Thai Journal of Mathematics
Volume 21 Number 3 (2023)
Pages 537–552

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

In memoriam Professor Charles E. Chidume (1947–2021)

Iterative Methods for Common Fixed Points of a Finite Family of Non-self Mappings

Abebe Regassa Tufa

Department of Mathematics, University of Botswana, Pvt. Bag 00704, Gaborone, Botswana
e-mail : abykabe@yahoo.com

Abstract In this paper, iterative processes involving a finite family of non-self mappings defined on a nonempty closed convex subset of a complete CAT(0) space are constructed. Some strong convergence and $\triangle$-convergence results for approximating common fixed points of a finite family of k-strictly pseudo-contractive mappings are obtained. In addition, strong convergence results for approximating a common fixed point of a finite family of demicontractive non-self mappings are obtained under some appropriate conditions. The results obtained in this paper improve, generalize and complement many of the results in the literature.

MSC: 37C25; 47H04; 47H09

Keywords: fixed points; demicontractive mappings; k-strictly pseudocontractive mappings; non-self mappings

Submission date: 12.06.2022 / Acceptance date: 27.04.2023

1. Introduction

Let X be a complete CAT(0) space and C be a nonempty subset of X. A mapping $T : C \to X$ is said to be *L-Lipschitz* if there exists $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y), \text{ for all } x, y \in C, \tag{1.1}$$

where d is a metric on X. When $L = 1$, the mapping T is said to be *nonexpansive*. If there exists $k \in [0, 1)$ such that

$$d^2(Tx, Ty) \leq d^2(x, y) + 4kd^2(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}Tx \oplus \frac{1}{2}y), \forall x, y \in C, \tag{1.2}$$

then the mapping T is said to be *k-strictly pseudocontractive*.

One can easily see that a nonexpansive mapping is 0-strictly pseudocontractive mapping. In addition, every k-strictly pseudocontractive mapping is Lipschitz with Lipschitz constant $L = \frac{1+k}{1-k}$ (see, e.g., [1]).

Published by The Mathematical Association of Thailand.

Given a mapping $T : C \to X$, a point $x \in C$ is called a *fixed point* of T if $x = Tx$. The set of all fixed points of the mapping T is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$. The mapping T is said to be

(a) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \forall x \in C, \forall p \in F(T).$$

(b) *demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in (0, 1)$ such that

$$d^2(Tx, p) \leq d^2(x, p) + kd^2(x, Tx), \forall x \in C, \forall p \in F(T).$$

We observe that every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive mapping and every quasi-nonexpansive mapping is demicontractive mapping. Moreover, if the mapping T is k-strictly pseudocontractive with $F(T) \neq \emptyset$, then it is demicontractive.

Because of its immense applications in several disciplines of sciences and engineering, the theory of fixed points of nonlinear mappings have been studied by several authors (see, e.g., [2–13] and the references therein). In particular, fixed point results in CAT(0) spaces are applicable in graph theory, biology and computer sciences (see, e.g., [14–17]). Related works can also be found in [18–24].

Several iterative methods have been constructed and studied for approximating fixed points (when they exist) of nonlinear mappings. One of the most known iterative methods studied by several authors (see, e.g., [7, 25, 26]) in Banach spaces is the following.

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and initial guess x_1 is arbitrarily chosen. The sequence $\{x_n\}$ generated by (1.3) is generally referred to as Mann iterative process in light of [27].

In 2001, Xu and Ori [28] proposed and studied the following Mann type implicit iteration process for common fixed points of a finite family of nonexpansive self mappings $\{T_i\}_{i=1}^N$ in a real Hilbert space H. Given $x_0 \in C \subset H$, let

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, n \in \mathbb{N},$$

where $T_1, T_2, \cdots, T_N$ are N nonexpansive self-mappings of C and $T_n = T_{n \ mod \ N}$. They proved that the proposed algorithm converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

A year later, Osilike [29] extended the results of Xu and Ori [28] from the class of nonexpansive mappings to the more general class of strictly pseudocontractive self mappings and proved some strong and weak convergence results.

In 2010, Chidume and Shahzad [30] constructed and studied an explicit Mann type iterative scheme for a finite family of mappings in a real Banach space which is more general than the Hilbert space. They proved the following Theorem.

Theorem 1.1. *Let X be a uniformly smooth real Banach space which is also uniformly convex and C be a nonempty closed convex subset of X. For each $1 \leq i \leq N$, let $T_i : C \to C$ be a λ_i-strict pseudocontraction for some $0 \leq \lambda_i < 1$ with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. For a fixed $x_0 \in C$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_{[n]} x_n,$$

where $T_{[n]} = T_i$ *with* $i = n(modN), 1 \leq i \leq N$ *and* $\{\alpha_n\}$ *is a real sequence in* $[0,1]$ *satisfying the following conditions: (i)* $\sum_{n=1}^{\infty} \alpha_n = \infty$; *(ii)* $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. *Let* $\lambda := \min\{\lambda_i : 1 \leq i \leq N\}$. *Then,* $\{x_n\}$ *converges weakly to a common fixed point of the family* $\{T_i\}_{i=1}^N$.

Many iterative methods have also been studied for approximating fixed points of nonlinear family of mappings in a CAT(0) space setting (see, e.g., [31, 32]). Note that all the above results are valid only for self mappings. For non-self mappings the concepts of metric projection or sunny nonexpansive retraction mappings have been used by several authors (see, e.g., [33–35]). It is pointed out in [36] that in many real world applications, the process of calculating these auxiliary operators can be a resource consumption task and it may require an approximating algorithm by itself. In an attempt to overcome this problem Colao and Marino [36] introduced a new searching strategy for the coefficient α_n which makes the Mann algorithm well defined for non-self mappings in the setting of a real Hilbert space H. In fact, they studied the following scheme:

$$\begin{cases} x_0 \in C, \\ \alpha_0 = \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}, n \geq 0, \end{cases} \tag{1.4}$$

where $h(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}, \forall x \in C \subseteq H$ and T is single-valued non-self mapping of C into H. They obtained weak and strong convergence results of the algorithm to a fixed point of nonexpansive non-self mappings under appropriate conditions.

However, how to adapt algorithm (1.4) to produce a converging sequence to a common fixed point for a family of nonself mappings was an open question. In order to answer this question, Guo et al. [37], introduced and studied an iterative process for approximating common fixed points of a countably infinite family of non-self nonexpansive mappings in the setting of Hilbert spaces. They obtained some weak and strong convergence results under suitable conditions. In [38] Tufa and Zegeye have extended this results of Guo et al. [37] to the setting of CAT(0) spaces. They have established weak and strong convergence result for approximating common fixed points of a countably infinite family of quasi-nonexpansive mappings under some mild conditions. Moreover, they proved strong convergence result for a family of demicontractive mappings. Indeed they proved the following theorem.

Theorem 1.2. *Let* C *be a nonempty, closed and convex subset of a complete* $CAT(0)$ *space* X *and* $T_i : C \to X$ *be an inward and demicontractive mapping with constant* k_i*, for each* $i = 1, 2, 3, \cdots$. *Let* $\{x_n\}$ *be generated from arbitrary initial point* $x_1 \in C$ *by*

$$\begin{cases} \alpha_1 = \max\{k_1, h_1(x_1)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T_n x_n, \\ \alpha_{n+1} = \max\{\alpha_n, k_{n+1}, h_{n+1}(x_{n+1})\}, n \geq 1, \end{cases} \tag{1.5}$$

where $h_n(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)T_n x_n \in C\}$.
Suppose that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* (F, C) *satisfies S-condition. If* $k := \sup\{k_n\} < 1$ *and* $\sum(1-\alpha_n) < \infty$, *then* $\{x_n\}$ *converges strongly to a point in* F.

How to adapt the algorithm for a finite family of mappings remained open until it is modified by Tufa et al. [39] in 2021 for approximating a common fixed point of a finite family of k-strictly pseudocontractive mappings in the setting of Hilbert spaces.

It is our purpose, in this paper, to construct an iterative scheme for approximating a common fixed point of a finite family of k-strictly pseudocontractive non-self mappings in a CAT(0) space. We prove $\triangle$-convergence or strong convergence results of the scheme based on the nature of the iteration parameter. Moreover, we construct and study an iterative scheme for approximating common fixed points of a finite family of demicontractive non-self mappings under some mild conditions. Our results fill some of the aforementioned gaps.

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A mapping $r : [0, l] \subset \mathbb{R} \to X$ with $r(0) = x, r(l) = y$ and $d(r(t), r(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$ is said to be a geodesic path joining x to y. The image of r is called a *geodesic segment* joining x and y. The geodesic segment between x and y, when it is unique, is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. In this case we write $z = tx \oplus (1 - t)y$.

If every two points of X are joined by a geodesic, then the metric space (X, d) is said to be a *geodesic space.* It is said to be *uniquely geodesic* space if there is exactly one geodesic joining x and y for each $x, y \in X$. A uniquely geodesic space (X, d) is said to be an $\mathbb{R}$-*tree,* if $x, y, z \in X$ with $[x, y] \cap [y, z] = \{y\}$ implies $[x, z] = [x, y] \cup [y, z]$. When there is no ambiguity, we simply denote a geodesic space (X, d) by X.

A geodesic triangle denoted by $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points $x_1, x_2, x_3 \in X$ (vertices) and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for all $i, j = 1, 2, 3$.

A geodesic space X is said to be a *CAT(0) space* if every geodesic triangle $\triangle$ in X and its comparison triangle $\bar{\triangle}$ in $\mathbb{R}^2$ satisfy the following condition:

$$d(x, y) \le d_{\mathbb{R}^2}(\bar{x}, \bar{y}),\ \forall x, y \in \triangle, \bar{x}, \bar{y} \in \bar{\triangle}. \tag{2.1}$$

It is well known that a CAT(0) space X is uniquely geodesic space. Pre-Hilbert spaces, $\mathbb{R}-$trees and Euclidean buildings are examples of $CAT(0)$ spaces (see, e.g., [40, 41]).

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known [42] that in a CAT(0) space X, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ consists of exactly one point. The sequence $\{x_n\}$ is said to be $\triangle$-*convergent* to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. The uniqueness of asymptotic center implies that a CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subseteq X$ such that $\{x_n\}$ $\triangle$-converges to x and given $y \in X$ with $y \neq x$, $\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$.

Let C be a nonempty subset of a CAT(0) space X and $T : C \to X$ be a mapping. Since it is not possible to formulate the concept of demiclosedness in a CAT(0) space setting, as stated in linear spaces, we shall say that the map T has demiclosedness-type property if for any sequence $\{x_n\} \subseteq C$ such that $\{x_n\}$ $\triangle-$converges to p and $d(x_n, Tx_n) \to 0$, then $p = Tp$ (see [1, 32]).

For any $x \in C$, the set

$$I_C(x) = \{w \in X : w = x \text{ } or \text{ } y = (1 - \frac{1}{\lambda})x \oplus \frac{1}{\lambda}w, \text{ for some y } \in C, \lambda \geq 1\}$$

is called an *inward* set at x. The mapping $T : C \to X$ is said to be *inward* on C if $Tx \in I_C(x)$.

We shall use the following lemmas in the sequel.

Lemma 2.1 ([43]). *Let X be a CAT(0) space. Then the following inequality holds true for all $x, y, z \in X$ and $\lambda \in [0, 1]$.*

$$d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y).$$

Lemma 2.2 ([1, 32]). *Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to X$ be k-strictly pseudocontractive mapping. Then the following hold.*

(i) $F(T)$ is closed and convex,

(ii) T has demiclosedness-type property.

Lemma 2.3 ([44]). *Every bounded sequence in a complete CAT(0) space always has a $\triangle-$convergent subsequence.*

The following Lemma is obtained from Lemma 3.3 of Tufa and Zegeye [45].

Lemma 2.4. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $T : C \to X$ be a mapping. Define $h : C \to \mathbb{R}$ by*

$$h(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1 - \lambda)Tx \in C\}.$$

Then for any $x \in C$, the following hold:

1) $h(x) \in [0, 1]$ and $h(x) = 0$ if and only if $Tx \in C$;

2) if $\beta \in [h(x), 1]$, then $\beta x \oplus (1 - \beta)Tx \in C$;

3) if T is inward, then $h(x) < 1$;

4) if $Tx \notin C$, then $h(x)x \oplus (1 - h(x))Tx \in \partial C$.

We may also use the following notions in the sequel.

Let C be a subset of a complete CAT(0) space X. A sequence $\{x_n\}$ in C is called Fejér-monotone with respect to a subset D of C if for any $x \in D$, we have

$$d(x_{n+1}, x) \leq d(x_n, x), \forall n \geq 1.$$

Definition 2.5 ([37, 38]). Let D and C be two closed, convex and nonempty subsets of a CAT(0) space X with $D \subseteq C$. For any sequence $\{x_n\}$ in C if $\{x_n\}$ converges strongly to an element $x^* \in \partial C \backslash D, x_n \neq x^*$ implies that $\{x_n\}$ is not Fejer-monotone with respect to the set D, we say that the pair (D, C) satisfies S-condition.

3. Main Results

We start this section by constructing an algorithm for a finite family of nonself mappings in a complete CAT(0) space X. Let C be a nonempty, closed and convex subset of X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of inward mappings. Given $x_1 \in C$, let

$$h_1(x_1) = \inf\{\alpha \geq 0 : \alpha x_1 \oplus (1-\alpha)T_1x_1) \in C\}.$$

Take $\alpha_1 = \max\{\beta, h_1(x_1)\}$, where β is any arbitrary fixed element of $(0,1)$. Then let

$$x_2 := \alpha_1 x_1 \oplus (1-\alpha_1)T_1x_1 \in C.$$

Now, let $h_2(x_2) = \inf\{\alpha \geq 0 : \alpha x_2 \oplus (1-\alpha)T_2x_2 \in C\}$ and
$\alpha_2 = \max\{\alpha_1, h_2(x_2)\}$. Then set

$$x_3 := \alpha_2 x_2 \oplus (1-\alpha_2)T_2x_2 \in C.$$

Continuing the process in the same fashion, we obtain:

$$\begin{array}{rcl} x_4 & := & \alpha_3 x_3 \oplus (1-\alpha_3)T_3x_3 \in C, \\ \vdots & & \vdots \\ x_{N+1} & := & \alpha_N x_N \oplus (1-\alpha_N)T_N x_N \in C, \\ x_{N+2} & := & \alpha_{N+1} x_{N+1} \oplus (1-\alpha_{N+1})T_1 x_{N+1} \in C, \\ \vdots & & \vdots \end{array}$$

This iterative process can be expressed in compact form as follows:

$$\begin{cases} x_1 \in C, \\ \alpha_1 = \max\{\beta, h_1(x_1)\} \in C, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T_n x_n, \\ \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\}, \end{cases} \tag{3.1}$$

where $h_n(x) := \inf\{\alpha \geq 0 : \alpha x \oplus (1-\alpha)T_n x \in C\}, T_n = T_{n \ mod \ N}$, $h_n = h_{n \ mod \ N}$ and β is any arbitrary fixed element of $(0,1)$.

Next we state and prove our main results.

Lemma 3.1. *Let C be a nonempty convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of demicontractive inward mappings. Let $k = \max\{k_i : i = 1, 2, \cdots, N\}$, where k_i is the demicontractive constant of T_i for each i. Let $\{x_n\}$ be a sequence as defined in (3.1) with $\beta = k + \epsilon < 1$, for some $\epsilon > 0$ and assume that $F = \cap_{n=1}^N F(T_n) \neq \emptyset$. Then $\lim\limits_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.*

Proof. Since T_n is demicontractive mapping for each $n = 1, 2, \cdots, N$, from (3.1) and Lemma 2.1, we have

$$\begin{array}{rcl} d^2(x_{n+1}, p) & = & d^2(\alpha_n x_n \oplus (1-\alpha_n)T_n x_n, p) \\ & \leq & \alpha_n d^2(x_n, p) + (1-\alpha_n)d^2(T_n x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, T_n x_n) \\ & \leq & \alpha_n d^2(x_n, p) + (1-\alpha_n)[d^2(x_n, p) + kd^2(x_n, T_n x_n)] \\ & & -\alpha_n(1-\alpha_n)d^2(x_n, T_n x_n) \\ & = & d^2(x_n, p) - (1-\alpha_n)(\alpha_n - k)d^2(x_n, T_n x_n) \\ & \leq & d^2(x_n, p). \end{array} \tag{3.2}$$

Then $\{d(x_n, p)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, p)$ exists. ∎

We observe that if $T_i : C \to X$ is k_i-strictly pseudocontractive mapping for each $i = 1, 2, \cdots, N$, then each T_i is k-strictly pseudocontractive mapping, where $k = \max\{k_i : i = 1, 2, \cdots, N\}$.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X, $\{T_i\}_{i=1}^N : C \to X$ be a finite family of k_i-strictly pseudocontractive inward mappings and $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence as defined in (3.1) with $\beta = k + \epsilon < 1$, for some $\epsilon > 0$, where $k = \max\{k_i : i = 1, 2, \cdots N\}$. Then the following conclusions hold:*

1) If there exists $b \in (0,1)$ such that $\alpha_n \leq b, \forall n \geq 1$, then $\{x_n\}$ $\triangle$-converges to a point in F.

2) If $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F, C) satisfies the S-condition then $\{x_n\}$ converges strongly to a point in F.

Proof.

1) Suppose that there exists $b \in (0,1)$ such that $\alpha_n \leq b, \forall n \geq 1$. Then from (3.2), we have:

$$d^2(x_{n+1}, p) \quad \leq \quad d^2(x_n, p) - (1-\alpha_n)(\alpha_n - k)d^2(x_n, T_n x_n), \tag{3.3}$$

which yields

$$(\alpha_n - k)(1-\alpha_n)d^2(x_n, T_n x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p).$$

This implies that

$$\sum_{n=1}^{\infty} \epsilon(1-b)d^2(x_n, T_n x_n) \leq \sum_{n=1}^{\infty}(\alpha_n - k)(1-\alpha_n)d^2(x_n, T_n x_n) < \infty.$$

Thus, we obtain that

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0.$$

Now, from (3.1), it follows that

$$d(x_{n+1}, x_n) = (1-\alpha_n)d(x_n, T_n x_n) \to 0 \text{ as } n \to \infty.$$

Also, for each i, we have

$$d(x_{n+i}, x_n) \leq d(x_{n+i}, x_{n+i-1}) + d(x_{n+i-1}, x_{n+i-2}) + \cdots + d(x_{n+1}, x_n),$$

which implies

$$d(x_{n+i}, x_n) \to 0 \text{ as } n \to \infty.$$

Moreover, we observe that each T_n is Lipschitz with Lipschitz constant $L = \frac{1+k}{1-k}$. Then we have

$$d(x_n, T_{n+i}x_n) \leq (1+L)d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i}x_{n+i}).$$

Hence, we have

$$d(x_n, T_{n+i}x_n) \to 0 \text{ as } n \to \infty.$$

Then since $T_n = T_{n \ mod \ N}$, we obtain

$$\lim_{n\to\infty} d(x_n, T_j x_n) = 0, \text{ for each } j = 1, 2, \cdots, N.$$

On the other hand, since $\{x_n\}$ is bounded, by Lemma 2.3, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that Δ-converges to $x \in C$. In addition, by Lemma 2.2,

T_i has demiclosedness-type property for each $i = 1, 2, \cdots, N$. This implies that $x \in F$. To show that such a $\triangle$-limit point is unique, let $\{x_{n_j}\}$ be subsequence of $\{x_n\}$ that $\triangle$-converges to $y \in C$. Suppose $x \neq y$. Then from the fact that $\lim_{n\to\infty} d(x_n, x)$ exists for all $x \in F$ (see Lemma 3.1) and CAT(0) space satisfies Opial's property (see also [23], Theorem 3.3), we have

$$\begin{aligned}\lim_{n\to\infty} d(x_n, x) &= \lim_{i\to\infty} d(x_{n_i}, x) < \lim_{i\to\infty} d(x_{n_i}, y) \\ &= \lim_{n\to\infty} d(x_n, y) = \lim_{j\to\infty} d(x_{n_j}, y) \\ &< \lim_{j\to\infty} d(x_{n_j}, x) = \lim_{n\to\infty} d(x_n, x),\end{aligned}$$

which is a contradiction and hence $x = y$. Therefore, $\{x_n\}$ Δ-converges to a fixed point of T.

2) Suppose that $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F, C) satisfies the S-condition. Since $\{x_n\}$ and $\{T_n x_n\}$ are bounded and $d(x_n, x_{n+1}) = (1-\alpha_n)d(x_n, T_n x_n)$, it follows that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Thus, $\{x_n\}$ is strongly Cauchy sequence and hence $x_n \to x^* \in C$ as $n \to \infty$. Since T_n is inward mapping for each n, it follows by Lemma 2.4(3) that $h_n(x^*) < 1$. Hence, by Lemma 2.4(2), we obtain that

$$\alpha_n x^* \oplus (1-\alpha_n)T_n x^* \in C, \text{ for any } \alpha_n \in (h_n(x^*), 1).$$

Also, since $\lim_{n\to\infty} \alpha_n = 1$ and $\alpha_n = \max\{\alpha_{n-1}, h_n(x_n)\}$, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{h_{n_j}(x_{n_j})\}$ is non-decreasing and $\lim_{j\to\infty} h_{n_j}(x_{n_j}) = 1$. Hence, we have

$$\left(\frac{j}{j+1}h_{n_j}(x_{n_j})x_{n_j} \oplus \left(1 - \frac{j}{j+1}h_{n_j}(x_{n_j})\right)T_{n_j}x_{n_j}\right) \notin C \text{ and}$$

$$\lim_{j\to\infty}\left(\frac{j}{j+1}h_{n_j}(x_{n_j})x_{n_j} \oplus \left(1 - \frac{j}{j+1}h_{n_j}(x_{n_j})\right)T_{n_j}x_{n_j}\right) = x^*,$$

which imply that $x^* \in \partial C$. Then since $\{x_n\}$ is Fejér-monotone with respect to F and (F, C) satisfies the S-condition, it follows that $x^* \in F$.

The proof is complete. ∎

In Theorem 3.2, if we assume that T_i is nonexpansive mapping for each i, then we obtain the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X, $\{T_i\}_{i=1}^N : C \to X$ be a finite family of nonexpansive inward mappings with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence as defined in (3.1). Then the following conclusions hold:*

(i) If there exists $b \in (0, 1)$ such that $\alpha_n \leq b, \forall n \geq 1$, then $\{x_n\}$ $\triangle$-converges to a point in F.

(ii) If $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ *and* (F, C) *satisfies the S-condition then* $\{x_n\}$ *strongly converges to a point in* F.

Next, we prove strong convergence results using condition (I) or hemicompactness condition. Recall that a mapping $T : C \to X$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in C$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$. Now, we modify this definition for a finite family of mappings as follows.

We say that a finite family of mappings $\{T_i\}_{i=1}^N : C \to X$ satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0, \infty)$ such that $d(x, T_i x) \geq f(d(x, F))$, for all $x \in C$ and for some $i \in \{1, 2, \cdots, N\}$, where $F = \cap_{i=1}^N F(T_i)$ and $d(x, F) = \inf\{d(x, p) : p \in F\}$. Now, we define an iterative process $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ \alpha_1 = \max\{\beta, h_1(x_1)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T_n x_n \\ \alpha_{n+1} \in [\max\{\alpha_n, h_{n+1}(x_{n+1})\}, 1), \end{cases} \tag{3.4}$$

where $h_n(x) := \inf\{\alpha \geq 0 : \alpha x \oplus (1-\alpha) T_n x) \in C\}, T_n = T_{n \bmod N},\ h_n = h_{n \bmod N}$ and β is any arbitrarily fixed element of $(0, 1)$.

Theorem 3.4. *Let* C *be a nonempty, closed and convex subset of a complete CAT(0) space* X, $\{T_i\}_{i=1}^N : C \to X$ *be a finite family of Lipschitz demicontractive inward mappings. Let* $k = \max\{k_i : i = 1, 2, \cdots, N\}$, *where* k_i *is the demicontractive constant of* T_i *and let* $\{x_n\}$ *be a sequence as defined in* (3.4) *with* $\beta = k + \epsilon < 1$ *for some* $\epsilon > 0$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Suppose that* $\{T_i\}_{i=1}^N$ *satisfies condition (I) and* $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. *Then* $\{x_n\}$ *converges strongly to a point in* F.

Proof. Let $p \in F$. Then by Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists and from inequality (3.2), we have:

$$\sum_{n=1}^{\infty}(1-\alpha_n)(\alpha_n - k) d^2(x_n, T_n x_n) < \infty. \tag{3.5}$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\alpha_n \geq k + \epsilon$ for each n, it follows that

$$\sum_{n=1}^{\infty}(1-\alpha_n)(\alpha_n - k) = \infty.$$

This and (3.5) imply that

$$\liminf_{n\to\infty} d(x_n, T_n x_n) = 0.$$

Also, since $d(x_{n+1}, x_n) = (1-\alpha_n) d(x_n, T_n x_n)$, it follows that

$$\liminf_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Moreover, for each i, we have

$$d(x_{n+i}, x_n) \leq d(x_{n+i}, x_{n+i-1}) + d(x_{n+i-1}, x_{n+i-2}) + \cdots + d(x_{n+1}, x_n).$$

This yields

$$\liminf_{n\to\infty} d(x_{n+i}, x_n) = 0.$$

Now, let L_i be the Lipschtiz constant of T_i for each $i = 1, 2, 3, \cdots, N$ and $L = \max\{L_i : i = 1, 2, \cdots, N\}$. Then we have

$$d(x_n, T_{n+i}x_n) \leq (1 + L)d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i}x_{n+i}).$$

Thus, since $T_n = T_{n \ mod \ N}$, we obtain

$$\liminf_{n\to\infty} d(x_n, T_j x_n) = 0, \text{ for each } j = 1, 2, \cdots, N.$$

Then since T_j satisfies condition (I) for some j, we have $\liminf_{n\to\infty} f(d(x_n, F)) = 0$ for some increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ and $r \in (0, \infty)$. This gives $\liminf_{n\to\infty} d(x_n, F) = 0$. Moreover, since $d(x_{n+1}, p) \leq d(x_n, p)$, taking infimum over all $p \in F$, we obtain that

$$d(x_{n+1}, F) \leq d(x_n, F).$$

Then the sequence $\{d(x_n, F)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, for any $n, m \geq 1$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(x_n, p) \leq 2d(x_n, p),$$

which implies that

$$d(x_{n+m}, x_n) \leq 2d(x_n, F).$$

Then $\{x_n\}$ is a Cauchy sequence and hence $x_n \to x^* \in C$. Thus, we have

$$d(x^*, F) \leq d(x^*, x_n) + d(x_n, F) \to 0.$$

Then it follows from Lemma 2.2 that $x^* \in F$. This completes the proof. ∎

Since every quasi-nonexpansive mapping is demicontractive mapping, we have the following result for a finite family of quasi-nonexpansive mappings.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of Lipschitz quasi-nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (3.4) such that $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Suppose that $F = \cap_{n=1}^N F(T_n) \neq \emptyset$ and $\{T_i\}_{i=1}^N$ satisfies condition (I). Then $\{x_n\}$ converges strongly to a point in F.*

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of k_i-strictly pseudocontractive inward mappings. Let $k = \max\{k_i : i = 1, 2, \cdots, N\}$ and $\{x_n\}$ be a sequence as defined in (3.4) with $a = k + \epsilon < 1$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Suppose that $\{T_i\}_{i=1}^N$ satisfies condition (I) and $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point in F.*

Proof. Note that T_i is Lipschitz and demicontractive mapping for each i. Then the proof follows immediately from Theorem 3.4. ∎

Corollary 3.7. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (3.4) such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that $\{T_i\}_{i=1}^N$ satisfies condition (I) and $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point in F.*

Next, we prove strong convergence results assuming that at least one of the mappings in $\{T_i\}_{i=1}^N$ is hemicompact. We recall that a mapping $T : C \to X$ is called hemicompact if for any sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$. Note that if C is compact, then the mapping $T : C \to X$ is hemicompact.

Now, we prove the following theorem.

Theorem 3.8. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of k_i-strictly pseudocontractive inward mappings. Let $k = \max\{k_i : i = 1, 2, \cdots, N\}$ and $\{x_n\}$ be a sequence as defined in (3.4) with $\beta = k+\epsilon < 1$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and T_i is hemicompact for some i. Then $\{x_n\}$ converges strongly to a point in F.*

Proof. Using the method of the proof of Theorem 3.4, we obtain

$$\liminf_{n\to\infty} d(x_n, T_j x_n) = 0, \text{ for each } j = 1, 2, \cdots, N.$$

Then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty} d(x_m, T_j x_m) = 0, \text{ for each } j = 1, 2, \cdots, N.$$

Since T_j is hemicompact for some j, there is a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k} \to x^* \in C$ as $k \to \infty$. Then since T_j is Lipschitz continuous for each $j = 1, 2, \cdots, N$, we have

$$d(x^*, T_j x^*) = \lim_{k\to\infty} d(x_{m_k}, T_j x_{m_k}) = 0.$$

Therefore, $x^* \in F$ and hence the proof. ∎

If, in Theorem 3.8, we assume that C is Compact, then each T_i is hemicompact and so we have the following corollary.

Corollary 3.9. *Let C be a nonempty, compact and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of k_i-strictly pseudocontractive inward mappings. Let $k = \max\{k_i : i = 1, 2, \cdots, N\}$ and $\{x_n\}$ be a sequence as defined in (3.4) with $\beta = k+\epsilon < 1$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and T_i is hemicompact for some i. Then $\{x_n\}$ converges strongly to a point in F.*

Corollary 3.10. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and $\{T_i\}_{i=1}^N : C \to X$ be a finite family of nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (3.4) with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that $F =$*

$\cap_{i=1}^N F(T_i) \neq \emptyset$ and T_i is hemicompact for some i. Then $\{x_n\}$ strongly converges to a point in F.

Example 3.11. Now, we give a finite family of mappings that satisfy the conditions of Theorem 3.4 and then we present some numerical experiment results to explain the conclusion of the theorem. Let $X = \mathbb{R}^2$ and d be a metric defined on X by

$$d(x, y) = \begin{cases} |x_1| + |y_1| + |x_2 - y_2|, & x_2 \neq y_2, \\ |x_1 - y_1|, & x_2 = y_2, \end{cases}$$

where $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then (X, d) is a complete CAT(0) space (see, e.g., [45]). Let $C = \{x = (x_1, x_2) \in \mathbb{R}^2 : d((x_1, x_2), (0, 0)) \leq 1\}$. Then C is nonempty, closed and convex subset of X. Define $\{T_i\}_{i=1}^3 : C \to \mathbb{R}^2$ by

$$T_1 x = -2x, \ T_2 x = -5x$$

and

$$T_3 x = \begin{cases} x, & x \in D, \\ -3x, & x \in C \setminus D, \end{cases}$$

where $D = \{x = (x_1, x_2) \in C : 0 \leq x_1 \leq 1\}$. Then we can easily observe that each T_i is inward and Lipschitz mapping with

$$F = F(T_1) \cap F(T_2) \cap F(T_3) = \{(0, 0)\}.$$

Moreover, $d(x, F) = |x_1| + |x_2|$ and $d(x, T_1 x) = 3|x_1| + 3|x_2|$. So, if we consider the function $f(r) = r, r \geq 0$, then it follows that

$$d(x, T_1 x) \geq f(d(x, F)).$$

Therefore, $\{T_i\}_{i=1}^3$ satisfies the condition (I). Now let $p \in F$, i.e., $p = (0, 0)$. Then one can easily verify that

$$d^2(T_1 x, T_1 p) \leq d^2(x, p) + k d^2(x, T_1 x), \ \text{for } \frac{1}{3} \leq k < 1.$$
$$d^2(T_2 x, T_2 p) \leq d^2(x, p) + k d^2(x, T x), \ \text{for } \frac{2}{3} \leq k < 1.$$
$$d^2(T_3 x, T_3 p) \leq d^2(x, p) + k d^2(x, T_3 x), \ \text{for } \frac{1}{2} \leq k < 1.$$

If we set $k := \frac{2}{3}$, then the mappings T_1, T_2 and T_3 are demicontractive mappings with a common constant $k = \frac{2}{3}$. Now, let the initial guess $x_1 = (1, 0)$ and $\epsilon = \frac{1}{6}$. Then $\beta = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$ and Algorithm 3.4 reduces to

$$\begin{cases} \alpha_1 = \max\{\frac{5}{6}, h_1(x_1)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T_n x_n, \\ \alpha_{n+1} \in [\max\{\alpha_n, h_{n+1}(x_{n+1})\}, 1), \end{cases}$$

where $h_n(x) := \inf\{\alpha \geq 0 : \alpha x \oplus (1 - \alpha) T_n x) \in C\}, T_n = T_{n \ mod \ 3}, \ h_n = h_{n \ mod \ 3}$. For convenience purpose, we denote the sequence $\{x_n\}$ by $\{x_n\} = \{(x_n^{(1)}, x_n^{(2)})\}$. Now, we have

$$\begin{aligned} h_1(x_1) &= \inf\{\alpha \geq 0 : \alpha x_1 \oplus (1 - \alpha) T_1 x_1 \in C\} \\ &= \inf\{\alpha \geq 0 : \alpha(1, 0) \oplus (1 - \alpha)(-2, 0) \in C\} \\ &= \frac{1}{3}. \end{aligned}$$

Hence, $\alpha_1 = \max\{\beta, h_1(x_1)\} = \frac{5}{6}$. Then we have

$$\begin{aligned} x_2 &= \alpha_1 x_1 \oplus (1-\alpha_1)T_1 x_1 \\ &= \frac{5}{6}\Big(1,0\Big) \oplus \Big(1-\frac{5}{6}\Big)(-2,0) \\ &= (\frac{1}{2},0) \end{aligned}$$

and hence we have

$$\begin{aligned} h_2(x_2) &= \inf\{\alpha \geq 0 : \alpha x_2 \oplus (1-\alpha)T_2 x_2 \in C\} \\ &= \inf\{\alpha \geq 0 : \alpha\Big(\frac{1}{2},0\Big) \oplus (1-\alpha)\Big(-\frac{5}{2},0\Big) \in C\} \\ &= \frac{1}{2}. \end{aligned}$$

Since $\alpha_2 \in [\frac{5}{6}, 1)$, if we take $\alpha_2 = \frac{5}{6}$, then we obtain

$$\begin{aligned} x_3 &= \alpha_2 x_2 \oplus (1-\alpha_2)T_2 x_2 \\ &= \frac{5}{6}\Big(\frac{1}{2},0\Big) \oplus \frac{1}{6}\Big(\frac{-5}{2},0\Big) \\ &= (0,0). \end{aligned}$$

This implies that $h_3(x_3) = 0$. Hence, $\alpha_3 \in [\frac{5}{6}, 1)$. Take $\alpha_3 = \frac{5}{6}$. Then it follows that $x_4 = (0,0)$, which implies that $T_4 x_4 = T_1 x_4 = (0,0)$ and $h_4(x_4) = 0$. Continuing the process in the same manner we obtain that $x_n = (0,0)$ for all $n \geq 3$. Hence, $x_n \to (0,0) \in F(T)$ as $n \to \infty$.

4. Conclusion

In this paper, Mann type iterative methods for approximating common fixed points of a finite family of non-self mappings are studied in the setting of a complete CAT(0) space. $\triangle$-convergence and strong convergence results are obtained under appropriate conditions. Our results extend, unify and complement many of the results in the literature (see, e.g., [25, 28, 29, 36–39, 46]). In particular, Theorem 3.2 extends Theorem 3.2 of [39] to a CAT(0)) space which is more general than the Hilbert space and hence it extends Theorem 8 of [46] in the sense that it is true for a finite family of k-strictly pseudocontractive mappings in a CAT(0)) space. Corollary 3.3 extends Theorem 1 of [36] to a finite family of non-expansive mappings in a space more general than the Hilbert space and complements Corollary 3.4 of [38] and Theorem 7 of [37]. Theorems 3.4 and 3.8 extend Theorem 2 of [29] from a finite family of self mappings to a finite family of non-self mappings in a more general space than the Hilbert space.

Note that Theorem 7 of [38] is proved for a countably infinitely family of demicontractive mappings under the assumption that $\sum(1-\alpha_n) < \infty$. But, Theorem 3.2 of the current paper establishes Δ- convergence for a finite family of k-strictly pseudocontractive mappings when $\{\alpha_n\}$ is bounded away from 1. Furthermore, Theorem 3.4 and Theorem 3.8 of the current paper provide strong convergence results for a finite family of k-strictly pseudocontractive mappings and demicontractive mappings (respectively) when $\sum(1-\alpha_n) = \infty$. We also remark that the S-condition which is essential in the proof of

Theorem 7 of [38] is not required in some of our results (see, for instance Theorem 3.4 and Theorem 3.8).

Acknowledgements

We would like to thank the referees and the editor for their comments and suggestions on the manuscript.

References

[1] A. Gharajelo, H. Dehghan, Convergence theorems for strict pseudo-contractions in CAT(0) metric spaces, Filomat 31 (7) (2017) 1967–1971.

[2] C. Chidume, H. Zegeye, Convergence theorems for fixed points of demicontinuous pseudocontractive mappings, Fixed Point Theory Appl. 2005 (2005) Article no. 475373.

[3] G.L. Acedo, H.K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 67 (2007) 2258–2271.

[4] O.A. Daman, H. Zegeye, Strong Convergence theorems for a common fixed point of a finite family of pseudocontractive mappings, Int. J. Math. Math. Sci. 2012 (2012) 18 pages.

[5] T.L. Hicks, J.R. Kubicek, On the Mann iterative process in Hilbert spaces, J. Math. Anal. Appl. 59 (1977) 498–504.

[6] Y. Kimura, W. Takahashi, M. Toyoda, Convergence to common fixed points of a finite family of nonexpansive mappings, Arch. Math. 84 (4) (2005) 350–363.

[7] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (2) (1979) 274–276.

[8] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025–1033.

[9] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc. 65 (2002) 109–113.

[10] H. Zegeye, N. Shahzad, An algorithm for a common fixed point of a family of pseudocontractive mappings, Fixed Point Theory and Appl. 2013 (2013) Article no. 234.

[11] C.E. Chidume, C.O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, J. Math. Anal. Appl. 318 (1) (2006) 288–295.

[12] S. Temir, Convergence of three-step iteration scheme for common fixed point of three Berinde nonexpansive mappings, Thai J. Math. 20 (2) (2022) 971–979.

[13] A.R. Tufa, H. Zegeye, Convergence theorems for Lipschitz pseudocontractive non-self mappings in Banach spaces, J. Nonlinear Anal. Optim. 6 (2) (2015) 1–17.

[14] M. Bestvina, $\mathbb{R}$-Trees in topology, geometry, and group theory, In Handbook of Geometric Topology, North-Holland, Amsterdam, The Netherlands (2002), 55–91.

[15] R. Espinola, W.A. Kirk, Fixed point theorems in $\mathbb{R}$-trees with applications to graph theory, Topol. Appl. 153 (7) (2006) 1046–1055.

[16] W.A. Kirk, Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees, Fixed Point Theory Appl. 2004 (4) (2004) 309–316.

[17] C. Semple, M. Steel, Phylogenetics, Oxford Lecture Series in Mathematics and Its Applications: Oxford University Press, Oxford, UK, 2003.

[18] C. Khunpanuk, C. Garodia, I. Uddin, N. Pakkaranang, On a proximal point algorithm for solving common fixed point problems and convex minimization problems in Geodesic spaces with positive curvature, AIMS Math. 7 (5) (2022) 9509–9523.

[19] N. Ekkarntrong, N. Pakkaranang, B. Panyanak, P. Yotkaew, On a sequence of quasi-nonexpansive mappings in a Geodesic space with curvature bounded above, J. Nonlinear Convex Anal. 23 (1) (2022) 113–127.

[20] N. Pakkaranang, P. Kumam, C. Wen, J. Yao, Y. Cho, On modified proximal point algorithms for solving minimization problems and fixed point problems in CAT(k) spaces, Math. Methods Appl. Sci. 44 (17) (2019) 12369–12382.

[21] T. Bantaojai, C. Garodia, I. Uddin, N. Pakkaranang, P. Yimmuang, A novel iterative approach for solving common fixed point problems in Geodesic spaces with convergence analysis, Carpathian J. Math. 37 (2) (2021) 145–160.

[22] C. Klangpraphany, B. Panyanak, Fixed point theorems for some generalized multi-valued nonexpansive mappings in Hadamard spaces, Thai J. Math. 17 (2) (2019) 543–555.

[23] S. Salisu, M.S. Minjibir, P. Kumam, Convergence theorems for fixed points in $\mathrm{CAT}_p(0)$ spaces, J. Appl. Math. Comput. 69 (2022) 631–650.

[24] D. Yambangwai, T. Thianwan, Δ-Convergence and strong convergence for asymptotically nonexpansive mappings on a CAT(0) space, Thai J. Math. 19 (3) (2021) 813–826.

[25] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–346.

[26] J. Chen, D. Wu, C. Zhang, A new iterative scheme of modified Mann iteration in Banach space, Abstr. Appl. Anal. 2014 (2014) Article ID 264909.

[27] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[28] H.K. Xu, R.G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. Optim. 22 (2001) 767–773.

[29] M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl. 294 (2004) 73–81.

[30] C.E. Chidumea, N. Shahzad, Weak convergence theorems for a finite family of strict pseudocontractions, Nonlinear Anal. Theory Methods Appl. 72 (2010) 1257–1265.

[31] H. Fukhar-ud-din, A. Khana, N. Hussain, Approximating common fixed points of total asymptotically nonexpansive mappings in CAT(0) spaces, J. Nonlinear Sci. Appl. 10 (2017) 771–779.

[32] M.S. Minjibir, S. Salisu, Strong and Δ-convergence theorems for a countable family of multi-valued demicontractive maps in Hadamard spaces, Nonlinear Funct. Anal. Appl. 27 (1) (2022) 45–58.

[33] C.E. Chidume, H. Zegeye, N. Shahzad, Convergence theorems for a common fixed point of a finite family of nonself nonexpansive mappings, Fixed Point Theory Appl. 2005 (2005) 233–241.

[34] S. Matsushita, W. Takahashi, Strong convergence theorems for nonexpansive nonself mappings without boundary conditions, Nonlinear Anal. Theory Methods Appl. 68 (2) (2008) 412–419.

[35] Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, J. Math. Anal. Appl. 321 (1) (2006) 316–326.

[36] V. Colao, G. Marino, Krasnoselskii-Mann method for non-self mappings, Fixed Point Theory Appl. 2015 (2015) 1–7.

[37] M. Guo, X. Li, Y. Su, On an open question of V. Colao and G. Marino presented in the paper "Krasnoselskii-Mann method for non-self mappings", SpringerPlus (2016) Article no. 1328.

[38] A.R. Tufa, H. Zegeye, Approximating common fixed point of a family of non-self mappings in CAT(0) spaces, Bol. Soc. Mat. Mex. 28 (1) (2022) Article no. 3.

[39] A.R. Tufa, O. Daman, T. Motsumi, Approximating common fixed points of a finite family of non-self mappings in Hilbert spaces, J. Anal. 29 (2021) 947–961.

[40] K.S. Brown, Buildings, Springer, New York, 1989.

[41] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer, Berlin, 1999.

[42] S. Dhompongsa, W.A. Kirk, B. Sims, Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal. Theory, Methods Appl. 65 (4) (2006) 762–772.

[43] S. Dhompongsa, B. Panyanak, On $\triangle$-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56 (10) (2008) 2572–2579.

[44] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. Theory, Methods Appl. 68 (12) (2008) 3689–3696.

[45] A.R. Tufa, H. Zegeye, Krasnoselskii-Mann method for multi-valued non-self mappings in CAT(0) spaces, Filomat 31 (14) (2017) 4629–4640.

[46] V. Colao, G. Marino, N. Hussain, On the approximation of fixed points of non-self strict pseudocontractions, RACSAM 111 (1) (2017) 159–165.