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In memoriam Professor Charles E. Chidume (1947–2021)

# On the Recursive Sequence $x_{n+1} = \frac{x_{n-2k-3}}{1+\prod_{m=1}^{k+1} x_{n-2m+1}}$

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**Abstract** In this paper, a solution of the following difference equation was investigated

$$x_{n+1} = \frac{x_{n-2k-3}}{1+\prod_{m=1}^{k+1} x_{n-2m+1}},\ n = 0, 1, 2, \ldots$$

where $x_{-2k-3}, x_{-2k-2}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers and $k = 0, 1, 2, \ldots$.

**MSC:** 39A10
**Keywords:** difference equation; period $2k + 4$ solution; recursive sequence

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## 1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in physics, ecology, biology, etc.

Recently, a high attention to studying the periodic nature of nonlinear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations, see the references [1–15].

Cinar [2, 3] studied the following problem with positive initial values,

$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}$$
$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}$$

for $n = 0, 1, 2, \ldots$

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Simsek et al. [8–10, 12, 13] investigated the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$$
$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}}$$
$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$$
$$x_{n+1} = \frac{x_{n-17}}{1+x_{n-5}x_{n-11}}$$
$$x_{n+1} = \frac{x_{n-7}}{1+x_{n-3}}$$

for $n = 0, 1, 2, \ldots$

Ogul et al. [7] studied the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-7}}{x_{n-1}x_{n-3}x_{n-5}}, n = 0, 1, 2, \ldots$$

where $x_{-7}, x_{-6}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

In this paper, we investigate the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-2k-3}}{1+\prod_{m=1}^{k+1} x_{n-2m+1}}, \; n = 0, 1, 2, \ldots \tag{1.1}$$

where $x_{-2k-3}, x_{-2k-2}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers and $k = 0, 1, 2, \ldots$

## 2. Main Results

**Theorem 2.1.** *Consider the difference equation (1.1). Then the following statements are true.*

*(a) The sequence $x_{(2k+4)n-(2k+4)+s}$ are decreasing and there exist $a_s \geq 0$ such that $\lim_{n\to\infty} x_{(2k+4)n-(2k+4)+s} = a_s$ for $s = 1, 2, \ldots, 2k+4$.*

*(b) $(a_1, a_2, \ldots, a_{2k+3}, a_{2k+4}, \ldots)$ is a solution of equation (1.1) of period $2k+4$.*

*(c) $\prod_{m=1}^{k+2} a_{2m+u-2} = 0$ for $u = 1, 2$.*

*(d) There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-2k-1}$ for all $n \geq n_0$, then $\lim_{n\to\infty} x_n = 0$.*

*(e) The following formulas*

$$x_{(2k+4)n+2t+u-2} = x_{-(2k+4)+2t+u-2}\left(1 - \frac{\prod_{l=1}^{k+2} x_{-(2l-u)}}{x_{-(2k+4)+2t+u-2}\left(1+\prod_{l=1}^{k+1} x_{-(2l-u)}\right)} \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1+\prod_{l=1}^{k+1} x_{2i-(2l-u)}}\right)$$

*for $t = 1, 2, \ldots, k+2$ and $u = 1, 2$ hold.*

*(f) If* $x_{(2k+4)n+2t+u-2} \to a_{2t+u-2} \neq 0$ *then* $x_{(2k+4)n+2k+u+2} \to 0$ *for* $t = 1, 2, \ldots, k+1$ *and* $u = 1, 2$

*Proof.* (a) Firstly, we consider the equation (1.1). From this equation, we obtain

$$x_{n+1}\left(1 + \prod_{m=1}^{k+1} x_{n-2m+1}\right) = x_{n-2k-3}.$$

Since $\prod_{m=1}^{k+1} x_{n-2m+1} > 0$ then $1 + \prod_{m=1}^{k+1} x_{n-2m+1} > 1$. Thus $x_{n+1} < x_{n-2k-3}, n \in \mathbb{N}$, we obtain that there exist $\lim_{n\to\infty} x_{(2k+4)n-(2k+4)+s} = a_s$ for $s = 1, 2, \ldots, 2k+4$.

(b) By (a), thus $(a_1, a_2, \ldots, a_{2k+3}, a_{2k+4}, \ldots)$ is a solution of equation (1.1) of period $2k+4$.

(c) In view of the equation (1.1), we obtian

$$x_{(2k+4)n+u} = \frac{x_{(2k+4)n-(2k+4)+u}}{1 + \prod_{m=1}^{k+1} x_{(2k+4)n-2m+u}}.$$

The limits as $n \to \infty$ are put on both sides of the above equality

$$\lim_{n\to\infty} x_{(2k+4)n+u} = \lim_{n\to\infty} \frac{x_{(2k+4)n-(2k+4)+u}}{1 + \prod_{m=1}^{k+1} x_{(2k+4)n-2m+u}}.$$

Then

$$a_u = \frac{a_u}{1 + \prod_{m=1}^{k+1} a_{(2k+4)-2m+u}}$$

$$a_u + a_u \prod_{m=1}^{k+1} a_{(2k+4)-2m+u} = a_u$$

$$a_u \prod_{m=1}^{k+1} a_{(2k+4)-2m+u} = 0$$

$$\prod_{m=1}^{k+2} a_{2m+u-2} = 0$$

for $u = 1, 2$.

(d) Suppose there exist $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-2k-1}$ for all $n \geq n_0$, then $a_u \leq a_{u+2} \leq \ldots \leq a_{u+2k+2} \leq a_u$. By (c) we have $\prod_{m=1}^{k+2} a_{2m+u-2} = 0$ for $u = 1, 2$, the results are obtained above.

(e) Subtracting $x_{n-2k-3}$ from both sides of equation (1.1), we obtain

$$x_{n+1} - x_{n-2k-3} = (x_{n-1} - x_{n-2k-5})\frac{1}{1 + \prod_{m=1}^{k+1} x_{n-2m+1}},$$

and the following formula is produced below, for $n \geq 2$

$$x_{2n-4+u} - x_{2n-2k-8+u} = (x_u - x_{-(2k+4)+u}) \prod_{i=1}^{n-2} \frac{1}{1 + \prod_{m=1}^{k+1} x_{2i-2m+u}}$$

for $u = 1, 2$ hold.
Replacing $n$ by $(k+2)j + t - 1$ and summing from $j = 0$ to $j = n$, we obtain

$$x_{(2k+4)n+2t+u-2} - x_{-(2k+4)+2t+u-2} = (x_{2t+u-2} - x_{-(2k+4)+2t+u-2}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1 + \prod_{m=1}^{k+1} x_{2i-2m+u}}$$

for $t = 1, 2, \ldots, k+2$ and $u = 1, 2$.
Now, we obtained of the above formulas

$$x_{(2k+4)n+2t+u-2} = x_{-(2k+4)+2t+u-2} \left( 1 - \frac{\prod_{l=1}^{k+2} x_{-(2l-u)}}{x_{-(2k+4)+2t+u-2} \left( 1 + \prod_{l=1}^{k+1} x_{-(2l-u)} \right)} \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1 + \prod_{l=1}^{k+1} x_{2i-(2l-u)}} \right)$$

for $t = 1, 2, \ldots, k+2$ and $u = 1, 2$.

(f) Assume that $a_{2t+u-2} = 0$ for $t = 1, 2, \ldots, k+2$ and $u = 1, 2$. By (e) we have

$$\lim_{n\to\infty} x_{(2k+4)n+2t+u-2} = \lim_{n\to\infty} x_{-(2k+4)+2t+u-2} \left( 1 - \frac{\prod_{l=1}^{k+2} \frac{x_{-(2l-u)}}{x_{-(2k+4)+2t+u-2}}}{1 + \prod_{l=1}^{k+1} x_{-(2l-u)}} \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1 + \prod_{l=1}^{k+1} x_{2i-(2l-u)}} \right)$$

$$a_{2t+u-2} = x_{-(2k+4)+2t+u-2} \left( 1 - \frac{\prod_{l=1}^{k+2} \frac{x_{-(2l-u)}}{x_{-(2k+4)+2t+u-2}}}{1 + \prod_{l=1}^{k+1} x_{-(2l-u)}} \sum_{j=0}^{\infty} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1 + \prod_{l=1}^{k+1} x_{2i-(2l-u)}} \right)$$

From $a_{2t+u-2} = 0$ then

$$\frac{x_{-(2k+4)+2t+u-2}\left(1+\prod_{l=1}^{k+1} x_{-(2l-u)}\right)}{\prod_{l=1}^{k+2} x_{-(2l-u)}} = \sum_{j=0}^{\infty} \prod_{i=1}^{(k+2)j+t-1} \frac{1}{1+\prod_{l=1}^{k+1} x_{2i-(2l-u)}}$$

Since

$$\prod_{i=1}^{(k+2)j+k+1} \frac{1}{1+\prod_{l=1}^{k+1} x_{2i-(2l-u)}} < \cdots < \prod_{i=1}^{(k+2)j} \frac{1}{1+\prod_{l=1}^{k+1} x_{2i-(2l-u)}}$$

Thus, $x_{u-2} < x_{u-4} < \ldots < x_{u-(2k+4)}$ for $u = 1, 2$. This contradicts our assumption. Which completes the proof of the theorem. ∎

## 3. Numerical Results

In this section, we demonstrate some results of equation (1.1) with $k = 0, 1, 2$ and 3.

**Example 3.1.** [8] Consider the recursive sequence $x_{n+1} = \dfrac{x_{n-3}}{1+x_{n-1}}$, which is a special case of (1.1) for $k = 0$. The initial conditions are selected as follows, $x_{-3} = 0.9, x_{-2} = 0.8, x_{-1} = 0.7, x_0 = 0.6$. Then the graph of solution is given below.

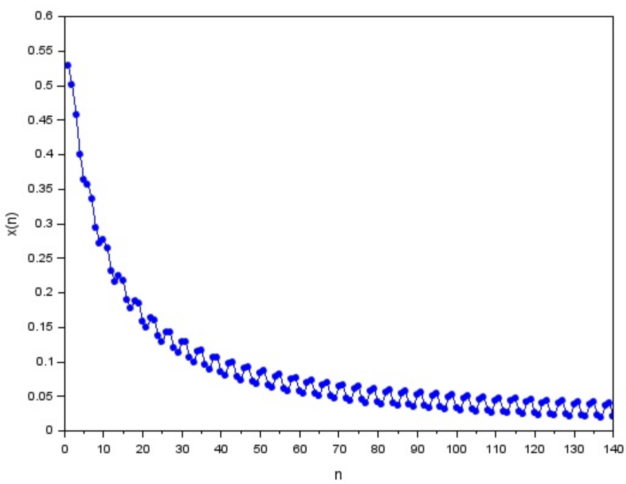


Figure 1. $x_n$ graph of the solution of equation (1.1) of period 4.

**Example 3.2.** [10] Consider the recursive sequence $x_{n+1} = \dfrac{x_{n-5}}{1+x_{n-1}x_{n-3}}$, which is a special case of (1.1) for $k = 1$. The initial conditions are selected as follows, $x_{-5} = 0.9, x_{-4} = 0.8, \ldots, x_0 = 0.4$. Then the graph of solution is given below.

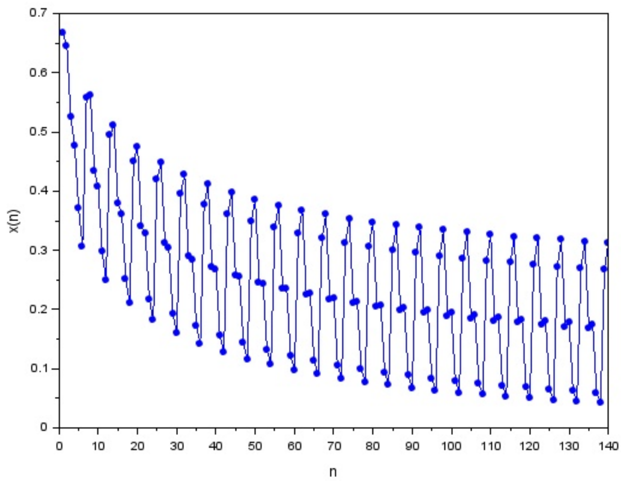


FIGURE 2. $x_n$ graph of the solution of equation (1.1) of period 6.

**Example 3.3.** [7] Consider the recursive sequence $x_{n+1} = \dfrac{x_{n-7}}{1 + x_{n-1}x_{n-3}x_{n-5}}$, which is a special case of (1.1) for $k = 2$. The initial conditions are selected as follows, $x_{-7} = 0.9, x_{-6} = 0.8, \ldots, x_0 = 0.2$. Then the graph of solution is given below.

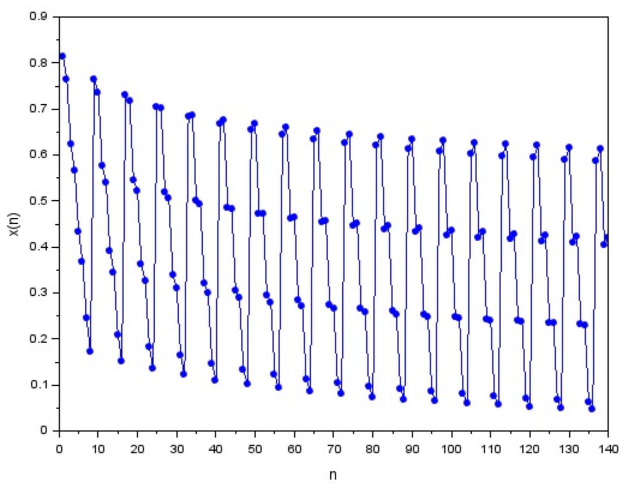


FIGURE 3. $x_n$ graph of the solution of equation (1.1) of period 8.

**Example 3.4.** Consider the recursive sequence $x_{n+1} = \dfrac{x_{n-9}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$, which is a special case of (1.1) for $k = 3$. The initial conditions are selected as follows, $x_{-9} = 0.99, x_{-4} = 0.89, \ldots, x_0 = 0.09$. Then the graph of solution is given below.

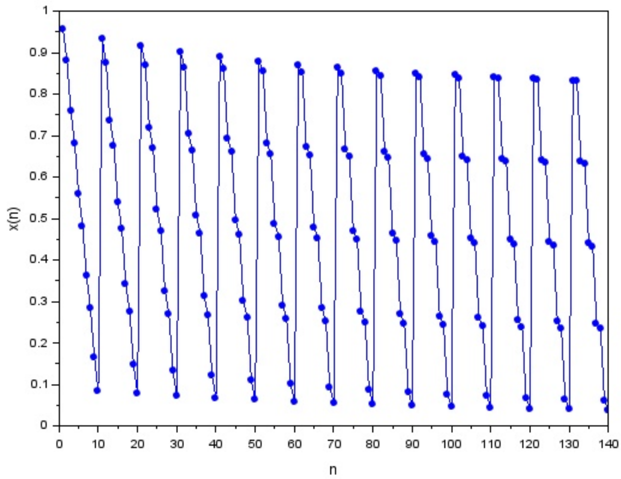


FIGURE 4. $x_n$ graph of the solution of equation (1.1) of period 10.


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