2.4 one has

$${}_HI^{\gamma}_{1^+}({}_HD^{\gamma}_{1^+})u(t) + k\,{}_HI^{\alpha}_{1^+}({}_HD^{\alpha-1,\beta}_{1^+})u(t) = {}_H\,I^{\alpha}_{1^+}h(t)$$

and

$$u(t) - \sum_{j=0}^{1}\frac{\left(\delta^{(2-j-1)}({}_HI^{2-\gamma}_{1^+}u)\right)(1)}{\Gamma(\gamma-j)}(\log t)^{\gamma-j-1} + k\,{}_HI^{\alpha}_{1^+}({}_HD^{\alpha-1,\beta}_{1^+})u(t) = {}_H\,I^{\alpha}_{1^+}h(t). \tag{3.3}$$

By Definition 2.3, we note that

$$
\begin{aligned}
{}_H D_{1^+}^{\alpha-1,\beta} u(t) &= {}_H I_{1^+}^{\beta(1-(\alpha-1))} \big({}_H D_{1^+}^{(\alpha-1)+\beta-(\alpha-1)\beta}\big) u(t) \\
&= {}_H I_{1^+}^{\beta(2-\alpha)} \big({}_H D_{1^+}^{\alpha-1+\beta-\alpha\beta+\beta}\big) u(t) \\
&= {}_H I_{1^+}^{(2\beta-\alpha\beta)} \big({}_H D_{1^+}^{\alpha+2\beta-\alpha\beta-1}\big) u(t) \\
&= {}_H I_{1^+}^{(2\beta-\alpha\beta)} \big({}_H I_{1^+}^{-\alpha-2\beta+\alpha\beta+1}\big) u(t) \\
&= {}_H I_{1^+}^{2\beta-\alpha\beta-\alpha-2\beta+\alpha\beta+1} u(t) \\
&= {}_H I_{1^+}^{1-\alpha} u(t).
\end{aligned}
$$

Then, we have

$$
u(t) - \frac{\delta({}_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma)} (\log t)^{\gamma-1} - \frac{({}_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma-1)} (\log t)^{\gamma-2} + k\,{}_H I_{1^+} u(t) = {}_H I_{1^+}^{\alpha} h(t). \tag{3.4}
$$

The equation (3.4) can be written as follows

$$
u(t) = c_0(\log t)^{\gamma-1} + c_1(\log\ t)^{\gamma-2} - k\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \frac{h(s)}{s}\Big(\log\ \frac{t}{s}\Big)^{\alpha-1} ds \tag{3.5}
$$

where c_0, c_1 are arbitrary constants. Now, the first boundary condition $u(1) = 0$ together with (3.5) yield $c_1 = 0$. The equation (3.5) can be written as follows

$$
u(t) = c_0(\log\ t)^{\gamma-1} - k\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{h(s)}{s}ds. \tag{3.6}
$$

Next, the second boundary conditions $u(e) = \lambda u(\theta)$ together with (3.6) yield

$$
\begin{aligned}
c_0 &= \frac{1}{1-\lambda(\log\ \theta)^{\gamma-1}}\Bigg\{k\Bigg[\int_1^e \frac{u(s)}{s}ds - \lambda\int_1^\theta \frac{u(s)}{s}ds\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg[\lambda\int_1^\theta \Big(\log\ \frac{\theta}{s}\Big)^{\alpha-1}\frac{h(s)}{s}ds - \int_1^e \Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{h(s)}{s}ds\Bigg]\Bigg\}.
\end{aligned}
$$

Substituting the value of c_0 in (3.6), we get equation (3.2).

The converse follows by direct computation. The proof is completed. ■

Let us introduce the Banach space $X = C\big([1,e],\mathbb{R}\big)$ endowed with the norm defined by $\|u\| := \max_{t\in[1,e]}|u(t)|$.

In view of Lemma 3.1, we define an operator $\mathcal{F} : X \to X$ where

$$
\begin{aligned}
\mathcal{F}(u)(t) &= \frac{(\log t)^{\gamma-1}}{1-\lambda(\log\ \theta)^{\gamma-1}}\Bigg\{k\Bigg[\int_1^e \frac{u(s)}{s}ds - \lambda\int_1^\theta \frac{u(s)}{s}ds\Bigg] + \frac{1}{\Gamma(\alpha)} \\
&\quad \Bigg[\lambda\int_1^\theta \Big(\log\ \frac{\theta}{s}\Big)^{\alpha-1}\frac{f(s,u(s))}{s}ds - \int_1^e \Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{f(s,u(s))}{s}ds\Bigg]\Bigg\} \\
&\quad -k\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{f(s,u(s))}{s}ds.
\end{aligned} \tag{3.7}
$$

We need the following hypotheses in the sequal:

(H_1) There exists a constant $l > 0$ such that for all $t \in [1, e]$ and $u_i \in \mathbb{R},\ i = 1, 2$

$$|f(t, u_1) - f(t, u_2)| \ \leq \ l|u_1 - u_2|.$$

(H_2) There exists a continuous nonnegative function $\phi \in C([1, e], \mathbb{R}^+)$ such that

$$|f(t, u)| \leq \phi(t), \text{ for each } (t, u) \in [1, e] \times \mathbb{R}.$$

(H_3) There exists a real constant $M > 0$ such that for all $t \in [1, e], u \in \mathbb{R}$

$$|f(t, u)| \ \leq M.$$

(H_4) There exist $p \in C([1, e], \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, u)| \leq p(t)\psi(\|u\|) \text{ for each } (t, u) \in [1, e] \times \mathbb{R}.$$

(H_5) There exists a constant $K > 0$ such that

$$\frac{\left\{1 - k\left[\dfrac{1 + \lambda(\log\ \theta)}{|1 - \lambda(\log\ \theta)^{\gamma-1}|} + 1\right]\right\}K}{\dfrac{\|p\|\psi(K)}{\Gamma(\alpha+1)}\left[\dfrac{1 + \lambda(\log\ \theta)^{\alpha}}{|1 - \lambda(\log\ \theta)^{\gamma-1}|} + 1\right]} > 1.$$

3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem

We prove an existence and uniqueness result based on Banachs contraction mapping principle.

Theorem 3.2. *Assume that (H_1) holds. Then the boundary value problem (1.1)-(1.2) has a unique solution on $[1, e]$, provided that*

$$\Pi := k\left[\frac{1 + \lambda(\log\ \theta)}{|1 - \lambda(\log\ \theta)^{\gamma-1}|} + 1\right] + \frac{l}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha} + 1}{|1 - \lambda(\log\ \theta)^{\gamma-1}|} + 1\right] < 1. \qquad (3.8)$$

Proof. We will use Banachs fixed point theorem to prove that $\mathcal{F}$, defined by (3.7) has a unique fixed point. Fixing $N = \max_{t \in [1,e]} |f(t, 0)| < \infty$ and using the hypothesis (H_1), we obtain

$$|f(t, u(t))| \leq |f(t, u(t)) - f(t, 0)| + |f(t, 0)| \leq l|u(t)| + |f(t, 0)| \leq l\|u\| + N. \qquad (3.9)$$

Choose

$$r \geq \frac{N}{(1 - \Pi)\Gamma(\alpha+1)}\left(\frac{\lambda(\log\ \theta)^{\alpha} + 1}{|1 - \lambda(\log\ \theta)^{\gamma-1}|} + 1\right).$$

We divide the proof into two steps.

Step I : We show that $\mathcal{F}(B_r) \subset B_r$, where $B_r = \{u \in X : \|u\| \leq r\}$. Let $u \in B_r$. Then we have

$$\begin{aligned}
|\mathcal{F}(u)(t)| &\leq \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\Bigg[\int_1^e \frac{|u(s)|}{s}ds + \lambda\int_1^\theta \frac{|u(s)|}{s}ds\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg[\lambda\int_1^\theta\Big(\log\frac{\theta}{s}\Big)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds + \int_1^e\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\Bigg]\Bigg\} \\
&\quad + k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
&\leq \frac{1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\|u\|(1+\lambda\log\theta) + \frac{(l\|u\|+N)}{\Gamma(\alpha+1)}(\lambda(\log\theta)^\alpha+1)\Bigg\} \\
&\quad + k\|u\| + \frac{(l\|u\|+N)}{\Gamma(\alpha+1)} \\
&= k\|u\|\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] + \frac{(l\|u\|+N)}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^\alpha+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] \\
&\leq kr\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] + \frac{(lr+N)}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^\alpha+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] \\
&\leq r.
\end{aligned}$$

Thus

$$\|\mathcal{F}(u)\| = \max_{t\in[1,e]}|\mathcal{F}(u)(t)| \leq r.$$

Step II : To show that the operator $\mathcal{F}$ is a contraction, let $u_1, u_2 \in X$. Then, for any $t \in [1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}(u_2)(t) - \mathcal{F}(u_1)(t)| \\
&\leq \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\Bigg[\int_1^e \frac{|u_2(s)-u_1(s)|}{s}ds + \lambda\int_1^\theta \frac{|u_2(s)-u_1(s)|}{s}ds\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u_2(s))-f(s,u_1(s))|}{s}ds \\
&\quad + \int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u_2(s))-f(s,u_1(s))|}{s}ds\Bigg]\Bigg\} + k\int_1^t\frac{|u_2(s)-u_1(s)|}{s}ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u_2(s))-f(s,u_1(s))|}{s}ds \\
&\leq \|u_2-u_1\|\Bigg\{k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] + \frac{l}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^\alpha+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\Bigg\} \\
&= \Pi\|u_2-u_1\|.
\end{aligned}$$

Thus

$$\|\mathcal{F}(u_2)-\mathcal{F}(u_1)\| = \max_{t\in[1,e]}|\mathcal{F}(u_2)(t)-\mathcal{F}(u_1)(t)| \leq \Pi\|u_2-u_1\|,$$

which in view of (3.8), shows that the operator $\mathcal{F}$ is a contraction. By theorem 3.2, we get that the operator $\mathcal{F}$ has a unique fixed point. Therefore the problem (1.1)-(1.2) has a unique solution on $[1, e]$. The proof is completed. ∎

3.2. Existence Result via Krasnoselskii's Fixed Point Theorem

In this subsection, we prove an existence result based on Krasnoselskii's fixed point theorem.

Theorem 3.3. *Assume that* (H_2) *holds. Then, the problem* (1.1)-(1.2) *has at least one solution on* $[1, e]$*, provided that*

$$k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]<1. \tag{3.10}$$

Proof. By assumption (H_2), we can fix

$$\rho \geq \frac{\dfrac{\|\phi\|}{\Gamma(\alpha+1)}\left[\dfrac{\lambda(\log\theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]}{1-k\left[\dfrac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]},$$

where $\|\phi\| = \sup_{t\in[1,e]} |\phi(t)|$ and consider $B_\rho = \{u \in C([1, e], \mathbb{R}) : \|u\| \leq \rho\}$. We split the operator $\mathcal{F} : C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ defined by (3.7) as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$(\mathcal{F}_1 u)(t) = \frac{(\log t)^{\gamma-1}}{1-\lambda(\log\ \theta)^{\gamma-1}}k\left[\int_1^e \frac{u(s)}{s}ds - \lambda\int_1^\theta \frac{u(s)}{s}ds\right] - k\int_1^t \frac{u(s)}{s}ds,$$

and

$$\begin{aligned}(\mathcal{F}_2 u)(t) &= \frac{(\log t)^{\gamma-1}}{1-\lambda(\log\ \theta)^{\gamma-1}}\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds\right.\\ &\quad \left.-\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds\right] + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds.\end{aligned}$$

For any $u, v \in B_\rho$, we have

$$\begin{aligned}|(\mathcal{F}_1 u)(t) + (\mathcal{F}_2 v)(t)| &\leq \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{k\left[\int_1^e \frac{|u(s)|}{s}ds + \lambda\int_1^\theta \frac{|u(s)|}{s}ds\right]\right.\\ &\quad \left.+\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,v(s))|}{s}ds + \int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,v(s))|}{s}ds\right]\right\}\\ &\quad + k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,v(s))|}{s}ds\\ &\leq \frac{1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{k\|u\|(1+\lambda\log\theta) + \frac{\|\phi\|}{\Gamma(\alpha+1)}(\lambda(\log\theta)^{\alpha}+1)\right\}\end{aligned}$$

$$
\begin{aligned}
&+ k\|u\| + \frac{\|\phi\|}{\Gamma(\alpha+1)} \\
&= k\|u\|\left[\frac{1+\lambda(\log\ \theta)}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right] + \frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^{\alpha}+1}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right] \\
&\le k\rho\left[\frac{1+\lambda(\log\ \theta)}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right] + \frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^{\alpha}+1}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right] \\
&\le \rho.
\end{aligned}
$$

Hence, $\|\mathcal{F}_1 u + \mathcal{F}_2 v\| \le \rho$, which shows that $\mathcal{F}_1 u + \mathcal{F}_2 v \in B_\rho$. It is easy to prove, using condition (3.10) that the operator $\mathcal{F}_1$ is a contraction mapping. The operator $\mathcal{F}_2$ is continuous by the continuity of f. Also $\mathcal{F}_2$ is uniformly bounded on B_ρ, since

$$
\|\mathcal{F}_2 u\| \le \frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^{\alpha}+1}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right].
$$

Finally we prove the compactness of the operator $\mathcal{F}_2$. We define $\sup_{(t,u)\in[1,e]\times B_\rho}|f(t,u)| = \overline{f}$ and take $t_1, t_2 \in [1,e], t_1 < t_2$. Then we have

$$
\begin{aligned}
|\mathcal{F}_2 u(t_2) - \mathcal{F}_2 u(t_1)| \le{}& \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^{\theta}\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right. \\
&\left.+\int_1^{e}\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right] \\
&+\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\ \frac{t_2}{s}\right)^{\alpha-1} - \left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right]\frac{|f(s,u(s))|}{s}ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
\le{}& \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{\frac{\overline{f}}{\Gamma(\alpha+1)}\Big[\lambda(\log\theta)^{\alpha}+1\Big]\right\} \\
&+\frac{\overline{f}}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\ \frac{t_2}{s}\right)^{\alpha-1} - \left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds + \frac{\overline{f}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
\le{}& \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\frac{\overline{f}}{\Gamma(\alpha+1)}\Big[\lambda(\log\theta)^{\alpha}+1\Big] \\
&+\frac{\overline{f}}{\Gamma(\alpha+1)}\Big[(\log t_2)^{\alpha} - (\log t_1)^{\alpha}\Big],
\end{aligned}
$$

which tends to zero, independently of $u \in B_\rho$, as $t_1 \to t_2$. Thus, $\mathcal{F}_2$ is equicontinuous. From the Arzelá-Ascoli theorem we conclude that the operator $\mathcal{F}_2$ is compact on B_ρ. Thus, the hypotheses of Krasnoselskii fixed point theorem are satisfied, and therefore there exists at least one solution on $[1,e]$. The proof is finished. ■

3.3. Existence Result via Schaefer's Fixed Point Theorem

Our second existence result is based on Schaefer's fixed point theorem.

Theorem 3.4. *Assume that* (H_3) *holds and*

$$k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\right]<1.$$

Then, the boundary value problem (1.1)-(1.2) *has at least one solution on* $[1,e]$.

Proof. We will prove that the operator $\mathcal{F}$, defined by (3.7), has a fixed point, by using Schaefer's fixed point theorem. We divide the proof into two steps.

Step I : We show that the operator $\mathcal{F}: X \to X$ is completely continuous.
We show first that $\mathcal{F}$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in X. Then for each $t \in$ [1,e], consider

$$\begin{aligned}
&|\mathcal{F}(u_n)(t)-\mathcal{F}(u)(t)|\\
&\le \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\left[\int_1^e \frac{|u_n(s)-u(s)|}{s}ds+\lambda\int_1^\theta \frac{|u_n(s)-u(s)|}{s}ds\right]\\
&\quad+\frac{1}{\Gamma(\alpha)}\Bigg[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s))-f(s,u(s))|}{s}ds\\
&\quad+\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s))-f(s,u(s))|}{s}ds\Bigg]\Bigg\}\\
&\quad+k\int_1^t\frac{|u_n(s)-u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s))-f(s,u(s))|}{s}ds.
\end{aligned}$$

Since f is continuous, we get

$$|f(s,u_n(s))-f(s,u(s))|\to 0 \quad as \quad u_n\to u.$$

Then

$$\|\mathcal{F}(u_n(t))-\mathcal{F}(u(t))\|\to 0 \quad as \quad u_n\to u.$$

Hence $\mathcal{F}$ is continuous.

Secondly, we show that the operator $\mathcal{F}$ maps bounded sets into bounded sets in X. For a positive number R, let $B_R=\{u\in X:\|u\|\le R\}$ be a bounded ball in X. Then, for $t\in[1,e]$, we have,

$$\begin{aligned}
|\mathcal{F}(u)(t)| &\le \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\left[\int_1^e\frac{|u(s)|}{s}ds+\lambda\int_1^\theta\frac{|u(s)|}{s}ds\right]\\
&\quad+\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds+\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right]\Bigg\}\\
&\quad+k\int_1^t\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\\
&\le kR\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]+\frac{M}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^\alpha+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right],
\end{aligned}$$

and

$$\|\mathcal{F}(u)\| \leq kR\left[\frac{1+\lambda(\log\ \theta)}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right]+\frac{M}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\theta)^{\alpha}+1}{\left|1-\lambda(\log\ \theta)^{\gamma-1}\right|}+1\right].$$

Thirdly, we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $u \in B_R$. Then we have

$$\begin{aligned}
|\mathcal{F}(u)(t_2)-\mathcal{F}(u)(t_1)| &\leq \frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\left[\int_1^e \frac{|u(s)|}{s}ds+\lambda\int_1^\theta \frac{|u(s)|}{s}ds\right]\\
&+\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds+\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right]\Bigg\}\\
&+k\int_{t_1}^{t_2}\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}-\left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right]\frac{|f(s,u(s))|}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\\
&\leq \frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{k\|u\|\Big[1+\lambda\log\theta\Big]+\frac{M}{\Gamma(\alpha+1)}\Big[\lambda(\log\theta)^\alpha+1\Big]\right\}\\
&+k\int_{t_1}^{t_2}\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}-\left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right]\frac{|f(s,u(s))|}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\\
&\leq \frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{k\|u\|\Big[1+\lambda\log\theta\Big]+\frac{M}{\Gamma(\alpha+1)}\Big[\lambda(\log\theta)^\alpha+1\Big]\right\}\\
&+k\|u\||\log t_2-\log t_1|+\frac{M}{\Gamma(\alpha+1)}\Big[2(\log t_2-\log t_1)^\alpha+|(\log t_2)^\alpha-(\log t_1)^\alpha|\Big]\\
&\leq \frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{kR\Big[1+\lambda\log\theta\Big]+\frac{M}{\Gamma(\alpha+1)}\Big[\lambda(\log\theta)^\alpha+1\Big]\right\}\\
&+kR|\log t_2-\log t_1|+\frac{M}{\Gamma(\alpha+1)}\Big[(\log t_2)^\alpha-(\log t_1)^\alpha\Big],
\end{aligned}$$

which tends to zero, independently of $u \in B_R$, as $t_1 \to t_2$. Thus, the Arzelá-Ascoli theorem applies and hence $\mathcal{F}: X \to X$ is completely continuous.

Step II : We show that the set $\mathcal{E} = \{u \in X \mid u = \eta\mathcal{F}(u), 0 \leq \eta \leq 1\}$ is bounded. Let $u \in \mathcal{E}$, then $u = \eta\mathcal{F}(u)$. For any $t \in [1, e]$, we have $u(t) = \eta\mathcal{F}(u)(t)$. Then, in view of the hypothesis (H_3), as in Step I, we obtain

$$\begin{aligned}
|u(t)| \quad &\leq \quad \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\Bigg\{k\left[\int_1^e \frac{|u(s)|}{s}ds+\lambda\int_1^\theta \frac{|u(s)|}{s}ds\right]\\
&+\frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds+\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right]\Bigg\}
\end{aligned}$$

$$+ k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds$$
$$\leq k\|u\|\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] + \frac{M}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right],$$

or

$$\left\{1-k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\right\}\|u\| \leq \frac{M}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right].$$

Thus

$$\|u\| \leq \frac{M}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\left\{1-k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\right]\right\}^{-1},$$

which shows that the set $\mathcal{E}$ is bounded. By Theorem 2.6, we get that the operator $\mathcal{F}$ has at least one fixed point. Therefore, the boundary value problem (1.1)-(1.2) has at least one solution on $[1, e]$. This completes the proof. ∎

3.4. Existence Result via Leray-Schauder Nonlinear Alternative

Our final existence result is proved via Leray-Schauder nonlinear alternative.

Theorem 3.5. *Assume that* (H_4), (H_5) *and* (3.10) *hold. Then, the boundary value problem* (1.1)-(1.2) *has at least one solution on* $[1, e]$.

Proof. As in Theorem 3.4 we can prove that the operator $\mathcal{F}$ is completely continuous. We will prove that there exists an open set $U \subseteq C([1, e], \mathbb{R})$ with $u \neq \mu\mathcal{F}(u)$ for $\mu \in (0, 1)$ and $u \in \partial U$.

Let $u \in C([1, e], \mathbb{R})$ be such that $u = \mu\mathcal{F}(u)$ for some $0 < \mu < 1$. Then, for each $t \in [1, e]$, we have

$$\begin{aligned}
|u(t)| \quad \leq \quad & \frac{(\log t)^{\gamma-1}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\left\{k\left[\int_1^e \frac{|u(s)|}{s}ds + \lambda\int_1^\theta \frac{|u(s)|}{s}ds\right]\right.\\
& \left. + \frac{1}{\Gamma(\alpha)}\left[\lambda\int_1^\theta\left(\log\ \frac{\theta}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds + \int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right]\right\}\\
& + k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\\
\leq \quad & k\|u\|\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] + \frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right],
\end{aligned}$$

or

$$\left\{1-k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\right\}\|u\| \leq \frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}\left[\frac{\lambda(\log\ \theta)^{\alpha}+1}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right].$$

Consequently

$$\frac{\left\{1-k\left[\dfrac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\right\}\|u\|}{\dfrac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}\left[\dfrac{1+\lambda(\log\ \theta)^{\alpha}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]}\leq 1.$$

In view of (H_5), there is no solution u such that $\|u\|\neq K$. Let us set

$$U=\{u\in C([1,e],\mathbb{R}):\|u\|<K\}.$$

The operator $\mathcal{F}:\overline{U}\to C([1,e],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u=\mu\mathcal{F}(u)$ for some $\mu\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [20], we deduce that $\mathcal{F}$ has a fixed point $u\in\overline{U}$ which is a solution of the boundary value problem (1.1)-(1.2). The proof is completed. ∎

4. Examples

In order to illustrate our results, we give in this section examples.

Example 4.1. Consider the following boundary value problem

$$\begin{cases}\left({}_HD_{1^+}^{\frac{5}{4},\frac{1}{2}}+\dfrac{1}{7}\,{}_HD_{1^+}^{\frac{1}{4},\frac{1}{2}}\right)u(t)=\dfrac{(1+\log\ t)|u(t)|}{100+|u(t)|}, & t\in[1,e],\\[2ex] u(1)=0,\quad u(e)=\dfrac{1}{3}u(2).\end{cases}\tag{4.1}$$

Here $\alpha=\frac{5}{4}$, $\beta=\frac{1}{2}$, $k=\frac{1}{7}$, $\lambda=\frac{1}{3}$, $\gamma=\frac{13}{8}$ and $\theta=2$. For each $u_1,u_2\in\mathbb{R}$ we have $|f(t,u_1)-f(t,u_2)|\leq\frac{1}{50}|u_1-u_2|$, and thus (H_1) is satisfied. Using the given data, we find that $\Pi\approx 0.4289<1$. Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, it follows by the conclusion of Theorem 3.2 that the boundary value problem (4.1) has a unique solution on $[1,e]$.

Example 4.2. Consider the following boundary value problem

$$\begin{cases}\left({}_HD_{1^+}^{\frac{3}{2},\frac{1}{4}}+\dfrac{1}{6}\,{}_HD_{1^+}^{\frac{1}{2},\frac{1}{4}}\right)u(t)=e^t\sin\ u(t), \quad t\in[1,e],\\[2ex] u(1)=0,\quad u(e)=\dfrac{1}{2}u(2).\end{cases}\tag{4.2}$$

Here $\alpha=\frac{3}{2}$, $\beta=\frac{1}{4}$, $k=\frac{1}{6}$, $\lambda=\frac{1}{2}$, $\gamma=\frac{13}{8}$ and $\theta=2$. For each $u\in\mathbb{R}$ we have $|f(t,u)|\leq e^t$, and thus (H_2) is satisfied. Using the given data, we find that

$$k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\approx 0.5392<1.$$

Hence, all the conditions of Theorem 3.3 are satisfied. Therefore, it follows by the conclusion of Theorem 3.3 that the boundary value problem (4.2) has at least one solution on $[1,e]$.

Example 4.3. Consider the following boundary value problem

$$\begin{cases} \left({}_HD_{1^+}^{\frac{3}{2},\frac{1}{2}} + \frac{2}{5}\,{}_HD_{1^+}^{\frac{1}{2},\frac{1}{2}}\right)u(t) = e^{|u(t)|}\log\, t, \quad t \in [1,e], \\ u(1) = 0, \quad u(e) = \frac{1}{4}u(\frac{3}{2}). \end{cases} \tag{4.3}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $k = \frac{2}{5}$, $\lambda = \frac{1}{4}$, $\gamma = \frac{7}{4}$ and $\theta = \frac{3}{2}$. For each $u \in \mathbb{R}$ we have $|f(t,u)| \leq 1$, and thus (H_3) is satisfied. Using the given data, we find that

$$k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}\right] \approx 0.5047 < 1.$$

Therefore, all the conditions of Theorem 3.4 are satisfied. Therefore, it follows by the conclusion of Theorem 3.4 that the boundary value problem (4.3) has at least one solution on $[1,e]$.

Example 4.4. Consider the following boundary value problem

$$\begin{cases} \left({}_HD_{1^+}^{\frac{3}{2},\frac{1}{4}} + \frac{1}{6}\,{}_HD_{1^+}^{\frac{1}{2},\frac{1}{4}}\right)u(t) = [u(t)]^2\cos\, t, \quad t \in [1,e], \\ u(1) = 0, \quad u(e) = \frac{1}{2}u(2). \end{cases} \tag{4.4}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{4}$, $k = \frac{1}{6}$, $\lambda = \frac{1}{2}$, $\gamma = \frac{13}{8}$ and $\theta = 2$. For each $u \in \mathbb{R}$, there exists a constant function $p(t) = 1$ and continuous nondecreasing function $\psi(x) = x^2$ for all $x \in \mathbb{R}^+$ such that $|f(t,u)| \leq p(t)\psi(\|u\|) = \|u\|^2$, and then (H_4) is satisfied. Using the given data, we find that

$$k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right] \approx 0.5392 < 1,$$

and there exists a constant $K = 0.1892$ such that

$$\frac{\left\{1-k\left[\frac{1+\lambda(\log\ \theta)}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]\right\}K}{\frac{\|p\|\psi(K)}{\Gamma(\alpha+1)}\left[\frac{1+\lambda(\log\ \theta)^{\alpha}}{|1-\lambda(\log\ \theta)^{\gamma-1}|}+1\right]} \approx 1.00041 > 1.$$

That is (H_5) is satisfied. Thus, all the conditions of Theorem 3.5 are satisfied. Therefore, it follows by the conclusion of Theorem 3.5 that the boundary value problem (4.4) has at least one solution on $[1,e]$.

5. Conclusion

In this paper, existence and uniqueness results are established for a boundary value problem for Hilfer-Hadamard sequential fractional differential equation, with three-point boundary conditions. The existence and uniqueness result is proved via Banach contraction mapping principle, while for the existence results the Schaefer and Krasnoselskii fixed point theorems as well and Leray-Schauder nonlinear alternative are used.

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