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Bivariate Quadratic Transformations of Quasi-copulas

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Abstract In this study, we construct a bivariate quadratic transformation of trivariate quasi-copulas. This construction yields new quasi-copulas by composing two priori given quasi-copulas with a quadratic polynomial function. We show that the constructed transformation is not a bivariate transformation of semi-copulas which show that these two classes of transformations are different. The constructed transformation is also not symmetric which implies similar transformations can be constructed via change of variables. This also implies that these transformations cannot be constructed from univariate transformations which show that the class of bivariate transformations is much larger and more complicated than that of the univariate case.

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1. Introduction

Aggregation functions have gained attention in the last few decades. Because of their properties, they are usually used for finding a data representation. Aggregation functions can be applied in many fields, especially, data analysis and decision making, for example, statistics, engineering sciences, and economics and finance, see [1–4]. Thus, a quasi-copula, a type of aggregation functions, is also gained interest. The construction of quasi-copulas becomes an interesting topic, after all, the more choices we have, the more tools we can select.

In this work, we are interested in the construction of quasi-copulas transformations in the form of

$$T_P(f_1,\ldots,f_k)(x_1,\ldots,x_n) = P(x_1,\ldots,x_n,f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n))$$

where P is a quadratic polynomial and $f_1, \ldots, f_k$ are real-value functions. This concept was first introduced in 2013 by Kolesárová et al. [5]. They introduced this idea to construct univariate transformations of bivariate (quasi-) copulas based on a given quadratic polynomial. After that, the characterization of a quadratic polynomial P is studied for

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finding necessary and sufficient conditions of (quasi, semi-) copulas transformations T_P. In the case of univariate transformations of bivariate functions, we refer [6, 7]. The bivariate quadratic transformation of bivariate copulas was examined in [8] and more detail about transformations of k bivariate copulas based on polynomial functions has been introduced in [9]. Boonmee and Tasena [10] characterized a quadratic polynomial P such that T_P is a univariate transformation of multivariate (semi-copulas) quasi-copulas. They also show that multivariate quadratic transformations of aggregation functions must come from quadratic aggregation functions themselves. See also [11] for characterizations of quadratic aggregation functions. Recently, the concept of transformations of multivariate semi-copulas was presented in [12].

We observe that the study of transformations of multivariate quasi-copulas remains opened. In this work, we construct a bivariate transformation of trivariate quasi-copulas using the same idea with in the first literature [5], that is, we construct this transformation based on a given quadratic polynomial.

The next section, we present some notations, basic terminologies, and theorems which are useful for this work. In Section 3, we will show our construction of a bivariate quadratic transformation of trivariate quasi-copulas. We also prove some special properties of this transformation. In addition, some quasi-copulas, which are constructed by this transformation, are provided in this section. The conclusion and discussion are discussed in the last section.

2. Preliminary

The closed unit interval $[0, 1]$ is denoted by a symbol I.

Definition 2.1. [3] A non-decreasing function $A : I^n \to I$ is said to be an *aggregation function* if $A(\vec{0}) = 0$ and $A(\vec{1}) = 1$.

For instance, well-known aggregation functions are the arithmetic mean, the maximum, the minimum, the median, and the product function. Moreover, semi-copulas and quasi-copulas, the functions that we concentrate in this study, are also aggregation functions.

Definition 2.2. [13] A function $S : I^n \to I$ is said to be a *semi-copula* if

(i) S is non-decreasing, and
(ii) $S(x_1, \dots, x_n) = x_i$ if $x_j = 1$ for all $j \neq i$.

The minimum function and the product function are semi-copulas. Besides, quasi-copulas are also semi-copulas.

Recall that a function $f : I^n \to I$ is called *Lipschitz* if

$$|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)| \leq \sum_{i=1}^{n} |x_i - y_i|$$

for all $x_i, y_i \in I$.

For example, the minimum function and the product function are Lipschitz.

Definition 2.3. [14] A function $Q : I^n \to I$ is said to be a *quasi-copula* if

(i) Q is non-decreasing,
(ii) $Q(x_1, \dots, x_n) = x_i$ if $x_j = 1$ for all $j \neq i$, and
(iii) Q is Lipschitz.

This implies that the minimum function and the product function are also quasi-copulas. Other well-known quasi-copulas are the FGM-copula $C_\theta : I^n \to I$ defined by

$$C_\theta(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i + \theta \prod_{i=1}^{n} x_i (1 - x_i)$$

where $\theta \in [-1, 1]$ and the function $W_n : I^n \to I$ defined as follows

$$W_n(x_1, \dots, x_n) = \max\left\{0, \sum_{i=1}^{n} x_i - n + 1\right\}$$

for all $x_i \in I$. A function W_n is the lower bound of the class of quasi-copulas, that is,

$$W_n(x_1, \dots, x_n) \le Q(x_1, \dots, x_n) \le \min(x_1, \dots, x_n)$$

for all quasi-copula Q. For convenience, let M and W stand for the minimum function and the function W_n in the $3-$dimension, respectively.

There is another way to prove quasi-copula properties instead of the definition.

Theorem 2.4. [15] *Let $Q : I^n \to I$ be a function satisfying*

$$Q(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$$

and the second property of the quasi-copula definition. Then Q is a quasi-copula if and only if Q is absolutely continuous in each coordinate and for any $j \in \{1, 2, \dots, n\}$ and $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in I^{n-1}$, the partial derivative

$$\frac{\partial}{\partial x_j} Q(x_1, \dots, x_j, \dots, x_n)$$

exists for almost all $x_j \in I$ and belongs to the interval I.

Next, we state the definitions of semi-copulas transformations and quasi-copulas transformations. Let P be a quadratic polynomial of $n + k$ variables and $f_1, \dots, f_k$ be real-valued functions on I^n. A transformation $T_P(f_1, \dots, f_k) : I^n \to \mathbb{R}$ defined via

$$T_P(f_1, \dots, f_k)(x_1, \dots, x_n) = P(x_1, \dots, x_n, f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n)). \tag{2.1}$$

The transformation T_P in the form of (2.1) is called a *transformation of k semi-copulas* if $T_P(S_1, \dots, S_k)$ is a semi-copula for all semi-copulas $S_1, \dots, S_k$. Also, if $T_P(Q_1, \dots, Q_k)$ is a quasi-copula for all quasi-copulas $Q_1, \dots, Q_k$, then T_P is said to be a *transformation of k quasi-copulas.*

The following theorems are useful results for proving our main results.

Theorem 2.5. [12] *Let P be a quadratic polynomial of $n + k$ variables of the form*

$$P(x_1, \dots, x_n, z_1, \dots, z_k) = \sum_{p=1}^{k} \sum_{q=1}^{k} a_{pq} z_p z_q + \sum_{i=1}^{n} \sum_{q=1}^{k} b_{iq} x_i z_q + \sum_{q=1}^{k} c_q z_q + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_i x_j + \sum_{i=1}^{n} e_i x_i + f$$

where $a_{pq} = a_{qp}$ and $d_{ij} = d_{ji}$ for all p, q and i, j. Then T_P is a transformation of k semi-copulas if and only if the quadratic polynomial P satisfies the following properties:

(1) $f = 0 = e_i = d_{ij}$ for all i, j,

(2) $\sum_{q=1}^{k} b_{iq} = - \sum_{p=1}^{k} \sum_{q=1}^{k} a_{pq}$ for all i,

(3) $(n-1)\sum_{p=1}^{k}\sum_{q=1}^{k} a_{pq} - \sum_{q=1}^{k} c_q + 1 = 0,$
(4) $b_{iq} \geq 0$ *for all* i, q,
(5) $c_q \geq 0$ *for all* q,
(6) $2\sum_{p=1}^{k}(a_{pq} \wedge 0) + \sum_{i=1}^{n} b_{iq} + c_q \geq 0$ *for all* q.

Theorem 2.6. [10] *Let P be a quadratic polynomial of $n+1$ variables. For any quasi-copula Q, T_P is a quasi-copula transformation if and only if it is given in the form*

$$T_P(Q)(x_1,\dots,x_n) = aQ^2(x_1,\dots,x_n) - aQ(x_1,\dots,x_n)\sum_{i=1}^{n} x_i + (an-a+1)Q(x_1,\dots,x_n)$$

where $a \in \left[-\frac{1}{n-1}, 0\right]$.

3. Main Results

We construct the bivariate transformation of trivariate quasi-copulas in the form of

$$T(Q_1,Q_2)(x_1,x_2,x_3) = P(x_1,x_2,x_3,Q_1(x_1,x_2,x_3),Q_2(x_1,x_2,x_3))$$

where

$$P(x_1,x_2,x_3,z_1,z_2) = \frac{3}{16}z_1^2 - \frac{3}{4}z_1z_2 + \frac{3}{16}z_2^2 + \frac{3}{8}x_1z_1 + \frac{3}{8}x_3z_1 + \frac{3}{8}x_2z_2 + \frac{1}{4}z_2$$

and Q_1, Q_2 are any quasi-copulas. Then the transformation T is in the following form

$$\begin{aligned} T(Q_1,Q_2)(x_1,x_2,x_3) = {} & \frac{3}{16}Q_1^2(x_1,x_2,x_3) - \frac{3}{4}Q_1(x_1,x_2,x_3)Q_2(x_1,x_2,x_3) \\ & + \frac{3}{16}Q_2^2(x_1,x_2,x_3) + \frac{3}{8}x_1Q_1(x_1,x_2,x_3) \\ & + \frac{3}{8}x_3Q_1(x_1,x_2,x_3) + \frac{3}{8}x_2Q_2(x_1,x_2,x_3) \\ & + \frac{1}{4}Q_2(x_1,x_2,x_3). \end{aligned} \tag{$*$}$$

Theorem 3.1. *The transformation T is a quasi-copula transformation, but it is not a semi-copula transformation.*

Proof. We first show that T is a quasi-copula transformation. Let Q_1, Q_2 be trivariate quasi-copulas and let $x \in I$. Without loss of generality, we prove boundary conditions by only showing that $T(Q_1,Q_2)(x,1,1) = x$. By ($*$), we have that

$$\begin{aligned} T(Q_1,Q_2)(x,1,1) &= \frac{3}{16}Q_1^2(x,1,1) - \frac{3}{4}Q_1(x,1,1)Q_2(x,1,1) + \frac{3}{16}Q_2^2(x,1,1) \\ &\quad + \frac{3}{8}xQ_1(x,1,1) + \frac{3}{8}Q_1(x,1,1) + \frac{3}{8}Q_2(x,1,1) + \frac{1}{4}Q_2(x,1,1) \\ &= \frac{3}{16}x^2 - \frac{3}{4}x^2 + \frac{3}{16}x^2 + \frac{3}{8}x^2 + \frac{3}{8}x + \frac{3}{8}x + \frac{1}{4}x \\ &= x. \end{aligned}$$

Next, we show that $0 \le \frac{\partial}{\partial x_i}T(Q_1,Q_2)(x_1,x_2,x_3) \le 1$ for all $(x_1,x_2,x_3) \in I^3$ such that $\frac{\partial}{\partial x_i}Q_1(x_1,x_2,x_3)$ and $\frac{\partial}{\partial x_i}Q_2(x_1,x_2,x_3)$ exist for each $i \in \{1,2,3\}$. For convenience, we denote (x_1,x_2,x_3) and $\frac{\partial}{\partial x_i}f(x_1,x_2,x_3)$ by $\vec{x}$ and $\partial_i f(\vec{x})$, respectively.

Let $\vec{x} \in I^3$. Suppose that $\partial_i Q_1(\vec{x})$, $\partial_i Q_2(\vec{x})$ exist for each $i \in \{1,2,3\}$.

Case $i = 1$**:** Consider

$$\begin{aligned}
\partial_1 T(Q_1,Q_2)(\vec{x}) &= \frac{3}{8}Q_1(\vec{x})\partial_1 Q_1(\vec{x}) - \frac{3}{4}Q_1(\vec{x})\partial_1 Q_2(\vec{x}) - \frac{3}{4}Q_2(\vec{x})\partial_1 Q_1(\vec{x}) \\
&\quad + \frac{3}{8}Q_2(\vec{x})\partial_1 Q_2(\vec{x}) + \frac{3}{8}x_1\partial_1 Q_1(\vec{x}) + \frac{3}{8}Q_1(\vec{x}) \\
&\quad + \frac{3}{8}x_3\partial_1 Q_1(\vec{x}) + \frac{3}{8}x_2\partial_1 Q_2(\vec{x}) + \frac{1}{4}\partial_1 Q_2(\vec{x}) \\
&= \left[\frac{3}{8}Q_1(\vec{x}) - \frac{3}{4}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3\right]\partial_1 Q_1(\vec{x}) \\
&\quad + \left[-\frac{3}{4}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4}\right]\partial_1 Q_2(\vec{x}) + \frac{3}{8}Q_1(\vec{x}).
\end{aligned}$$

We now consider signs of coefficients of $\partial_1 Q_1(\vec{x})$ and $\partial_1 Q_2(\vec{x})$.

First, by $Q_2(\vec{x}) \le M(\vec{x}) \le x_1, x_3$ and

$$\frac{3}{8}Q_1(\vec{x}) - \frac{3}{4}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3 = \frac{3}{8}Q_1(\vec{x}) - \frac{3}{8}Q_2(\vec{x}) - \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3,$$

we have that a coefficient of $\partial_1 Q_1(\vec{x})$ is non-negative.

Moreover, we can compute that $M(\vec{x}) - W(\vec{x}) \le \frac{2}{3}$ by programming. This implies that

$$\begin{aligned}
-\frac{3}{4}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4} &= -\frac{3}{8}Q_1(\vec{x}) - \frac{3}{8}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4} \\
&= \frac{3}{8}\left(Q_2(\vec{x}) - Q_1(\vec{x})\right) - \frac{3}{8}Q_1(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4} \\
&\ge \frac{3}{8}\left(-\frac{2}{3}\right) - \frac{3}{8}Q_1(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4} \\
&\ge 0,
\end{aligned}$$

that is, a coefficient of $\partial_1 Q_2(\vec{x})$ is also non-negative. Thus, it is obvious that $\partial_1 T(Q_1,Q_2)(\vec{x}) \ge 0$.

For proving $\partial_1 T(Q_1,Q_2)(\vec{x}) \le 1$, consider that

$$\begin{aligned}
\partial_1 T(Q_1,Q_2)(\vec{x}) &\le \frac{3}{8}Q_1(\vec{x}) - \frac{3}{4}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3 - \frac{3}{4}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 \\
&\quad + \frac{1}{4} + \frac{3}{8}Q_1(\vec{x}) \\
&= -\frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}\sum_{i=1}^{3} x_i + \frac{1}{4} \\
&\le \frac{3}{4} + \frac{1}{4} \\
&= 1
\end{aligned}$$

by the facts that $\partial_1 Q_1(\vec{x}), \partial_1 Q_2(\vec{x}) \le 1$ and $\sum_{i=1}^{3} x_i - 2 \le W(\vec{x}) \le Q_2(\vec{x})$.

Case $i = 2$**:** Consider

$$\begin{aligned}
\partial_2 T(Q_1, Q_2)(\vec{x}) &= \frac{3}{8}Q_1(\vec{x})\partial_2 Q_1(\vec{x}) - \frac{3}{4}Q_1(\vec{x})\partial_2 Q_2(\vec{x}) - \frac{3}{4}Q_2(\vec{x})\partial_2 Q_1(\vec{x}) \\
&\quad + \frac{3}{8}Q_2(\vec{x})\partial_2 Q_2(\vec{x}) + \frac{3}{8}x_1\partial_2 Q_1(\vec{x}) + \frac{3}{8}x_3\partial_2 Q_1(\vec{x}) \\
&\quad + \frac{3}{8}x_2\partial_2 Q_2(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{1}{4}\partial_2 Q_2(\vec{x}) \\
&= \left[\frac{3}{8}Q_1(\vec{x}) - \frac{3}{4}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3\right]\partial_2 Q_1(\vec{x}) \\
&\quad + \left[-\frac{3}{4}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 + \frac{1}{4}\right]\partial_2 Q_2(\vec{x}) + \frac{3}{8}Q_2(\vec{x}).
\end{aligned}$$

We note that both coefficients of $\partial_1 Q_1(\vec{x})$ and $\partial_1 Q_2(\vec{x})$ are the same as the previous case. Thus, $\partial_2 T(Q_1, Q_2)(\vec{x}) \geq 0$.

Similarly to the previous case,

$$\begin{aligned}
\partial_2 T(Q_1, Q_2)(\vec{x}) &\leq \frac{3}{8}Q_1(\vec{x}) - \frac{3}{4}Q_2(\vec{x}) + \frac{3}{8}x_1 + \frac{3}{8}x_3 - \frac{3}{4}Q_1(\vec{x}) + \frac{3}{8}Q_2(\vec{x}) + \frac{3}{8}x_2 \\
&\quad + \frac{1}{4} + \frac{3}{8}Q_2(\vec{x}) \\
&= -\frac{3}{8}Q_1(\vec{x}) + \frac{3}{8}\sum_{i=1}^{3} x_i + \frac{1}{4} \\
&\leq 1.
\end{aligned}$$

Case $i = 3$**:** Consider

$$\begin{aligned}
\partial_3 T(Q_1, Q_2)(\vec{x}) &= \frac{3}{8}Q_1(\vec{x})\partial_3 Q_1(\vec{x}) - \frac{3}{4}Q_1(\vec{x})\partial_3 Q_2(\vec{x}) - \frac{3}{4}Q_2(\vec{x})\partial_3 Q_1(\vec{x}) \\
&\quad + \frac{3}{8}Q_2(\vec{x})\partial_3 Q_2(\vec{x}) + \frac{3}{8}x_1\partial_3 Q_1(\vec{x}) + \frac{3}{8}x_3\partial_3 Q_1(\vec{x}) \\
&\quad + \frac{3}{8}Q_1(\vec{x}) + \frac{3}{8}x_2\partial_3 Q_2(\vec{x}) + \frac{1}{4}\partial_3 Q_2(\vec{x}).
\end{aligned}$$

We can prove that $0 \leq \partial_3 T(Q_1, Q_2)(\vec{x}) \leq 1$ by the same way as the case $i = 1$.

Now, we can conclude that $T(Q_1, Q_2)$ is a non-decreasing Lipschitz function. Combining with the boundary conditions, it follows that $T(Q_1, Q_2)$ is a quasi-copula. In particular, T is a quasi-copula transformation.

After that, we can see that this T is not a semi-copula transformation by considering $q = 2$ in the condition 6 of Theorem 2.5. It gives that

$$\begin{aligned}
2\sum_{p=1}^{2}(a_{p2} \wedge 0) + \sum_{i=1}^{3} b_{i2} + c_2 &= 2\left((a_{12} \wedge 0) + (a_{22} \wedge 0)\right) + \sum_{i=1}^{3} b_{i2} + c_2 \\
&= 2\left(\left(-\frac{3}{8} \wedge 0\right) + \left(\frac{3}{16} \wedge 0\right)\right) + \frac{3}{8} + \frac{1}{4} \\
&= 2 \cdot -\frac{3}{8} + \frac{5}{8} \\
&= -\frac{1}{8} \\
&\ngeq 0.
\end{aligned}$$

∎

Although the class of all quasi-copulas is a subclass of the class of all semi-copulas, the last theorem show us that the class of all bivariate transformations of quasi-copulas is not a subclass of the class of all bivariate transformations of semi-copulas, that is, they are different. Moreover, we note that T does not transform symmetric quasi-copulas because coefficients of terms x_1, x_2, x_3 are not necessarily equal. Then interchanges of x_2 and x_1, x_3 give different values of the transformation T. This implies that the equation $(*)$ is not a symmetric function. By the same reason, T is also not interchangeable on Q_1 and Q_2. Thus, we can construct other bivariate transformations of quasi-copulas by swapping x_2 and x_1, x_3 or swapping Q_1 and Q_2 in the equation $(*)$ such as a following example.

Swapping x_1 and x_2 in $(*)$ gives a new bivariate transformation of quasi-copulas as follow

$$\begin{aligned} T^{'}(Q_1,Q_2)(x_2,x_1,x_3) = {} & \frac{3}{16}Q_1^2(x_2,x_1,x_3) - \frac{3}{4}Q_1(x_2,x_1,x_3)Q_2(x_2,x_1,x_3) \\ & + \frac{3}{16}Q_2^2(x_2,x_1,x_3) + \frac{3}{8}x_2Q_1(x_2,x_1,x_3) \\ & + \frac{3}{8}x_3Q_1(x_2,x_1,x_3) + \frac{3}{8}x_1Q_2(x_2,x_1,x_3) \\ & + \frac{1}{4}Q_2(x_2,x_1,x_3). \end{aligned}$$

Next, we consider whether univariate transformations and bivariate transformations of quasi-copulas are related.

Theorem 3.2. *The bivariate quasi-copulas transformation given by $(*)$ cannot be constructed using composition between univariate quasi-copula transformations and convex combinations.*

Proof. Assume that the equation $(*)$ can be constructed using composition between univariate quasi-copula transformations and convex combinations. For any $\vec{x} \in I^3$,

$$T(Q_1,Q_2)(\vec{x}) = \alpha T_{P_1}\left(\beta Q_1 + (1-\beta)\,Q_2\right)(\vec{x}) + (1-\alpha)\,T_{P_2}\left(\gamma Q_1 + (1-\gamma)\,Q_2\right)(\vec{x})$$

for some $\alpha, \beta, \gamma \in I$, univariate quasi-copula transformations T_{P_1}, T_{P_2} satisfying Theorem 2.6 and quasi-copulas Q_1, Q_2. Since a convex combination of quasi-copulas is also a quasi-copula and Theorem 2.6 holds, we have that

$$\begin{aligned} T(Q_1,Q_2)(\vec{x}) = {} & \alpha\Bigg[a_1\left(\beta Q_1 + (1-\beta)\,Q_2\right)^2(\vec{x}) - a_1\left(\beta Q_1 + (1-\beta)\,Q_2\right)(\vec{x})\sum_{i=1}^{3} x_i \\ & + (3a_1 - a_1 + 1)\left(\beta Q_1 + (1-\beta)\,Q_2\right)(\vec{x})\Bigg] \\ & + (1-\alpha)\Bigg[a_2\left(\gamma Q_1 + (1-\gamma)\,Q_2\right)^2(\vec{x}) - a_2\left(\gamma Q_1 + (1-\gamma)\,Q_2\right)(\vec{x})\sum_{i=1}^{3} x_i \\ & + (3a_2 - a_2 + 1)\left(\gamma Q_1 + (1-\gamma)\,Q_2\right)(\vec{x})\Bigg] \end{aligned}$$

for some $a_1, a_2 \in [-\frac{1}{2}, 0]$. We observe coefficients of x_1, x_2, and x_3. All coefficients are equal to

$$-\alpha a_1\left(\beta Q_1 + (1-\beta)\,Q_2\right)(\vec{x}) - (1-\alpha)\,a_2\left(\gamma Q_1 + (1-\gamma)\,Q_2\right)(\vec{x}),$$

that is, these three coefficients are same while the equation $(*)$ gives that coefficients of x_1 and x_2 are $\frac{3}{8}Q_1(\vec{x})$ and $\frac{3}{8}Q_2(\vec{x})$, respectively. It is a contradiction. Hence, $(*)$ cannot be constructed from univariate quasi-copula transformations. ∎

This means that there exists a bivariate transformation which is not in the class of univariate transformations. Thus, the class of bivariate transformations is more complicated and larger than the class of univariate transformations.

The last part of this section, we give some examples of new quasi-copulas by substituting two well-known quasi-copulas to Q_1 and Q_2 in $(*)$.

Example 3.3. Given Q_1, Q_2 are product functions. Then

$$T(Q_1,Q_2)(x_1,x_2,x_3) = -\frac{3}{8}x_1^2x_2^2x_3^2 + \frac{3}{8}x_1^2x_2x_3 + \frac{3}{8}x_1x_2^2x_3 + \frac{3}{8}x_1x_2x_3^2 + \frac{1}{4}x_1x_2x_3.$$

Example 3.4. Given Q_1 is a product function and $Q_2(x_1,x_2,x_3) = \prod_{i=1}^{3} x_i + \prod_{i=1}^{3} x_i\,(1-x_i)$.
Then

$$\begin{aligned}
T(Q_1,Q_2)(x_1,x_2,x_3) = {} & \frac{3}{16}x_1^4x_2^4x_3^4 - \frac{3}{8}x_1^4x_2^4x_3^3 + \frac{3}{16}x_1^4x_2^4x_3^2 - \frac{3}{8}x_1^4x_2^3x_3^4 + \frac{3}{4}x_1^4x_2^3x_3^3 \\
& - \frac{3}{8}x_1^4x_2^3x_3^2 + \frac{3}{16}x_1^4x_2^2x_3^4 - \frac{3}{8}x_1^4x_2^2x_3^3 + \frac{3}{16}x_1^4x_2^2x_3^2 - \frac{3}{8}x_1^3x_2^4x_3^4 \\
& + \frac{3}{4}x_1^3x_2^4x_3^3 - \frac{3}{8}x_1^3x_2^4x_3^2 + \frac{3}{4}x_1^3x_2^3x_3^4 - \frac{9}{8}x_1^3x_2^3x_3^3 + \frac{3}{8}x_1^3x_2^3x_3^2 \\
& - \frac{3}{8}x_1^3x_2^2x_3^4 + \frac{3}{8}x_1^3x_2^2x_3^3 + \frac{3}{16}x_1^2x_2^4x_3^4 - \frac{3}{8}x_1^2x_2^4x_3^3 + \frac{3}{16}x_1^2x_2^4x_3^2 \\
& - \frac{3}{8}x_1^2x_2^3x_3^4 + \frac{3}{8}x_1^2x_2^3x_3^3 - \frac{3}{8}x_1^2x_2^3x_3^2 + \frac{3}{8}x_1^2x_2^3x_3 + \frac{3}{16}x_1^2x_2^2x_3^4 \\
& - \frac{7}{16}x_1^2x_2^2x_3^2 - \frac{1}{8}x_1^2x_2^2x_3 + \frac{1}{4}x_1^2x_2x_3^2 + \frac{1}{8}x_1^2x_2x_3 + \frac{3}{8}x_1x_2^3x_3^2 \\
& - \frac{3}{8}x_1x_2^3x_3 - \frac{1}{8}x_1x_2^2x_3^2 + \frac{1}{2}x_1x_2^2x_3 + \frac{1}{8}x_1x_2x_3^2 + \frac{1}{2}x_1x_2x_3.
\end{aligned}$$

4. Conclusion and Discussion

This study shows a construction of a new bivariate transformation of trivariate quasi-copulas based on a given quadratic polynomial function. The class of new quasi-copulas is obtained by composing this priori given quadratic polynomial and any two quasi-copulas. The constructed transformation T is not a bivariate transformation of semi-copulas. In particular, it cannot be constructed from any convex combinations of univariate transformations. Moreover, many quasi-copulas can be constructed by replacing any quasi-copulas to the transformation T, for instance, we can choose FGM-copulas in many ways by picking $\theta \in [-1,1]$.

However, the characterization of the class of bivariate quadratic transformations of trivariate quasi-copulas is still an open problem. If we can solve this problem, then we will have the family of new quasi-copulas which is larger than the family of quasi-copulas that we obtained in this work. Therefore, it is an interesting topic that we will try to solve in the future.

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