Thai Journal of Mathematics
Volume 21 Number 3 (2023)
Pages 563–567

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On the Non-Linear Diophantine Equations $4^x - a^y = dz^2$ and $4^x + a^y = dz^2$

Suton Tadee[1] **and Umarin Pintoptang**[2,*]

[1] *Department of Mathematics, Faculty of Science and Technology, Thepsatri Rajabhat University, Lopburi 15000, Thailand*
e-mail : suton.t@lawasri.tru.ac.th

[2] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
e-mail : umarinp@nu.ac.th

Abstract In this article, we study Diophantine equations $4^x - a^y = dz^2$ and $4^x + a^y = dz^2$ where a, d, x, y, and z are non-negative integers. Under some conditions of integers a, d and by using congruence properties, we give all non-negative integer solutions of $4^x - a^y = dz^2$ and we show that $4^x + a^y = dz^2$ has no non-negative integer solution.

MSC: 11D61
Keywords: Diophantine equation; integer solutions; congruence

Submission date: 10.09.2022 / Acceptance date: 23.07.2023

1. Introduction

Solving Diophantine equation is one of the most interesting topics in number theory and it has been widely studied for many years. The knowledge of Diophantine equation can be applied to many areas such as balancing chemical equations, network flow and word problem on business [1–4]. The solutions of Diophantine equation are investigated by many researchers and the solutions of Diophantine equation usually mean non-negative integer solutions. In 2011, Suvarnamani et al. [5] proved that the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. Later Chotchaisthit [6] showed that the Diophantine equation $4^x + p^y = z^2$, where p is a prime, has all non-negative integer solutions of the form $(x, y, z, p) \in \{(2, 2, 5, 3)\} \cup \{(r, 1, 2^r + 1, 2^{r+1} + 1) : r \in \mathbb{N}_0\} \cup \{(r, 2r + 3, 3 \cdot 2^r, 2) : r \in \mathbb{N}_0\}$. Then Burshtein [7] found positive integer solutions of the Diophantine equation $p^x + q^y = z^2$ where p and q are odd primes with $q - p \in \{2, 4, 6, 8\}$. In 2018, Rabago [8] showed the set of all non-negative integer solutions of the Diophantine equation $4^x - p^y = z^2$ where a prime $p \equiv 3 \pmod 4$, that is, $(x, y, z, p) \in \{(0, 0, 0, p)\} \cup \{(q - 1, 1, 2^{q-1} - 1, 2^q - 1)\}$. Moreover, Rabago proved that the Diophantine equation $4^x - p^y = 3z^2$, where a prime $p \equiv 3 \pmod 4$,

*Corresponding author.

Published by The Mathematical Association of Thailand.

has only two non-negative integer solutions $(x, y, z) \in \{(0,0,0), (1,0,1)\}$. Recently, Elshahed and Kamarulhaili [9] studied the Diophantine equation $(4^n)^x - p^y = z^2$ where p is an odd prime and n is a positive integer. Elshahed and Kamarulhaili listed all of non-negative integer solutions of such equation, that is, $(x, y, z, p) \in \left\{\left(k, 1, 2^{nk}-1, 2^{nk+1}-1\right) : k \in \mathbb{N}\right\} \cup \{(0,0,0,p)\}$. For the Diophantine equation $4^x - p^y = 3z^2$ where a prime $p \equiv 3 \pmod 4$, all solutions of it are completely investigated in [8]. While the case $p \equiv 1 \pmod 4$, there was no result about its solutions and this case was left as an open problem in [8]. This therefore motivates us to do research in this paper. In this article, the Diophantine equation $4^x - p^y = 3z^2$ of [8] is extended to $4^x - a^y = dz^2$ where an integer $a \equiv 1, 3 \pmod 4$ does not need to be prime and an integer $d \equiv 3 \pmod 8$ does not need to equal to 3. In the special case when primes $a \equiv 3 \pmod 4$ and $d = 3$, we get the same result as in the main theorem of [8]. Moreover, we show that the Diophantine equation $4^x + a^y = dz^2$ has no non-negative integer solution with some assumptions of integers a and d. The proofs of our theorems base on properties of congruence and our proofs do not apply the previous results of Diophantine equations from any researchers.

2. Preliminaries

In this section, we recall some properties of congruence which is the background for proving our main results. Let $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{N}$. We say a is congruent to b modulo m if $m|(a-b)$, and denote by $a \equiv b \pmod m$. We collect some properties of congruence as follows.

1. If $a \equiv b \pmod m$ and $b \equiv c \pmod m$, then $a \equiv c \pmod m$.
2. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $a \pm c \equiv b \pm d \pmod m$.
3. If $a \equiv b \pmod m$, then $ac \equiv bc \pmod m$.
4. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $ac \equiv bd \pmod m$.
5. If $a \equiv b \pmod m$, then $a^n \equiv b^n \pmod m$ for all $n \in \mathbb{N}$.

3. Main Results

In this study, we focus on Diophantine equations $4^x - a^y = dz^2$ and $4^x + a^y = dz^2$. In the first part, we investigate the Diophantine equation

$$4^x - a^y = dz^2. \tag{3.1}$$

We start to give all non-negative integer solutions of (3.1) where $a \equiv 3 \pmod 4$ and $d \equiv 3 \pmod 8$ in Theorem 3.1.

Theorem 3.1. *Let $a, d \in \mathbb{N}$ be such that $a \equiv 3 \pmod 4$ and $d \equiv 3 \pmod 8$. Then the Diophantine equation (3.1) has non-negative integer solutions $(x, y, z) \in \{(0,0,0), (1,0,1)\}$ for $d = 3$ and $(x, y, z) = (0,0,0)$ for $d \neq 3$.*

Proof. Let $x, y, z \in \mathbb{N}_0$ be a solution of (3.1).

Case 1. $x = 0$. From (3.1), we consider equation $1 - a^y = dz^2$. Since $0 \leq dz^2 = 1 - a^y, a \geq 3$ and $d \geq 3$, we get $y = 0$ and $z = 0$. Then $(x, y, z) = (0,0,0)$ is a solution of (3.1) in this case.

Case 2. $x = 1$. From (3.1), we have equation $4 - a^y = dz^2$. Since $0 \leq dz^2 = 4 - a^y$ and $a \geq 3$, we have $y = 0$ or $y = 1$. Since $d \geq 3$ and $z \in \mathbb{N}_0$, we get $dz^2 \neq 1$. If $y = 1$ then $a = 3$ and $1 = dz^2$, a contradiction. If $y = 0$ then $3 = dz^2$ so $d = 3$ and $z = 1$. Thus, $(x, y, z) = (1,0,1)$ is a solution of (3.1) in this case.

Case 3. $x \geq 2$. Since a is odd, we see that a^y is odd and so $4^x - a^y$ is odd. From (3.1), it implies that z is odd, and hence $z^2 \equiv 1 \pmod 8$. By assumption $d \equiv 3 \pmod 8$, we get $dz^2 \equiv 3 \pmod 8$. Consequently, $dz^2 \equiv -1 \pmod 4$. From (3.1), we have $4^x - a^y \equiv -1 \pmod 4$ and $a^y \equiv 1 \pmod 4$. Since $a \equiv -1 \pmod 4$, we have y is even. Then $y = 2n$ for some $n \in \mathbb{N}_0$. Since a is odd, we see that $a^2 \equiv 1 \pmod 8$. Thus, $a^y = a^{2n} \equiv 1 \pmod 8$. This implies $4^x - a^y \equiv -1 \pmod 8$ and so $dz^2 \equiv -1 \pmod 8$ which contradicts $dz^2 \equiv 3 \pmod 8$. Hence, there is no solution of (3.1) when $x \geq 2$. ∎

The assumptions $a \equiv 3 \pmod 4$ and $d \equiv 3 \pmod 8$ in Theorem 3.1 make equation (3.1) a general of $4^x - p^y = 3z^2$ which studied in [8]. The special case of (3.1) when $y = 0$ is investigated in the next corollary and the proof of it is directly from Theorem 3.1.

Corollary 3.2. *If $d \in \mathbb{N}$ with $d \equiv 3 \pmod 8$, then the Diophantine equation $4^x - 1 = dz^2$ has non-negative integer solutions $(x, z) \in \{(0,0), (1,1)\}$ for $d = 3$ and $(x, z) = (0, 0)$ for $d \neq 3$.*

As we know that the Diophantine equation $4^x - p^y = 3z^2$ where a prime $p \equiv 1 \pmod 4$ is still left as an open problem in [8], so we try to find out the solutions of $4^x - p^y = 3z^2$ in this case $p \equiv 1 \pmod 4$. We remark that $a \equiv 1 \pmod 4$ iff $a \equiv 1 \pmod 8$ or $a \equiv 5 \pmod 8$. In another word, set of integers a, which $a \equiv 1 \pmod 4$, is separated into two parts, that is, $a \equiv 1 \pmod 8$ or $a \equiv 5 \pmod 8$. In the next theorem, we investigate the first part $a \equiv 1 \pmod 8$ and we give all non-negative integer solutions of general equation $4^x - a^y = dz^2$ when $a \equiv 1 \pmod 8$ and $d \equiv 3 \pmod 8$.

Theorem 3.3. *If $a, d \in \mathbb{N} - \{1\}$ with $a \equiv 1 \pmod 8$ and $d \equiv 3 \pmod 8$, then the Diophantine equation (3.1) has non-negative integer solutions $(x, y, z) \in \{(0,0,0), (1,0,1)\}$ for $d = 3$ and $(x, y, z) = (0, 0, 0)$ for $d \neq 3$.*

Proof. Let $x, y, z \in \mathbb{N}_0$ be a solution of (3.1).

Case 1. $x = 0$. From (3.1), we consider equation $1 - a^y = dz^2$. Note that $0 \leq dz^2 = 1 - a^y$. Since $a \neq 1$ and $a \equiv 1 \pmod 8$, we have $y = 0$ and so $z = 0$. Thus, $(x, y, z) = (0, 0, 0)$ is a solution of (3.1) in this case.

Case 2. $x = 1$. From (3.1), we consider equation $4 - a^y = dz^2$. Recall that $0 \leq dz^2 = 4 - a^y$. Since $a > 1$ and $a \equiv 1 \pmod 8$, we get $y = 0$. Then $3 = dz^2$. Since $z^2 \neq 3$, we have $d = 3$ and $z = 1$. Hence, $(x, y, z) = (1, 0, 1)$ is a solution of (3.1) when $d = 3$.

Case 3. $x \geq 2$. Remark that $4^x \equiv 0 \pmod 8$. Since $a \equiv 1 \pmod 8$, we obtain $a^y \equiv 1 \pmod 8$. Then we get $4^x - a^y \equiv -1 \pmod 8$ and $4^x - a^y$ is odd. From (3.1), $dz^2 \equiv -1 \pmod 8$ and dz^2 is odd. This means z is odd and hence $z^2 \equiv 1 \pmod 8$. Now we have $dz^2 \equiv 3 \pmod 8$ which contradicts $dz^2 \equiv -1 \pmod 8$. Therefore, there is no solution of (3.1) when $x \geq 2$. ∎

We only get a non-negative integer solution of $4^x - a^y = dz^2$ in the first two cases of the proof of Theorem 3.3 and we have $y = 0$ in these two cases. This leads to result in the next Corollary 3.4, which we describe the situation when the Diophantine equation $4^x - a^y = dz^2$ has no non-negative integer solution.

Corollary 3.4. *Let $a, d \in \mathbb{N} - \{1\}$ be such that $a \equiv 1 \pmod 8$ and $d \equiv 3 \pmod 8$. If $y > 0$, then (3.1) has no non-negative integer solution.*

In the second part, we study the Diophantine equation

$$4^x + a^y = dz^2. \tag{3.2}$$

We show that the equation (3.2) has no non-negative integer solution in various assumptions of integers a and d.

Theorem 3.5. *If $a, d \in \mathbb{N}$ with $a \equiv 1 \pmod 4$ and $d \equiv 3 \pmod 4$, then the Diophantine equation (3.2) has no non-negative integer solution.*

Proof. Suppose $x, y, z \in \mathbb{N}_0$ is a solution of (3.2).

Case 1. $x = 0$. From (3.2), we have $1 + a^y = dz^2$. Since $a \equiv 1 \pmod 4$, we see that $a^y \equiv 1 \pmod 4$ and $1 + a^y \equiv 2 \pmod 4$. Note that $1 + a^y$ is even. Then $dz^2 \equiv 2 \pmod 4$ and dz^2 is even. Since d is odd, we have z is even. Consequently, $z^2 \equiv 0 \pmod 4$ and $dz^2 \equiv 0 \pmod 4$ which contradicts $dz^2 \equiv 2 \pmod 4$.

Case 2. $x \geq 1$. Then $4^x \equiv 0 \pmod 4$. Since $a^y \equiv 1 \pmod 4$, we obtain $4^x + a^y \equiv 1 \pmod 4$. From (3.2), we have $dz^2 \equiv 1 \pmod 4$. This implies z is odd and hence $z^2 \equiv 1 \pmod 4$. Thus, $dz^2 \equiv 3 \pmod 4$ which contradicts $dz^2 \equiv 1 \pmod 4$. ∎

Theorem 3.6. *Let $a, d \in \mathbb{N}$ be such that $a \equiv 3 \pmod 4$ and $d \equiv 3 \pmod 4$. If y is even, then the Diophantine equation (3.2) has no non-negative integer solution.*

Proof. Suppose $x, y, z \in \mathbb{N}_0$ is a solution of (3.2) where y is even.

Case 1. $x = 0$. We consider $1 + a^y = dz^2$. Since $1 + a^y$ is even and d is odd, we have z is even and so $z^2 \equiv 0 \pmod 4$ and $dz^2 \equiv 0 \pmod 4$. Thus, $1 + a^y \equiv 0 \pmod 4$ and hence $a^y \equiv -1 \pmod 4$. Since $a \equiv -1 \pmod 4$, we get y is odd which contradicts y is even.

Case 2. $x \geq 1$. Since 4^x is even and a^y is odd, we have $4^x + a^y$ is odd. Then dz^2 is odd. This means z is odd and so $z^2 \equiv 1 \pmod 4$. Thus, $dz^2 \equiv 3 \pmod 4$. From (3.2), $4^x + a^y \equiv 3 \pmod 4$. Since $4^x \equiv 0 \pmod 4$, we have $a^y \equiv -1 \pmod 4$. Since $a \equiv -1 \pmod 4$ and $a^y \equiv -1 \pmod 4$, we get y is odd which contradicts y is even. ∎

Theorem 3.7. *Let $a, d \in \mathbb{N}$ be such that $a \equiv 7 \pmod 8$ and $d \equiv 3 \pmod 8$. If $x \geq 2$, then the Diophantine equation (3.2) has no non-negative integer solution.*

Proof. Suppose $x, y, z \in \mathbb{N}_0$ is a solution of (3.2) where $x \geq 2$. Since $a \equiv 7 \pmod 8$, we have $a^y \equiv 1, 7 \pmod 8$ and a^y is odd. From (3.2), we obtain dz^2 is odd. Since $d \equiv 3 \pmod 8$, we have d is odd and so z is odd. Thus, $z^2 \equiv 1 \pmod 8$ and hence $dz^2 \equiv 3 \pmod 8$. From (3.2) and $x \geq 2$, we get $a^y \equiv 3 \pmod 8$ which contradicts $a^y \equiv 1, 7 \pmod 8$. ∎

4. Conclusions

We provide conclusions after studying Diophantine equations $4^x - a^y = dz^2$ and $4^x + a^y = dz^2$ as follows. For the Diophantine equation $4^x - a^y = dz^2$ with condition $d \equiv 3 \pmod 8$, we try to find all non-negative integer solutions of it in both cases $a \equiv 3 \pmod 4$ and $a \equiv 1 \pmod 4$. In the case $a \equiv 3 \pmod 4$, we completely obtain all non-negative integer solutions of it in Theorem 3.1. While the case $a \equiv 1 \pmod 4$, we can not get completely non-negative integer solutions of it. We notice that $a \equiv 1 \pmod 4$ iff $a \equiv 1 \pmod 8$ or $a \equiv 5 \pmod 8$. In the first part $a \equiv 1 \pmod 8$, we find all non-negative integer solutions of it in Theorem 3.3. However, we can not conclude about its non-negative integer solutions in the latter part $a \equiv 5 \pmod 8$, and hence the case when $a \equiv 5 \pmod 8$ remains an open problem to be solved. For the Diophantine equation $4^x + a^y = dz^2$, we confirm that it has no non-negative integer solution under some conditions in Theorem 3.5 - Theorem 3.7.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. This work was supported by the Research and Development Institute and the Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

References

[1] J. Klaška, Real-world applications of number theory, South Bohemia Mathematical Letters 25 (1) (2017) 39–47.

[2] D. Kaur, M. Sambhor, Diophantine equations and its applications in real life, Int. J. Math. And Appl. 5 (2-B) (2017) 217–222.

[3] R. Anbuselvi, J. Sivasankari, Applications of Diophantine equations in chemical equations, J. Emerg. Technol. Innov. Res. 6 (6) (2019) 371–373.

[4] C.K. Dhurga, A linear Diophantine equation and its real life applications, Advances and Applications in Mathematical Sciences 20 (8) (2021) 1389–1394.

[5] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine equations $4^x+7^y = z^2$ and $4^x + 11^y = z^2$, Sci. & Tech. RMUTT J. 1 (1) (2011) 25–28.

[6] S. Chotchaisthit, On the Diophantine equation $4^x + p^y = z^2$ where p is a prime number, Amer. J. Math. Sci. 1 (1) (2012) 191–193.

[7] N. Burshtein, On solutions of the Diophantine equation $p^x + q^y = z^2$, Ann. Pure Appl. Math. 13 (1) (2017) 143–149.

[8] J.F.T. Rabago, On the Diophantine equation $4^x - p^y = 3z^2$ where p is a prime, Thai J. Math. 16 (3) (2018) 643–650.

[9] A. Elshahed, H. Kamarulhaili, On the Diophantine equation $(4^n)^x - p^y = z^2$, WSEAS Transactions on Mathematics 19 (2020) 349–352.