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Regularity and Finiteness Conditions on Transformation Semigroups with Invariant Sets

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Abstract Let X be a nonempty set and $T(X)$ denote the semigroup of transformations from X to itself under the composition of functions. For a fixed nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of total transformations of X which leave a subset Y of X invariant. In this paper, we characterize coregular elements of $S(X,Y)$ and give necessary and sufficient conditions for $S(X,Y)$ to be coregular. Moreover, we study some properties of regularity on $S(X,Y)$ and give necessary and sufficient conditions for $S(X,Y)$ to be left regular, right regular, completely regular, intra-regular and directly finite.

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1. Introduction

Regular semigroups play an important role in the semigroup theory and they have been studied from various aspects. An element a in a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$, *left regular* if $a = ba^2$ for some $b \in S$, *right regular* if $a = a^2b$ for some $b \in S$, *completely regular* if $a = aba$ and $ab = ba$ for some $b \in S$ and *intra-regular* if $a = ba^2c$ for some $b, c \in S$. In fact, a is both left and right regular if and only if a is completely regular. S is a *regular* [*left regular, right regular, completely regular and intra-regular*] *semigroup* if every element of S is regular [left regular, right regular, completely regular and intra-regular].

A special case of a regular element is a coregular element. An element a in a semigroup S is *coregular* if there exists $b \in S$ such that $a = aba = bab$, and S is *coregular* if every element in S is coregular. Clearly, an element a in a semigroup S is coregular if and only

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if $a^3 = a$, and every coregular element is regular, left regular, right regular and completely regular. Coregular semigroup was first introduced and studied by Bijev and Todorov [2].

We denote the set of all regular elements, left regular elements, right regular elements, completely regular elements, intra-regular elements and coregular elements of a semigroup S by $\mathrm{Reg}(S)$, $\mathrm{LReg}(S)$, $\mathrm{RReg}(S)$, $\mathrm{CReg}(S)$, $\mathrm{IReg}(S)$ and $\mathrm{CoReg}(S)$, respectively.

An element a in a semigroup S is said to be *idempotent* if $a^2 = a$. Then an idempotent of S is regular, left regular, right regular, completely regular, intra-regular and coregular. The set of all idempotents of S is denoted by $E(S)$.

A semigroup S is *factorizable* if $S = GE$ for some subgroup G of S and some set E of idempotents of S. We note that if a semigroup S is factorizable as GE, then $S = GE(S)$.

An element a of a monoid S is said to be *unit regular* if $a = aua$ for some unit u in S; and S is a *unit regular semigroup* if every element of S is unit regular. A monoid S with identity 1 is *directly finite* if for any a and b in S, $ab = 1$ implies $ba = 1$.

In 1980, Alarcao [1] characterized when a monoid S is unit regular and when it is directly finite. The author also gave relationships between a factorizable semigroup, a unit regular semigroup and a directly finite semigroup.

Let X be a nonempty set, and $T(X)$ denote the set of all transformations from X to itself. Then $T(X)$ is a semigroup under the composition of maps and it is called the *full transformation semigroup* on X. It is known that $T(X)$ is a regular semigroup and every semigroup can be embedded in $T(Z)$ for some set Z (see [7]).

In 1979, Tirasupa [10] showed that $T(X)$ is factorizable if and only if X is finite. Later, in 1980 Alarcao [1] gave necessary and sufficient conditions for $T(X)$ to be unit regular and directly finite.

For a fixed nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of total transformations on X which leave a subset Y of X invariant. The semigroup $S(X,Y)$ was first introduced and studied by Magill [8] in 1966. To the extent that $S(X,X) = T(X)$, we may regard $S(X,Y)$ as a generalization of $T(X)$. Note that the identity map on X, denoted by id_X, belongs to $S(X,Y)$. For many years, its concepts in semigroup theory such as regularity, automorphisms, factorization, Green's relations and ideals are studied. In fact, elements of $S(X,Y)$ need not be regular that means $S(X,Y)$ is not a regular semigroup in general. In 2005, Nenthein, Youngkhong and Kemprasit [9] showed that $S(X,Y)$ is a regular semigroup if and only if $X = Y$ or Y contains exactly one element, and

$$\mathrm{Reg}(S(X,Y)) = \{\alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha\}$$

is the set of all regular elements of $S(X,Y)$. Boonmee [3] characterized when $S(X,Y)$ is a factorizable semigroup in 2007. Later in 2011, Honyam and Sanwong [6] characterized when $S(X,Y)$ is isomorphic to $T(Z)$ for some set Z and proved that every semigroup A can be embedded in $S(A^1, A)$ where A^1 is a monoid obtained from A by adjoining an identity if necessary. Moreover, they also described Green's relations and ideals of $S(X,Y)$. In 2013, Choomanee, Honyam and Sanwong [4] characterized left regular, right regular and intra-regular elements of $S(X,Y)$ and considered the relationships between these elements. Also, they found the number of left regular elements of $S(X,Y)$ when X is a finite set.

In this paper, we characterize coregular elements on $S(X,Y)$ and give necessary and sufficient conditions for $S(X,Y)$ to be coregular in section 3. In section 4, we determine when $S(X,Y)$ is left regular, right regular, completely regular and intra-regular and we characterize when $\mathrm{LReg}(S(X,Y))$, $\mathrm{RReg}(S(X,Y))$, $\mathrm{CReg}(S(X,Y))$ and $\mathrm{IReg}(S(X,Y))$ to be subsemigroups of $S(X,Y)$. Moreover, in section 5, we give necessary and sufficient conditions for $S(X,Y)$ to be directly finite.

2. Preliminaries

In this section, we introduced some concepts and some results that will be used throughout this paper.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Also, we write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first.

According to Clifford and Preston [5], vol. 2, p. 241, we will use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$ for all $i \in I$. Given $i \in I$, if $X_i = \{x\}$ for some $x \in X$, then we simply write x instead of $\{x\}$.

We modify the convention as in $T(X)$, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X \setminus Y; \{a_i\} \subseteq Y, \{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, I is a nonempty set, but J or K can be empty. For example: if $\alpha \in \mathrm{Reg}(S(X,Y))$, then J is an empty set. And if $\alpha \notin \mathrm{Reg}(S(X,Y))$, then both I and J are nonempty sets but K can be an empty set.

We note that for any $\alpha \in S(X,Y)$, the symbol π_α will denote the partition of X induced by the map α, namely

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$$

and $\pi_\alpha(Y)$ will denote the subset of π_α which is defined by

$$\pi_\alpha(Y) = \{x\alpha^{-1} : x \in X\alpha \cap Y\}.$$

Green's relations on $S(X,Y)$ are given by Honyam and Sanwong [6], which are needed in characterizing some properties of regularity on $S(X,Y)$.

Theorem 2.1. [6, Lemmas 2-3 and Theorems 4-5] *Let $\alpha, \beta \in S(X,Y)$. Then the following statements hold.*

(1) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*
(2) *$\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.*
(3) *$\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta, Y\alpha = Y\beta, \pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.*
(4) *$\alpha\mathcal{D}\beta$ if and only if $|Y\alpha| = |Y\beta|, |X\alpha \setminus Y| = |X\beta \setminus Y|$ and*
$$|(X\alpha \cap Y) \setminus Y\alpha| = |(X\beta \cap Y) \setminus Y\beta|.$$
(5) *$\alpha\mathcal{J}\beta$ if and only if $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

For an element a in a semigroup S, D_a and H_a denote the equivalence class of $\mathcal{D}$ containing a and the equivalence class of $\mathcal{H}$ containing a, respectively, that is

$$D_a = \{b \in S : b\mathcal{D}a\} \text{ and } H_a = \{b \in S : b\mathcal{H}a\}.$$

In [6] the authors showed that H_{id_X} is the group of units of $S(X,Y)$. In this case

$$H_{id_X} = \left\{ \begin{pmatrix} a_i & b_j \\ a_i\sigma & b_j\sigma \end{pmatrix} : \sigma \in G(X,Y) \right\}$$

where $Y = \{a_i\}$ and $X \setminus Y = \{b_j\}$; and $G(X,Y) = \{\alpha \in S(X) : \alpha|_Y \in S(Y)\}$. Note that $S(X)$ and $S(Y)$ are the permutation group on X and the permutation group on Y, respectively.

In general, $\mathrm{Reg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$. In [6], the authors gave necessary and sufficient conditions for $\mathrm{Reg}(S(X,Y))$ to be a subsemigroup of $S(X,Y)$ as follows.

$\mathrm{Reg}(S(X,Y))$ is a regular subsemigroup of $S(X,Y)$ if and only if $Y = X$ or $|Y| = 1$.

Left regularity, right regularity and intra-regularity on $S(X,Y)$ were studied by Choomanee, Honyam and Sanwong [4] as shown in the following theorems.

Theorem 2.2. [4, Theorems 3.1, 3.2 and 3.4] *Let $\alpha \in S(X,Y)$. Then the following statements hold.*
(1) *α is left regular if and only if $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.*
(2) *α is right regular if and only if $\pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$.*
(3) *α is intra-regular if and only if $|X\alpha| = |X\alpha^2|$, $|Y\alpha| = |Y\alpha^2|$ and*

$$|X\alpha \setminus Y| = |X\alpha^2 \setminus Y|.$$

Theorem 2.3. [4, Theorem 3.11] *Let $\alpha \in S(X,Y)$ be such that $X\alpha$ is a finite set. Then the following statements are equivalent.*
(1) *α is left regular.*
(2) *α is right regular.*
(3) *α is intra-regular.*

Theorem 2.4. [4, Theorem 4.3] *The number of left regular elements in $S(X,Y)$ is*

$$\sum_{m=0}^{n-r} \sum_{k=1}^{r} \binom{r}{k} k! k^{r-k} \binom{n-r}{m} m! (k+m)^{n-r-m}$$

where $|X| = n$ and $|Y| = r$.

3. Coregular Elements on $S(X,Y)$

In this section, we characterize coregular elements of $S(X,Y)$ and give necessary and sufficient conditions for $S(X,Y)$ to be coregular. Also, we describe when $\mathrm{CoReg}(S(X,Y))$ is a subsemigroup of $S(X,Y)$.

For regularity on $S(X,Y)$ when $|X| \leq 2$, we obtain the following results.

Remark 3.1. If $|X| = 1$, then $S(X,Y) = T(X)$ contains exactly one element, and hence $S(X,Y)$ is regular, coregular, left regular, right regular, completely regular and intra-regular.

Lemma 3.2. *If $|X| = 2$, then $S(X,Y)$ is regular, coregular, left regular, right regular, completely regular and intra-regular.*

Proof. Let $|X| = 2$. We consider two cases.

Case 1: $|Y| = 1$. Let $X = \{a, b\}$ and $Y = \{a\}$. Then

$$S(X,Y) = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} \{a,b\} \\ a \end{pmatrix} \right\}.$$

Thus all elements in $S(X,Y)$ are idempotents and so each element is regular, coregular, left regular, right regular, completely regular and intra-regular.

Case 2: $|Y| = 2$. Then

$$S(X,Y) = T(X) = \left\{ \alpha_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \alpha_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \alpha_3 = \begin{pmatrix} \{a,b\} \\ a \end{pmatrix}, \alpha_4 = \begin{pmatrix} \{a,b\} \\ b \end{pmatrix} \right\}.$$

Thus $\alpha_1, \alpha_3, \alpha_4$ are idempotents and we obtain that $\alpha_2 = {\alpha_2}^3 = (\alpha_2){\alpha_2}^2 = {\alpha_2}^2(\alpha_2) = id_X({\alpha_2}^2)\alpha_2$. So $\alpha_1, \alpha_2, \alpha_3$ and α_4 are regular, coregular, left regular, right regular, completely regular and intra-regular.

Therefore, $S(X,Y)$ is regular, coregular, left regular, right regular, completely regular and intra-regular. ■

Now, we study coregularity on $S(X,Y)$. Recall that α in $S(X,Y)$ is coregular if and only if $\alpha^3 = \alpha$.

In general, $S(X,Y)$ is not a coregular semigroup that means there exists an element in $S(X,Y)$ which is not coregular as shown in the following example.

Example 3.3. Let $X = \{1,2,3,4,5,6\}$ and $Y = \{1,2,3\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & 5 & 6 \\ 2 & 1 & 4 & 5 \end{pmatrix}.$$

Then $Y\alpha = \{1,2\} \subseteq Y$ and hence $\alpha \in S(X,Y)$. We see that

$$\alpha^3 = \begin{pmatrix} \{1,2,3,4,5\} & 6 \\ 2 & 1 \end{pmatrix}.$$

So $\alpha^3 \neq \alpha$ and therefore, α is not coregular.

The following theorem is a characterization of coregular elements of $S(X,Y)$.

Theorem 3.4. *Let $\alpha \in S(X,Y)$. Then the following statements are equivalent.*
(1) *α is coregular.*
(2) *$x\alpha \in x\alpha^{-1}$ for all $x \in X\alpha$.*
(3) *$\alpha^2|_{X\alpha} = id_{X\alpha}$.*

Proof. (1) ⇒ (2) Assume that α is coregular. Thus $\alpha^3 = \alpha$. Let $x \in X\alpha$. Then $x = z\alpha$ for some $z \in X$. So

$$x = z\alpha = z\alpha^3 = (z\alpha)\alpha\alpha = (x\alpha)\alpha,$$

that means $x\alpha \in x\alpha^{-1}$.

(2) ⇒ (3) Assume that $x\alpha \in x\alpha^{-1}$ for all $x \in X\alpha$. For each $x \in X\alpha$,

$$x\alpha^2 = (x\alpha)\alpha = x = x\, id_{X\alpha}.$$

Thus $\alpha^2|_{X\alpha} = id_{X\alpha}$.

(3) ⇒ (1) Assume that $\alpha^2|_{X\alpha} = id_{X\alpha}$. Let $x \in X$. Then

$$x\alpha^3 = (x\alpha)\alpha^2 = (x\alpha)\, id_{X\alpha} = x\alpha$$

and so $\alpha^3 = \alpha$. Hence α is a coregular element. ■

Now, we give a simple example of coregular elements of $S(X,Y)$.

Example 3.5. Let $X = \mathbb{N}$ denote the set of all positive integers and $Y = \{1,2,3,4,5,6\}$. Define $\alpha, \beta \in S(X,Y)$ by

$$\alpha = \begin{pmatrix} \{1,2\} & 3 & \{4,5\} & \{6,7\} & 2n & 2n+1 \\ 6 & 4 & 3 & 1 & 2n+1 & 2n \end{pmatrix}_{n\geq 4}$$

and

$$\beta = \begin{pmatrix} \{1,2\} & \{3,4,5\} & \{6,7\} & n \\ 3 & 1 & 6 & n+1 \end{pmatrix}_{n\geq 8}.$$

Then

$$\alpha^2 = \begin{pmatrix} \{1,2\} & 3 & \{4,5\} & \{6,7\} & n \\ 1 & 3 & 4 & 6 & n \end{pmatrix}_{n\geq 8}.$$

Thus $\alpha^2|_{X\alpha} = id_{X\alpha}$ and hence α is a coregular element by Theorem 3.4 (3). However, β is not a coregular element since $9 \in X\beta$ and $9\beta = 10 \notin \{8\} = 9\beta^{-1}$.

As a consequence of Theorem 3.4, the necessary and sufficient condition for the semigroup $S(X,Y)$ to be a coregular semigroup given as follows.

Theorem 3.6. *$S(X,Y)$ is coregular if and only if $|X| \leq 2$.*

Proof. Assume that $|X| \leq 2$. By Remark 3.1 and Lemma 3.2, we have $S(X,Y)$ is a coregular semigroup. Conversely, assume that $|X| \geq 3$. We consider two cases.

Case 1: $|Y| = 1$. Then $|X \setminus Y| \geq 2$. Let $Y = \{y\}$ and $z \in X \setminus Y$. So $X \setminus \{y,z\} \neq \emptyset$. Define

$$\alpha = \begin{pmatrix} \{y,z\} & X \setminus \{y,z\} \\ y & z \end{pmatrix} \in S(X,Y).$$

Thus $z \in X\alpha$ and $z\alpha = y \notin X \setminus \{y,z\} = z\alpha^{-1}$. By Theorem 3.4 (2), we get α is not coregular.

Case 2: $|Y| > 1$. Let $a, b \in Y$ be such that $a \neq b$. Then $X \setminus \{a,b\} \neq \emptyset$ since $|X| \geq 3$. Define

$$\alpha = \begin{pmatrix} \{a,b\} & X \setminus \{a,b\} \\ a & b \end{pmatrix}.$$

We see that $X\alpha = \{a,b\} \subseteq Y$ and so $\alpha \in S(X,Y)$. From $b \in X\alpha$ and $b\alpha = a \notin X \setminus \{a,b\} = b\alpha^{-1}$, we conclude that α is not coregular by Theorem 3.4 (2).

Therefore, $S(X,Y)$ is not a coregular semigroup. ∎

From Theorem 3.4, we obtain that

$$\begin{aligned} \mathrm{CoReg}(S(X,Y)) &= \{\alpha \in S(X,Y) : x\alpha \in x\alpha^{-1} \text{ for all } x \in X\alpha\} \\ &= \left\{\alpha \in S(X,Y) : \alpha^2|_{X\alpha} = id_{X\alpha}\right\}. \end{aligned}$$

The following example shows that, in general, $\mathrm{CoReg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$.

Example 3.7. Let $X = \{1,2,3,4,5\}$ and $Y = \{1,2,3\}$. Define

$$\alpha = \begin{pmatrix} \{1,2,3\} & \{4,5\} \\ 1 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \{1,2\} & \{3,4,5\} \\ 3 & 2 \end{pmatrix}.$$

Then $\alpha, \beta \in S(X,Y)$ such that α is an idempotent and

$$\beta^2 = \begin{pmatrix} \{1,2\} & \{3,4,5\} \\ 2 & 3 \end{pmatrix}$$

Thus $\alpha^3 = \alpha$ and $\beta^2|_{X\beta} = id_{X\beta}$, and so $\alpha, \beta \in \text{CoReg}(S(X,Y))$. Consider

$$\alpha\beta = \begin{pmatrix} \{1,2,3\} & \{4,5\} \\ 3 & 2 \end{pmatrix}.$$

We see that $2(\alpha\beta) = 3 \notin \{4,5\} = 2(\alpha\beta)^{-1}$. By Theorem 3.4 (2), we have $\alpha\beta$ is not coregular, that means $\alpha\beta \notin \text{CoReg}(S(X,Y))$.

In order to give necessary and sufficient conditions for the set of all coregular elements is a subsemigroup of $S(X,Y)$, the following lemma is needed.

Lemma 3.8. *If $|X| \geq 3$, then* $\text{CoReg}(S(X,Y))$ *is not a subsemigroup of* $S(X,Y)$.

Proof. Let $|X| \geq 3$. We consider two cases.

Case 1: $X = Y$. Let x, y, z be distinct elements in X. Define $\alpha, \beta \in S(X,Y) = T(X)$ by

$$\alpha = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & z \end{pmatrix} \text{ and } \beta = \begin{pmatrix} y & X \setminus \{y\} \\ y & x \end{pmatrix}.$$

Then α, β are idempotents and so α, β are coregular elements. We see that

$$\alpha\beta = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & x \end{pmatrix}$$

and $x \in X\alpha\beta$ such that $x(\alpha\beta) = y \notin X \setminus \{x,y\} = x(\alpha\beta)^{-1}$. By Theorem 3.4 (2), we have $\alpha\beta$ is not a coregular element.

Case 2: $Y \subsetneq X$. Let $y \in Y$ and $z \in X \setminus Y$.

Subcase 2.1: $|Y| = 1$. Thus $Y = \{y\}$ and there exists $x \in X \setminus \{y,z\}$. Define

$$\alpha = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & z \end{pmatrix} \text{ and } \beta = \begin{pmatrix} y & X \setminus \{y\} \\ y & x \end{pmatrix}.$$

We see that α, β are idempotents in $S(X,Y)$ and so α, β are coregular. Then

$$\alpha\beta = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & x \end{pmatrix}$$

and $x(\alpha\beta) = y \notin X \setminus \{x,y\} = x(\alpha\beta)^{-1}$. Thus $\alpha\beta$ is not a coregular element.

Subcase 2.2: $|Y| \geq 2$. Then there is $x \in Y \setminus \{y\}$. Define

$$\alpha = \begin{pmatrix} Y & X \setminus Y \\ y & z \end{pmatrix} \text{ and } \beta = \begin{pmatrix} Y \setminus \{x\} & (X \setminus Y) \cup \{x\} \\ y & x \end{pmatrix}.$$

So $\alpha, \beta \in S(X,Y)$ and α, β are idempotents, which implies that α, β are coregular. We obtain that

$$\alpha\beta = \begin{pmatrix} Y & X \setminus Y \\ y & x \end{pmatrix}$$

and $x(\alpha\beta) = y \notin X \setminus Y = x(\alpha\beta)^{-1}$, thus $\alpha\beta$ is not coregular.

Therefore, $\text{CoReg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$. ∎

The following theorem is a direct consequence of Theorem 3.6 and Lemma 3.8.

Theorem 3.9. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is coregular.*
(3) $\mathrm{CoReg}(S(X,Y))$ *is a subsemigroup of* $S(X,Y)$.

4. Some Properties of Regularity of $S(X,Y)$

In this section, we characterize when $S(X,Y)$ is left regular, right regular, completely regular and intra-regular. Moreover, we give necessary and sufficient conditions for $\mathrm{LReg}(S(X,Y))$, $\mathrm{RReg}(S(X,Y))$, $\mathrm{CReg}(S(X,Y))$ and $\mathrm{IReg}(S(X,Y))$ to be subsemigroups of $S(X,Y)$.

Theorem 4.1. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is left regular.*
(3) $\mathrm{LReg}(S(X,Y))$ *is a subsemigroup of* $S(X,Y)$.

Proof. It is clear that (2) $\Rightarrow$ (3). To prove that (1) $\Rightarrow$ (2), assume that $|X| \leq 2$. Then $S(X,Y)$ is left regular by Remark 3.1 and Lemma 3.2. Now, we prove that (3) $\Rightarrow$ (1). Assume that $|X| \geq 3$. If $X = Y$, then we define α, β as in Lemma 3.8 (Case 1). So α, β are idempotent and hence α, β are left regular elements. We see that

$$\alpha\beta = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & x \end{pmatrix} \text{ and } (\alpha\beta)^2 = \begin{pmatrix} X \\ y \end{pmatrix},$$

so $X(\alpha\beta) \neq X(\alpha\beta)^2$. Thus $\alpha\beta$ is not left regular by Theorem 2.2 (1). If $Y \subsetneq X$ and $|Y| = 1$, then we define α, β as in Lemma 3.8 (Subcase 2.1). By the same prove as given for the case $X = Y$, we get α, β are left regular, but $\alpha\beta$ is not left regular. And, if $Y \subsetneq X$ and $|Y| \geq 2$, we define α, β as in Lemma 3.8 (Subcase 2.2). So α, β are idempotent and also left regular elements. We obtain that

$$\alpha\beta = \begin{pmatrix} Y & X \setminus Y \\ y & x \end{pmatrix} \text{ and } (\alpha\beta)^2 = \begin{pmatrix} X \\ y \end{pmatrix},$$

and hence $X(\alpha\beta) \neq X(\alpha\beta)^2$. Thus $\alpha\beta$ is not left regular by Theorem 2.2 (1). Therefore, $\mathrm{LReg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$. ∎

Theorem 4.2. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is right regular.*
(3) $\mathrm{RReg}(S(X,Y))$ *is a subsemigroup of* $S(X,Y)$.

Proof. Clearly (2) $\Rightarrow$ (3). By Remark 3.1 and Lemma 3.2, we conclude that (1) $\Rightarrow$ (2). Now, we prove that (3) $\Rightarrow$ (1). Assume that $|X| \geq 3$. Define $\alpha, \beta \in S(X,Y)$ as in Lemma 3.8. Then α, β are right regular elements. But, we obtain that $\pi_{\alpha\beta} \neq \pi_{(\alpha\beta)^2}$, thus $\alpha\beta$ is not right regular by Theorem 2.2 (2). So $\mathrm{RReg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$. ∎

Remark 4.3. $\alpha \in S(X,Y)$ is completely regular if and only if α is both left and right regular.

Theorem 4.4. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is completely regular.*
(3) $\mathrm{CReg}(S(X,Y))$ *is a subsemigroup of* $S(X,Y)$.

Proof. Obviously (2) $\Rightarrow$ (3). By Theorems 4.1, 4.2 and Remark 4.3, it follows immediately that (1) $\Rightarrow$ (2). Assume that $|X| \geq 3$. Define $\alpha, \beta \in S(X,Y)$ as in Lemma 3.8. Then α, β are completely regular elements. But, we get $\pi_{\alpha\beta} \neq \pi_{(\alpha\beta)^2}$, thus $\alpha\beta$ is not right regular by Theorem 2.2 (2) and hence $\alpha\beta$ is not completely regular by Remark 4.3. So $\mathrm{CReg}(S(X,Y))$ is not a subsemigroup of $S(X,Y)$. ∎

Theorem 4.5. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is intra-regular.*
(3) $\mathrm{IReg}(S(X,Y))$ *is a subsemigroup of* $S(X,Y)$.

Proof. From Remark 3.1 and Lemma 3.2, we obtain that (1) $\Rightarrow$ (2). To prove (3) $\Rightarrow$ (1), assume that $|X| \geq 3$. Then there exist $\alpha, \beta \in \mathrm{IReg}(S(X,Y))$ which are defined as in Lemma 3.8. But, we obtain that $|X\alpha\beta| = 2 \neq 1 = |X(\alpha\beta)^2|$ and hence $\alpha\beta \notin \mathrm{IReg}(S(X,Y))$ by Theorem 2.2 (3). And, it is clear that (2) $\Rightarrow$ (3), so the proof is complete. ∎

As a direct consequence of Theorems 3.9, 4.1, 4.2, 4.4 and 4.5, we have the following corollary.

Corollary 4.6. *The following statements are equivalent.*
(1) $|X| \leq 2$.
(2) $S(X,Y)$ *is coregular.*
(3) $S(X,Y)$ *is left regular.*
(4) $S(X,Y)$ *is right regular.*
(5) $S(X,Y)$ *is completely regular.*
(6) $S(X,Y)$ *is intra-regular.*

We know that every coregular element is both left and right regular, but there are left and right regular elements which are not coregular as shown in the following lemma.

Lemma 4.7. *If* $|Y| \geq 3$ *or* $|X \setminus Y| \geq 3$, *then there exists* $\alpha \in \mathrm{LReg}(S(X,Y)) \cap \mathrm{RReg}(S(X,Y))$ *such that* $\alpha \notin \mathrm{CoReg}(S(X,Y))$.

Proof. Assume that $|Y| \geq 3$ or $|X \setminus Y| \geq 3$. We consider two cases.

Case 1: $|Y| \geq 3$. Let a, b, c be distinct elements in Y. Define

$$\alpha = \begin{pmatrix} a & b & c & x \\ b & c & a & x \end{pmatrix}_{x \in X \setminus \{a,b,c\}}.$$

Then $\alpha \in S(X,Y)$ and we see that

$$\alpha^2 = \begin{pmatrix} a & b & c & x \\ c & a & b & x \end{pmatrix}_{x \in X \setminus \{a,b,c\}}.$$

So $X\alpha = X = X\alpha^2, Y\alpha = Y = Y\alpha^2, \pi_\alpha = \pi_{\alpha^2}$ and $\pi_\alpha(Y) = \pi_{\alpha^2}(Y)$. By Theorem 2.2, we obtain that α is left regular and right regular. But, since $c\alpha = a \notin \{b\} = c\alpha^{-1}$, we have $\alpha \notin \mathrm{CoReg}(S(X,Y))$ by Lemma 3.4 (2).

Case 2: $|X \setminus Y| \geq 3$. Let a, b, c be distinct elements in $X \setminus Y$. Define α as in Case 1. By the same proof, we obtain that α is left regular and right regular but not coregular. ∎

Recall that $\text{CoReg}(S(X,Y)) = \text{LReg}(S(X,Y)) = \text{RReg}(S(X,Y)) = \text{CReg}(S(X,Y)) = \text{IReg}(S(X,Y)) = S(X,Y)$ when $|X| \leq 2$. Here, there are some other cases that coregular elements, left regular elements, right regular elements, completely regular elements and intra-regular elements are coincide as the following theorem.

Theorem 4.8. *If* $(|X|,|Y|) \in \{(3,1),(3,2),(4,2)\}$, *then*

$$\text{CoReg}(S(X,Y)) = \text{LReg}(S(X,Y)) = \text{RReg}(S(X,Y)) = \text{CReg}(S(X,Y)) = \text{IReg}(S(X,Y)).$$

Proof. We know that every idempotent is coregular and every coregular element is left regular. So $E(S(X,Y)) \subseteq \text{CoReg}(S(X,Y)) \subseteq \text{LReg}(S(X,Y))$. Assume that $(|X|,|Y|) \in \{(3,1),(3,2),(4,2)\}$. We consider three cases.

Case 1: $(|X|,|Y|) = (3,1)$. Let $X = \{a,b,c\}$ and $Y = \{a\}$. Then

$$E(S(X,Y)) = \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & \{b,c\} \\ a & b \end{pmatrix}, \begin{pmatrix} a & \{b,c\} \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,b\} & c \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,c\} & b \\ a & b \end{pmatrix}, \begin{pmatrix} \{a,b,c\} \\ a \end{pmatrix} \right\}.$$

Let

$$\alpha = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \in S(X,Y).$$

Then $\alpha^2 = id_X$ and hence α is coregular by Theorem 3.4 (3). Hence $E(S(X,Y)) \cup \{\alpha\} \subseteq \text{CoReg}(S(X,Y)) \subseteq \text{LReg}(S(X,Y))$. By Theorem 2.4, we get

$$\begin{aligned}
|\text{LReg}(S(X,Y))| &= \sum_{m=0}^{2} \sum_{k=1}^{1} \binom{1}{k} k! k^{1-k} \binom{2}{m} m! (k+m)^{2-m} \\
&= \binom{1}{1} 1! 1^{1-1} \binom{2}{0} 0! (1+0)^{2-0} + \binom{1}{1} 1! 1^{1-1} \binom{2}{1} 1! (1+1)^{2-1} \\
&\quad + \binom{1}{1} 1! 1^{1-1} \binom{2}{2} 2! (1+2)^{2-2} \\
&= 1 + 4 + 2 \\
&= 7.
\end{aligned}$$

Since $|E(S(X,Y)) \cup \{\alpha\}| = 7 = |\text{LReg}(S(X,Y))|$, we obtain that $\text{CoReg}(S(X,Y)) = \text{LReg}(S(X,Y))$.

Case 2: $(|X|,|Y|) = (3,2)$. let $X = \{a,b,c\}$ and $Y = \{a,b\}$. Then

$$E(S(X,Y)) = \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} \{a,b\} & c \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,b\} & c \\ b & c \end{pmatrix}, \begin{pmatrix} \{a,c\} & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & \{b,c\} \\ a & b \end{pmatrix}, \begin{pmatrix} \{a,b,c\} \\ a \end{pmatrix}, \begin{pmatrix} \{a,b,c\} \\ b \end{pmatrix} \right\}.$$

Define $\alpha_1, \alpha_2, \alpha_3 \in S(X,Y)$ by

$$\alpha_1 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} \{a,c\} & b \\ b & a \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} a & \{b,c\} \\ b & a \end{pmatrix}.$$

Then ${\alpha_1}^2 = id_X$, ${\alpha_2}^2 = \begin{pmatrix} \{a,c\} & b \\ a & b \end{pmatrix}$ and ${\alpha_3}^2 = \begin{pmatrix} a & \{b,c\} \\ a & b \end{pmatrix}$ and so ${\alpha_2}^2|_{X\alpha_2} = id_{X\alpha_2}$ and ${\alpha_3}^2|_{X\alpha_3} = id_{X\alpha_3}$. By Theorem 3.4 (3), we see that $\alpha_1, \alpha_2, \alpha_3$ are coregular. Hence $E(S(X,Y)) \cup \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \text{CoReg}(S(X,Y)) \subseteq \text{LReg}(S(X,Y))$. By Theorem 2.4, we have

$$\begin{aligned}
|\text{LReg}(S(X,Y))| &= \sum_{m=0}^{1}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{1}{m}m!(k+m)^{1-m} \\
&= \sum_{m=0}^{1}\left[\binom{2}{1}1!1^{2-1}\binom{1}{m}m!(1+m)^{1-m}\right. \\
&\qquad \left. +\binom{2}{2}2!2^{2-2}\binom{1}{m}m!(2+m)^{1-m}\right] \\
&= \sum_{m=0}^{1}\left[(2)\binom{1}{m}m!(1+m)^{1-m} + (2)\binom{1}{m}m!(2+m)^{1-m}\right] \\
&= \left[(2)\binom{1}{0}0!(1+0)^{1-0} + (2)\binom{1}{0}0!(2+0)^{1-0}\right] \\
&\qquad + \left[(2)\binom{1}{1}1!(1+1)^{1-1} + (2)\binom{1}{1}1!(2+1)^{1-1}\right] \\
&= 2+4+2+2 \\
&= 10.
\end{aligned}$$

So $|E(S(X,Y)) \cup \{\alpha_1, \alpha_2, \alpha_3\}| = 10 = |\text{LReg}(S(X,Y))|$ and thus $\text{CoReg}(S(X,Y)) = \text{LReg}(S(X,Y))$.

Case 3: $(|X|,|Y|) = (4,2)$. Let $X = \{a,b,c,d\}$ and $Y = \{a,b\}$. Then

$$\begin{aligned}
E(S(X,Y)) = \Bigg\{ &\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & c & d \\ a & c & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & c & d \\ b & c & d \end{pmatrix}, \\
&\begin{pmatrix} \{a,c\} & b & d \\ a & b & d \end{pmatrix}, \begin{pmatrix} \{a,d\} & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & \{b,c\} & d \\ a & b & d \end{pmatrix}, \\
&\begin{pmatrix} a & \{b,d\} & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & \{c,d\} \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & \{c,d\} \\ a & b & d \end{pmatrix}, \\
&\begin{pmatrix} \{a,b\} & \{c,d\} \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,b\} & \{c,d\} \\ a & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & \{c,d\} \\ b & c \end{pmatrix}, \\
&\begin{pmatrix} \{a,b\} & \{c,d\} \\ b & d \end{pmatrix}, \begin{pmatrix} \{a,c\} & \{b,d\} \\ a & b \end{pmatrix}, \begin{pmatrix} \{a,d\} & \{b,c\} \\ a & b \end{pmatrix}, \\
&\begin{pmatrix} \{a,b,c\} & d \\ a & d \end{pmatrix}, \begin{pmatrix} \{a,b,c\} & d \\ b & d \end{pmatrix}, \begin{pmatrix} \{a,b,d\} & c \\ a & c \end{pmatrix}, \\
&\begin{pmatrix} \{a,b,d\} & c \\ b & c \end{pmatrix}, \begin{pmatrix} \{a,c,d\} & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & \{b,c,d\} \\ a & b \end{pmatrix}, \\
&\begin{pmatrix} \{a,b,c,d\} \\ a \end{pmatrix}, \begin{pmatrix} \{a,b,c,d\} \\ b \end{pmatrix} \Bigg\}.
\end{aligned}$$

Define

$$\alpha_1 = \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} \{a,b\} & c & d \\ a & d & c \end{pmatrix}, \quad \alpha_5 = \begin{pmatrix} \{a,b\} & c & d \\ b & d & c \end{pmatrix}, \quad \alpha_6 = \begin{pmatrix} \{a,c\} & b & d \\ b & a & d \end{pmatrix},$$

$$\alpha_7 = \begin{pmatrix} \{a,d\} & b & c \\ b & a & c \end{pmatrix}, \quad \alpha_8 = \begin{pmatrix} a & \{b,c\} & d \\ b & a & d \end{pmatrix}, \quad \alpha_9 = \begin{pmatrix} a & \{b,d\} & c \\ b & a & c \end{pmatrix},$$

$$\alpha_{10} = \begin{pmatrix} a & b & \{c,d\} \\ b & a & c \end{pmatrix}, \quad \alpha_{11} = \begin{pmatrix} a & b & \{c,d\} \\ b & a & d \end{pmatrix}, \quad \alpha_{12} = \begin{pmatrix} \{a,c\} & \{b,d\} \\ b & a \end{pmatrix},$$

$$\alpha_{13} = \begin{pmatrix} \{a,d\} & \{b,c\} \\ b & a \end{pmatrix}, \quad \alpha_{14} = \begin{pmatrix} \{a,c,d\} & b \\ b & a \end{pmatrix}, \quad \alpha_{15} = \begin{pmatrix} a & \{b,c,d\} \\ b & a \end{pmatrix}.$$

Then $\alpha_i \in S(X,Y)$ for all $i \in \{1,\dots,15\}$. It is easy to check that ${\alpha_i}^2|_{X\alpha_i} = id_{X\alpha_i}$ for all $i \in \{1,\dots,15\}$. Thus $\alpha_i \in \mathrm{CoReg}(S(X,Y))$ by Theorem 3.4 (3) and hence

$$E(S(X,Y)) \cup \{\alpha_i : i \in \{1,\dots,15\}\} \subseteq \mathrm{CoReg}(S(X,Y)) \subseteq \mathrm{LReg}(S(X,Y)).$$

By Theorem 2.4, we obtain that

$$\begin{aligned}
|\mathrm{LReg}(S(X,Y))| &= \sum_{m=0}^{2}\sum_{k=1}^{2}\binom{2}{k}k!k^{2-k}\binom{2}{m}m!(k+m)^{2-m} \\
&= \sum_{m=0}^{2}\Bigg[\binom{2}{1}1!1^{2-1}\binom{2}{m}m!(1+m)^{2-m} \\
&\qquad + \binom{2}{2}2!2^{2-2}\binom{2}{m}m!(2+m)^{2-m}\Bigg] \\
&= \sum_{m=0}^{2}\left[(2)\binom{2}{m}m!(1+m)^{2-m} + (2)\binom{2}{m}m!(2+m)^{2-m}\right] \\
&= \left[(2)\binom{2}{0}0!(1+0)^{2-0} + (2)\binom{2}{0}0!(2+0)^{2-0}\right] \\
&\quad + \left[(2)\binom{2}{1}1!(1+1)^{2-1} + (2)\binom{2}{1}1!(2+1)^{2-1}\right] \\
&\quad + \left[(2)\binom{2}{2}2!(1+2)^{2-2} + (2)\binom{2}{2}2!(2+2)^{2-2}\right] \\
&= 2+8+8+12+4+4 \\
&= 38.
\end{aligned}$$

So $|E(S(X,Y)) \cup \{\alpha_i : i \in \{1,\dots,15\}\}| = |E(S(X,Y))| + 15 = 23 + 15 = 38 = |\mathrm{LReg}(S(X,Y))|$. Hence $\mathrm{CoReg}(S(X,Y)) = \mathrm{LReg}(S(X,Y))$.

Since X is a finite set, we have $\mathrm{LReg}(S(X,Y)) = \mathrm{RReg}(S(X,Y)) = \mathrm{IReg}(S(X,Y))$ by Theorem 2.3. By Remark 4.3, we get $\mathrm{LReg}(S(X,Y)) = \mathrm{RReg}(S(X,Y)) = \mathrm{CReg}(S(X,Y))$. Therefore, $\mathrm{CoReg}(S(X,Y)) = \mathrm{LReg}(S(X,Y)) = \mathrm{RReg}(S(X,Y)) = \mathrm{CReg}(S(X,Y)) = \mathrm{IReg}(S(X,Y))$. ■

Moreover, we have the following theorem.

Theorem 4.9. $\mathrm{CoReg}(S(X,Y)) = \mathrm{LReg}(S(X,Y)) = \mathrm{RReg}(S(X,Y))$ *if and only if* $|X| \leq 2$ *or* $(|X|,|Y|) \in \{(3,1),(3,2),(4,2)\}$.

Proof. Assume that $|X| \leq 2$ or $(|X|,|Y|) \in \{(3,1),(3,2),(4,2)\}$. By Remark 3.1, Lemma 3.2 and Theorem 4.8, we get $\mathrm{CoReg}(S(X,Y)) = \mathrm{LReg}(S(X,Y)) = \mathrm{RReg}(S(X,Y))$.

Conversely, assume that $|X| \geq 3$ and $(|X|,|Y|) \notin \{(3,1),(3,2),(4,2)\}$. Then $|Y| \geq 3$ or $|X \setminus Y| \geq 3$. By Lemma 4.7, $\mathrm{CoReg}(S(X,Y)) \subsetneq \mathrm{LReg}(S(X,Y)) \cap \mathrm{RReg}(S(X,Y))$, that means $\mathrm{CoReg}(S(X,Y)) \neq \mathrm{LReg}(S(X,Y))$ and $\mathrm{CoReg}(S(X,Y)) \neq \mathrm{RReg}(S(X,Y))$. ■

5. Finiteness Conditions on $S(X,Y)$

In this section, we characterize when $S(X,Y)$ is unit regular and directly finite which depend on the finiteness conditions on sets.

Alarcao [1] characterized when a monoid S is directly finite and gave a relationship between a unit regular semigroup and a factorizable semigroup as follows.

Theorem 5.1. [1, Propositions 1-3] *Let S be a semigroup with identity 1. Then the following statements hold.*

(1) *S is unit regular if and only if S is factorizable.*

(2) *S is directly finite if and only if $H_1 = D_1$.*

Later, Boonmee [3] characterized when $S(X,Y)$ is factorizable as follows.

Theorem 5.2. [3, Theorem 3.3.13] *$S(X,Y)$ is factorizable if and only if the following conditions hold.*

(1) *X is finite.*

(2) *$X = Y$ or $|Y| = 1$.*

As a direct consequence of Theorems 5.1 and 5.2, we have the following theorem.

Theorem 5.3. *$S(X,Y)$ is unit regular if and only if the following statements hold.*

(1) *X is a finite set,*

(2) *$X = Y$ or $|Y| = 1$.*

The following example shows that if X is an infinite set, $S(X,Y)$ need not be directly finite.

Example 5.4. Let $X = \mathbb{N}$ and Y be the set of all even positive integers. Define

$$\alpha = \begin{pmatrix} 2 & 2n+2 & 2n-1 \\ 2 & 2n+4 & 2n-1 \end{pmatrix}_{n\geq 1}$$

and

$$\beta = \begin{pmatrix} \{2,4\} & 2n+4 & 2n-1 \\ 2 & 2n+2 & 2n-1 \end{pmatrix}_{n\geq 1}.$$

Then $\alpha, \beta \in S(X,Y)$ and

$$\alpha\beta = \begin{pmatrix} 2 & 2n+2 & 2n-1 \\ 2 & 2n+2 & 2n-1 \end{pmatrix}_{n\geq 1} = id_X,$$

but

$$\beta\alpha = \begin{pmatrix} \{2,4\} & 2n+4 & 2n-1 \\ 2 & 2n+4 & 2n-1 \end{pmatrix}_{n\geq 1} \neq id_X.$$

Thus $S(X,Y)$ is not directly finite.

Finally, we give the necessary and sufficient condition for $S(X,Y)$ to be directly finite.

Theorem 5.5. *$S(X,Y)$ is directly finite if and only if X is a finite set.*

Proof. Assume that X is a finite set. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha\beta = id_X$. Then α is injective. Since X is finite, we have α is surjective and hence $X\alpha = X$. Let $x \in X$. Thus $x \in X\alpha$ and there exists $z \in X$ such that $x = z\alpha$. So $x\beta\alpha = z\alpha\beta\alpha = z\, id_X\, \alpha = z\alpha = x$, and we conclude that $\beta\alpha = id_X$.

Conversely, assume that X is an infinite set. To prove that $S(X,Y)$ is not directly finite, we consider two cases.

Case 1: Y is infinite. Choose $a \in Y$. Then $|Y \setminus \{a\}| = |Y|$ and hence there is a bijection $\varphi : Y \setminus \{a\} \to Y$. Let $Y \setminus \{a\} = \{y_i : i \in I\}$. Fix $i_0 \in I$ and let $I' = I \setminus \{i_0\}$. Define $\alpha \in S(X,Y)$ by

$$\alpha = \begin{pmatrix} \{y_{i_0}, a\} & y_{i'} & x \\ y_{i_0}\varphi & y_{i'}\varphi & x \end{pmatrix}_{x \in X\setminus Y}.$$

Thus α is surjective and hence $|Y\alpha| = |Y| = |Y id_X|$, $|X\alpha \setminus Y| = |X \setminus Y| = |X id_X \setminus Y|$ and $|(X\alpha \cap Y) \setminus Y\alpha| = |Y \setminus Y| = |(X id_X \cap Y) \setminus Y id_X|$. So $\alpha \in D_{id_X}$ by Theorem 2.1 (4). We see that $\pi_\alpha \neq \pi_{id_X}$, that means $\alpha \notin H_{id_X}$. Thus $D_{id_X} \neq H_{id_X}$ and hence $S(X,Y)$ is not directly finite by Theorem 5.1 (2).

Case 2: Y is finite. Thus $X \setminus Y$ is infinite. Choose $b \in X \setminus Y$. Then $|X \setminus (Y \cup \{b\})| = |X \setminus Y|$ and there exists a bijection $\sigma : X \setminus (Y \cup \{b\}) \to X \setminus Y$. Let $X \setminus (Y \cup \{b\}) = \{x_j : j \in J\}$. Fix $j_0 \in J$ and let $J' = J \setminus \{j_0\}$. Define

$$\beta = \begin{pmatrix} y & \{x_{j_0}, b\} & x_{j'} \\ y & x_{j_0}\sigma & x_{j'}\sigma \end{pmatrix}_{y \in Y}.$$

Then $\beta \in S(X,Y)$ and $X\beta = X$. So $|Y\beta| = |Y| = |Y id_X|$, $|X\beta \setminus Y| = |X \setminus Y| = |X id_X \setminus Y|$ and $|(X\beta \cap Y) \setminus Y\beta| = |Y \setminus Y| = |(X id_X \cap Y) \setminus Y id_X|$. By Theorem 2.1 (4), we have $\beta \in D_{id_X}$. However, $\beta \notin H_{id_X}$ since $\pi_\beta \neq \pi_{id_X}$. Thus $D_{id_X} \neq H_{id_X}$ and therefore, $S(X,Y)$ is not directly finite. ∎

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