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Aggregation Function Constructed from Semi-copula

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Abstract In this work, we show that almost all aggregation functions can be constructed by composing semi-copulas and tuples of non-decreasing univariate functions. In particular, this construction method works for all strictly increasing aggregation functions. This method also resembles Sklar's construction of multivariate distribution functions. As a demonstration, construction examples via this method are also given.

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1. Introduction

One important tool for analysis and interpretation of data is a representative function. Aggregation functions are functions with special properties. They are usually used for data representation. In addition, they interpret the data contained in a k-tuple into a single-value representation. For a few decade, aggregation functions are extensively used in many fields such as applied mathematics, computer and engineering sciences, economics, and finance. One can refer to [1–3] for related literature. Each aggregation is used depending on a suitable situation. For example, we may use the mean or the median to aggregate some data, but it is not suitable for the set of data containing extremely different values. There are a lot of aggregation functions, then we should pick the appropriate functions for given data. Hence, the study of a classification is very important, the more different class, the better we get. Although, aggregation functions have used a long time ago, but construction methods are widely studied nowadays.

There are several methods to construct aggregation functions, for example, extensions of functions to aggregation functions [4–6], transformations of functions to aggregation functions [7–14], transformations of aggregation functions to other aggregation functions [15–17], and penalty-based constructions [18–23].

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We are interested in a method, which constructs aggregation functions based on copulas, Sklars copula-based method [24].

In 2008, Durante et al. [15] introduced new methods to construct aggregation functions. They applied Sklar's method, considered a multivariate distribution as a 2-increasing aggregation function and obtained similar results.

In this work, we show that almost all aggregation functions can be constructed by composing semi-copulas and tuples of non-decreasing univariate functions. This method also resembles Sklars construction of multivariate distribution functions. On the other hand, we obtain the necessary and sufficient conditions for representing aggregation functions in a form of our method. Moreover, we show that all continuous aggregation functions can be approximated by aggregation functions in this form.

In the next section, we will provide some preliminary concepts about aggregation functions, copulas, and semi-copulas. In Section 3, we will present our construction method with its properties. The proof of main theorems will be presented in Section 4. Finally, some conclusions are given.

2. Preliminaries

For reasons of completeness, we will assume that the data is lying in the interval $\mathbb{I} = [0, 1]$. The argument $\vec{u} = (u_1, \ldots, u_k)$ denotes an element of $\mathbb{I}^k$ with special cases for the zero vector $\vec{0} = (0, \ldots, 0)$ and the vector $\vec{1} = (1, \ldots, 1)$. The vector which has 1 at i-th coordinate and 0 otherwise is denoted by $\vec{e}_i$.

Definition 2.1. A function $A : \mathbb{I}^k \to \mathbb{I}$ is called an *aggregation function* if A is non-decreasing (in each variable) with the boundary conditions, $A(\vec{0}) = 0$ and $A(\vec{1}) = 1$.

Note that the notation $\vec{u} < \vec{v}$ stand for $u_i < v_i$ for all i. Similarly, $\vec{u} \leq \vec{v}$ we mean $u_i \leq v_i$.

Definition 2.2. For any aggregation function $A : \mathbb{I}^k \to \mathbb{I}$ and all $a_i, b_i \in \mathbb{I}$ in which $a_i \leq b_i$. The *volume* of A, denoted by V_A, is defined by

$$V_A\left(\prod_{i=1}^{k}(a_i, b_i]\right) := \sum_{\vec{x} \in \prod_{i=1}^{k}\{a_i, b_i\}} (-1)^{N(\vec{x}, \vec{a})} A(\vec{x}),$$

where $N(\vec{x}, \vec{a})$ is the number of i such that $x_i = a_i$.

A k-dimentional aggregation function is called a $k-$*increasing* if the volume of A is non-negative.

Example 2.3. The arithmetic mean, Mean : $\mathbb{I}^k \to \mathbb{I}$ defined by $\mathrm{Mean}(\vec{u}) = \dfrac{1}{k}\sum_{i=1}^{k} u_i$, is an aggregation function.

Definition 2.4. A function $S : \mathbb{I}^k \to \mathbb{I}$ is called a *k-semi-copula* (or a *semi-copula*) if

(i) S is non-decreasing (in each variable),
(ii) $S(\vec{u}) = u_i$ if $u_j = 1$ for all $j \neq i$.

Note that $S(\vec{u}) = 0$ when $u_i = 0$ for some i.

Clearly, every semi-copula is an aggregation function, but aggregation functions are not semi-copulas by Example 2.3.

Example 2.5. The function $L : \mathbb{I}^k \to \mathbb{I}$ defined by

$$L(\vec{u}) = \begin{cases} u_i & \text{if } u_j = 1 \text{ for all } i \neq j, \\ 0 & \text{otherwise,} \end{cases}$$

and the minimum function M in k-dimentional defined by

$$M(\vec{u}) = \min\{u_1, \ldots, u_k\}$$

are semi-copulas. Moreover, $L \leq S \leq M$ for all semi-copula S.

Definition 2.6. A *k-copula* (or a *copula*) is a function $C : \mathbb{I}^k \to \mathbb{I}$ satisfying the following properties:

(i) C is a k-increasing,
(ii) $C(\vec{u}) = 0$ whenever $u_i = 0$ for some $i = 1, \ldots, k$,
(iii) $C(\vec{u}) = u_i$ whenever $u_j = 1$ for all $j \neq i$.

It can be shown that a copula is a semi-copula satisfying a $k-$increasing condition. The function L is a semi-copula but is not a copula.

One important theorem in copula theory is the Sklar's theorem[24]. The theorem is demonstated that every multivariate distribution function can be constructed by a copula and marginal distribution functions. In 2008, Durante et al.[15] shown that a 2-increasing aggregation function can be represented by a copula and univariate non-decreasing functions using Sklar's method.

In this work, we are motivated by Sklar's method and results of Durante et al. We will show that this method also works for almost all aggregation functions if we choose semi-copulas instead of a copula and then compose them with univariate non-decreasing functions.

The aim of the next section is to present our construction method together with some construction examples.

3. Main Results

In this section, we will present our construction method and some examples. The construction part is actually simple and natural as stated in the following theorem.

Theorem 3.1. *Let S be a semi-copula and F_i be a non-decreasing function on $\mathbb{I}$ with $F_i(0) = 0$ and $F_i(1) = 1$ for all $i = 1, 2, \ldots, k$. Then the function A defined by*

$$A(\vec{u}) = S(F_1(u_1), F_2(u_2), \ldots, F_k(u_k)), \tag{3.1}$$

for all $\vec{u} \in \mathbb{I}^k$, is an aggregation function. Moreover, the aggregation function A satisfies the following condition for all $i = 1, 2, \ldots, k$.

$$F_i(u) = F_i(v) \text{ implies } A(\vec{u} + (u - u_i)\vec{e}_i) = A(\vec{u} + (v - u_i)\vec{e}_i) \text{ for all } \vec{u} \in \mathbb{I}^k, u, v \in \mathbb{I}. \tag{3.2}$$

Example 3.2. Let $S : \mathbb{I}^2 \to \mathbb{I}$ be given by $S(u, v) = u^2v + uv^2 - u^2v^2$.
We khown that the function S is a semi-copula [14]. Let F_1, F_2 be functions on $\mathbb{I}$ given by $F_1(u) = u$ and $F_2(v) = v^2$ for all $u, v \in \mathbb{I}$. Then the function A constructed via

$$A(u, v) = S(u, v^2) = u^2v^2 + uv^4 - u^2v^4$$

is an aggregation function.

The proof of Theorem 3.1 is obvious and simple that follows from a non-decreasing property and extreme-value conditions. A more complicated question is whether this construction is widely adaptable. In other words, whether we can use this method to construct many aggregation functions. Obviously, not all aggregation functions can be constructed via (3.1) because such aggregation functions also satisfy (3.2), it turns out that (3.2) is also sufficient for an aggregation function represented in the form of (3.1). Therefore, (3.2) can be considered as a necessary and sufficient condition yielding (3.1).

Theorem 3.3. *Let A be an aggregation function, $F_j(u) = A(\vec{1} - (1-u)\vec{e_j})$ for all $u \in \mathbb{I}, j \in \{1, 2, \dots, k\}$. If (3.2) holds, then there is a semi-copula S such that*

$$A(\vec{u}) = S(F_1(u_1), \dots, F_k(u_k)).$$

Even though the above condition seems natural, but the proof is not easy. Thus, we will postpone the proof to the next section. It should be mentioned that condition (3.2) fails for disjunctive aggregation functions, e.g., those that have $F_j(u) = 1$ for all $u \in \mathbb{I}$. This can also happen for an aggregation function A such that F_j is not constant.

Example 3.4. Let $A : \mathbb{I}^2 \to \mathbb{I}$ be defined by

$$A(u,v) = \begin{cases} \frac{u+v}{2} & \text{if } (u,v) \in \mathbb{I} \times [0, \frac{1}{2}] \\ \max\{u,v\} & \text{if } (u,v) \in \mathbb{I} \times (\frac{1}{2}, 1] \end{cases}$$

for all $u, v \in \mathbb{I}$.
Let $F_1(u) = A(u, 1)$ for all u and $F_2(v) = A(1, v)$ for all v.
First, we will show that A is an aggregation function.
Let us look at the boundary conditions of A. It can be easily seen that $A(0,0) = 0$ and $A(1,1) = 1$. Now, we will show a non-decreasing property in the first coordinate. Let $u_1, u_2 \in \mathbb{I}$ be such that $u_1 \leq u_2$.
Case 1 $v \in [0, \frac{1}{2}]$.

$$A(u_1, v) = \frac{u_1 + v}{2} \leq \frac{u_2 + v}{2} = A(u_2, v).$$

Case 2 $v \in (\frac{1}{2}, 1]$.

$$A(u_1, v) = \max\{u_1, v\} \leq \max\{u_2, v\} = A(u_2, v).$$

Next, we will show a non-decreasing property in the second coordinate.
Let $v_1, v_2 \in \mathbb{I}$ be such that $v_1 \leq v_2$. It is obvious in the case of $v_1, v_2 \in [0, \frac{1}{2}]$ and $v_1, v_2 \in (\frac{1}{2}, 1]$.
Consider in the case of $v_1 \in [0, \frac{1}{2}]$ and $v_2 \in (\frac{1}{2}, 1]$.
If $u \leq v_1 < v_2$, then

$$A(u, v_1) = \frac{u + v_1}{2} < \frac{2v_1}{2} = v_1 < v_2 = \max\{u, v_2\} = A(u, v_2).$$

Therefore, $A(u, v_1) \leq A(u, v_2)$.
If $u > v_1$, then we consider 2 cases.
Case 1 $v_1 < u \leq v_2$.

$$A(u, v_1) = \frac{u + v_1}{2} < \frac{2u}{2} = u \leq v_2 = \max\{u, v_2\} = A(u, v_2).$$

Therefore, $A(u, v_1) \leq A(u, v_2)$.
Case 2 $v_1 < v_2 < u$.

$$A(u, v_1) = \frac{u + v_1}{2} < \frac{2u}{2} = u = \max\{u, v_2\} = A(u, v_2).$$

Therefore, $A(u, v_1) \leq A(u, v_2)$. Now, we can conclude that A is an aggregation function. Moreover, F_1 and F_2 are non-decreasing. Consider $F_1(\frac{1}{3}) = A(\frac{1}{3}, 1) = \max\{\frac{1}{3}, 1\} = 1$ and $F_1(\frac{2}{3}) = A(\frac{2}{3}, 1) = \max\{\frac{2}{3}, 1\} = 1$, but $A(\frac{1}{3}, \frac{1}{2}) = \frac{1}{2}(\frac{1}{3} + \frac{1}{2}) = \frac{5}{12} \neq \frac{7}{12} = \frac{1}{2}(\frac{2}{3} + \frac{1}{2}) = A(\frac{2}{3}, \frac{1}{2})$.

In most cases, aggregation functions satisfy (3.2). Thus, they can be constructed via (3.1). Also, this construction works for strictly increasing aggregation functions as the following result.

Corollary 3.5. *Let A be a strictly increasing aggregation function. Then there are semi-copula S and strictly increasing F_j where $j = 1, 2, \ldots, k$ such that*

$$A(\vec{u}) = S(F_1(u_1), \ldots, F_k(u_k)).$$

Proof. Let $A : \mathbb{I}^k \to \mathbb{I}$ be a strictly increasing aggregation function.
Define $F_j : \mathbb{I} \to \mathbb{I}$ by $F_j(u) = A(\vec{1} - (1 - u)\vec{e_j})$ for each $j \in \{1, 2, \ldots, k\}$.
We can see that F_j are non-decreasing for all j. Next, we will show that the above statement satisfies the sufficient condition of Theorem 3.3. Let $u, v, u_j \in \mathbb{I}$ for all $j = 1, 2, \ldots, k$.
For each i, assume that $F_i(u) = F_i(v)$. We have

$$A(\vec{1} - (1 - u)\vec{e}_i) = F_i(u) = F_i(v) = A(\vec{1} - (1 - v)\vec{e}_i).$$

Since A is strictly increasing, we get A is injective, that is, $u = v$. It implies that $A(u_1, \ldots, u, \ldots, u_k) = A(u_1, \ldots, v, \ldots, u_k)$. By Theorem 3.3, there is a semi-copula S such that

$$A(\vec{u}) = S(F_1(u_1), \ldots, F_k(u_k)).$$ ■

Observe that a strictly increasing property is stronger than (3.2). However, we can show that the class of strictly increasing aggrgation functions that represented in the form (3.1) is almost all aggregation functions via approximation method.

Theorem 3.6. *For any an aggregation function A, there is a sequence $\{A_n\}$ of (strictly) aggregation functions that can be written in the form (3.1) and that the sequence $\{A_n\}$ converges uniformly to A.*

Proof. Let A be an aggregation function. For each $n \in \mathbb{N}$, define

$$A_n = (1 - \frac{1}{n})A + \frac{1}{n}M.$$

For each n, let $F_{i,n}(u) = A_n(\vec{1} - (1 - u)\vec{e}_i)$ for all i. We can see that A_n is strictly increasing. It implies that $F_{i,n}$ are also strictly increasing for all i. By Corollary 3.5, we obtain that for each n, there is a strictly semi-copula S_n such that

$$A_n = S_n(F_{1,n}, \ldots, F_{k,n}).$$

Next, we prove that A_n converges uniformly to an aggregation function A.
Let $u_j \in \mathbb{I}$ for all $j = 1, 2, \ldots, k$.

$$\begin{aligned}|A_n(\vec{u}) - A(\vec{u})| &= |(1-\frac{1}{n})A(\vec{u}) + \frac{1}{n}M(\vec{u}) - A(\vec{u})| \\ &= |-\frac{1}{n}A(\vec{u}) + \frac{1}{n}M(\vec{u})| \\ &\leq \frac{1}{n}(|A(\vec{u})| + |M(\vec{u})|) \\ &\leq \frac{2}{n} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore, $\{A_n\}$ converges uniformly to an aggregation function A. ■

4. Proof of Theorem 3.3

To prove Theorem 3.3, we study some following results. Recall the standard conventions, $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

Proposition 4.1. *Let $X \subseteq \mathbb{I}$ be closed, $f : X \to \mathbb{I}$ be non-decreasing, let $u, v \in \mathbb{I}$ with $u \leq v$. Then*

(i) $u_X^- \leq u \leq u_X^+$ for all $u \in \mathbb{I}$,
(ii) If $u \in \mathbb{I} \backslash X$, then $u_X^-, u_X^+ \in X$,
(iii) If $u \in X, v \in \mathbb{I} \backslash X$, then $f(u) \leq f(v_X^-)$,
(iv) If $u \in \mathbb{I} \backslash X, v \in X$, then $f(u_X^+) \leq f(v)$

where $u_X^- := \sup\{t \in X : t < u\}$, $u_X^+ := \inf\{t \in X : t > u\}$.

Proof. (i) Let $u \in \mathbb{I}$. If $u \in X$, then $u_X^- = u = u_X^+$.
Suppose that $u \in \mathbb{I} \backslash X$. If there are $a_1, a_2 \in X$ such that $a_1 \leq u \leq a_2$, we have $u_X^- \leq u \leq u_X^+$. If $u \leq a$ for all $a \in X$, then $u_X^- = \sup \emptyset = 0 \leq u$ and $u \leq \inf X = u_X^+$.
In the case of $u \geq a$ for all $a \in X$, we have $u_X^- = \sup X \leq u$ and $u \leq 1 = \inf \emptyset = u_X^+$.
(ii) Let $u \in \mathbb{I} \backslash X$. The proof is similar to (i).
(iii) Let $u \in X, v \in \mathbb{I} \backslash X$. Also, we get $u < v$. Then we have $u \leq v_X^-$. Since f is non-decreasing and $v_X^- \in X$, we obtain that $f(u) \leq f(v_X^-)$.
(iv) Let $u \in \mathbb{I} \backslash X, v \in X$. Also, we get $u < v$ and hence $u_X^+ \leq v$. Since f is non-decreasing and $u_X^+ \in X$, we get $f(u_X^+) \leq f(v)$. ■

Proposition 4.2. *Let $X \subseteq \mathbb{I}$. Define $X^- = \{u \in \mathbb{I} : u_X^- = u\}$ and $X^+ = \{u \in \mathbb{I} : u_X^+ = u\}$. Then the closure of X is a subset of $X^* = X^- \cup X^+$.*

Proof. Denote the closure of X by $\bar{X}$. Let $a \in \bar{X}$. Then there is a sequence $\{a_n\} \subseteq X$ which converges to a. If $a \in X$, then $a = a_X^- = a_X^+$. Thus, $a \in \bar{X} \subseteq X^*$.
Consider in the case of $a \in \mathbb{I} \backslash X$, we have $a_n \neq a$ for all $n \in \mathbb{N}$.
Case 1 $a_n > a$ for all $n \in \mathbb{N}$.
There is a monotone subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} > a$ for all $k \in \mathbb{N}$.
Moreover, we have $a_{n_k} \to a$ as $k \to \infty$. Thus, $a = \inf_{k \in \mathbb{N}} a_{n_k}$.
Since $\{a_{n_k} \in X : k \in \mathbb{N}\} \subseteq \{t \in X : t > a\}$, we get $a = \inf_{k \in \mathbb{N}} a_{n_k} \geq \inf\{t \in X : t > a\}$.
We know that $a \leq \inf\{t \in X : t > a\}$. Then we have $a = \inf\{t \in X : t > a\} = a_X^-$, that is, $\bar{X} \subseteq X^- \subseteq X^*$.

Case 2 $a_n < a$ for all $n \in \mathbb{N}$. The proof of this case is similar to Case 1.
Case 3 There is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that for each $k \in \mathbb{N}$, either $a_{n_k} > a$ or $a_{n_k} < a$. There is a subsequence $\{a_{n_{k_l}}\}$ of $\{a_n\}$ such that either $a_{n_{k_l}} > a$ or $a_{n_{k_l}} < a$ for all $l \in \mathbb{N}$. By the proof of Case 1 and Case 2, we have $\bar{X} \subseteq X^*$. ■

Observe that if we have a non-decreasing function F on $\mathbb{I}$, we can define $F^{\downarrow}(u) = \inf\{x : F(x) \geq u\}$ and $F^{\uparrow}(u) = \sup\{x : F(x) \leq u\}$. It is easy to prove that $F^{\downarrow}$ and $F^{\uparrow}$ are non-decreasing. Moreover, $F^* := \frac{1}{2}F^{\downarrow} + \frac{1}{2}F^{\uparrow}$ is also non-decreasing.

Lemma 4.3. *Let $F : \mathbb{I} \to \mathbb{I}$ be non-decreasing. Then $FF^*(u) = u$ for all $u \in Range(F)$. Moreover, $FF^*F(u) = F(u)$ for all $u \in \mathbb{I}$.*

Proof. Suppose F is non-decreasing. Let $u \in Range(F)$. Then there is $u' \in \mathbb{I}$ such that $F(u') = u$.
Case 1 u' is the only one point such that $F(u') = u$. Then $F^{\uparrow}(u) = F^{\downarrow}(u) = u'$.
We have $F^*(u) = \frac{1}{2}F^{\downarrow}(u) + \frac{1}{2}F^{\uparrow}(u) = F^{\downarrow}(u) = u'$. Hence, $FF^*(u) = F(u') = u$.
Case 2 u' is not only one point such that $F(u') = u$.
Let $X = \{z \in \mathbb{I} : F(z) = u\}$. We can see that X is a bounded set. We set $l = \inf X$ and $m = \sup X$. Thus, $F^{\uparrow}(u) = m$. Similarly, $F^{\downarrow}(u) = l$. In this case, $F^{\downarrow}(u) = l < m = F^{\uparrow}(u)$. We have

$$F^*(u) = \frac{1}{2}F^{\downarrow}(u) + \frac{1}{2}F^{\uparrow}(u) = \frac{1}{2}l + \frac{1}{2}m > \frac{1}{2}l + \frac{1}{2}l = l.$$

Similarly, $F^*(u) < m$. Hence, there are $z_1, z_2 \in \mathbb{I}$ such that $F(z_1) = u = F(z_2)$ and $l < z_1 \leq F^*(u) \leq z_2 < m$. Therefore, $FF^*(u) = F(z_1) = F(z_2) = u$. Moreover, $FF^*F(u) = F(u)$ for all $u \in \mathbb{I}$. ■

Theorem 4.4. *Let A be an aggregation function, S be a semi-copula, F_i be a monotone function for all $i = 1, 2, \ldots, k$ such that*

$$A(\vec{u}) = A(u_1, \ldots, u_k) = S(F_1(u_1), \ldots, F_k(u_k))$$

where $u_i \in \mathbb{I}$ for all i. Then the following conditions are satisfied:

(i) $F_i(1) = 1$ *for all* $i = 1, 2, \ldots, k$,
(ii) *For each* $i \in \{1, 2, \ldots, k\}$, $F_i(u) = A(\vec{1} - (1-u)\vec{e_i})$ *for all* u,
(iii) $S(\vec{u}) = A(F_1^*(u_1), F_2^*(u_2), \ldots, F_k^*(u_k))$ *for all* $u_i \in Range(F_i)$,
(iv) $A(F_1^*(1), F_2^*(1), \ldots, F_k^*(1)) = 1$,
(v) *For fixed* i, $A(\sum_{j=1}^k F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e_i}) = u_i$ *for all* $u_i \in Range(F_i)$.

Proof. (i) By assumption, $1 = A(\vec{1}) = S(F_1(1), F_2(1), \ldots, F_k(1))$.
Since S is non-decreasing, for each i, $1 \leq S(\vec{1} - (1 - F_i(1))\vec{e_i}) = F_i(1)$.
Thus, $F_i(1) = 1$ for each $i \in \{1, 2, \ldots, k\}$.
(ii) For each i, we have $A(\vec{1} - (1-u)\vec{e_i}) = S(\vec{1} - (1 - F_i(u))\vec{e_i}) = F_i(u)$.
(iii) Let $u_i \in Range(F_i)$ for all $i = 1, 2, \ldots, k$. By Lemma 4.3, we have $F_iF_i^*(u_i) = u_i$. Thus, $A(F_1^*(u_1), F_2^*(u_2), \ldots, F_k^*(u_k)) = S(F_1F_1^*(u_1), F_2F_2^*(u_2), \ldots, F_kF_k^*(u_k)) = S(\vec{u})$.
(iv) From (iii) holds and S is a semi-copula, we get $A(F_1^*(1), F_2^*(1), \ldots, F_k^*(1)) = S(\vec{1}) = 1$.
(v) For fixed i, let $u_i \in Range(F_i)$. From (iii) holds and S is a semi-copula, we get $u_i = S(\vec{1} - (1-u_i)\vec{e_i}) = A(\sum_{j=1}^k F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e_i})$. ■

Lemma 4.5. *Let A be an aggregation function, let $u \in \mathbb{I}$. For each $j \in \{1, 2, \ldots, k\}$, define $F_j(u) = A(\vec{1} - (1-u)\vec{e_j})$. Assume that*

$$A(\sum_{j=1}^{k} F_j^*(1)\vec{e_j} + (u - F_i^*(1))\vec{e}_i) = F_i(u).$$

Let $u_i \in Range(F_i)$ for all i. Then

$$A(\sum_{j=1}^{k} F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e}_i) = u_i \text{ for all } i.$$

Proof. Let $u_i \in Range(F_i)$ for fixed $i \in \{1, 2, \ldots, k\}$.
By assumption, we have $A(\sum_{j=1}^{k} F_j^*(1)\vec{e_j} + (u - F_i^*(1))\vec{e}_i) = F_i(u)$ for all u.
By Lemma 4.3, we get $A(\sum_{j=1}^{k} F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e}_i) = F_i(F_i^*(u_i)) = u_i$. ■

Lemma 4.6. *Let A be an aggregation function, let $u \in \mathbb{I}$. For each $j \in \{1, 2, \ldots, k\}$, assume that $F_j(u) = A(\vec{1} - (1-u)\vec{e_j})$ for all j. For fixed $i \in \{1, 2, \ldots, k\}$ and $u_i \in Range(F_i)$, suppose that*

$$A(\sum_{j=1}^{k} F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e}_i) = u_i.$$

Define

$$S(\vec{u}) = S(u_1, \ldots, u_k) = A(F_1^*(u_1), \ldots, F_k^*(u_k))$$

for all $u_j \in Range(F_j)$. Then S is non-decreasing with $S(u_1, \ldots, u_k) = u_i$ for all $u_i \in Range(F_i)$ where $u_j = 1$ for all $j \neq i$.

Proof. Define $S : \prod_{j=1}^{k} Range(F_j) \to \mathbb{I}$ by

$$S(\vec{u}) = A(F_1^*(u_1), \ldots, F_k^*(u_k)) \text{ for all } u_j \in Range(F_j).$$

We will show that S is non-decreasing with $S(\vec{u}) = u_i$ for all $u_i \in Range(F_i)$ where $u_j = 1$ for all $j \neq i$. Let $u_i, v_i \in Range(F_i)$ be such that $u_i \leq v_i$ for all i. It implies that

$$S(\vec{u}) = A(F_1^*(u_1), \ldots, F_k^*(u_k)) \leq A(F_1^*(v_1), \ldots, F_k^*(v_k)) = S(\vec{v}).$$

Therefore, S is non-decreasing. Moreover, for $u_i \in Range(F_i)$,

$$S(\vec{1} - (1-u_i)\vec{e}_i) = A(F_1^*(1), \ldots, F_i^*(u_i), \ldots, F_k^*(1)) = u_i.$$ ■

Henceforth, for each $u \in \mathbb{I}$, we put $S(\vec{u}+(u-u_i)\vec{e}_i)$ instead of $S(u_1, ..., u_{i-1}, u, u_{i+1}, ..., u_k)$. For now, Lemmas 4.7-4.10 are the tools that help us to prove Theorem 4.11 which shows that a non-decreasing function S can be extended to a semi-copula under some conditions.

Lemma 4.7. *Let $S : \prod_{j=1}^{k} X_j \to \mathbb{I}$ be non-decreasing where $X_j \subseteq \mathbb{I}$ for all $j = 1, 2, \ldots, k$. For each j, define $\hat{X}_j = \{u \in \mathbb{I} : u_{X_j}^- = u\}$. For fixed $i \in \{1, 2, \ldots, k\}$, define*

$$\hat{S}(\vec{u}) = \begin{cases} S(\vec{u}) & \text{if } \vec{u} \in \prod_{j=1}^{k} X_j, \\ \sup_{t<u_i} S(\vec{u} + (t-u_i)\vec{e}_i) & \text{if } \vec{u} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j \neq i}}^{k} X_j. \end{cases}$$

Then $\hat{S} : \hat{X}_i \times \prod_{\substack{j=1 \\ j \neq i}}^{k} X_j \to \mathbb{I}$ is also non-decreasing. Moreover, if $S(\vec{1} - (1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$, then $\hat{S}(\vec{1} - (1-u_i)\vec{e}_i) = u_i$ for all $u_i \in \hat{X}_i$ also.

Proof. Let $\vec{u}, \vec{v} \in \hat{X}_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$ with $\vec{u} \leq \vec{v}$. It can be shown that $X_j \subseteq \hat{X}_j$ for all j. To prove $\hat{S}$ is non-decreasing on its domain, we can divide the situation into four cases.
Case 1 $\vec{u}, \vec{v} \in \prod_{j=1}^{k} X_j$. It is obvious.
Case 2 $\vec{u} \in \prod_{j=1}^{k} X_j$ and $\vec{v} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$. Then

$$\hat{S}(\vec{u}) = S(\vec{u}) \leq \sup_{u_i \leq t < v_i} S(\vec{u} + (t - u_i)\vec{e}_i) \leq \sup_{t<v_i} S(\vec{v} + (t - v_i)\vec{e}_i) = \hat{S}(\vec{v}).$$

Case 3 $\vec{u} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$ and $\vec{v} \in \prod_{j=1}^{k} X_j$. We have

$$\hat{S}(\vec{u}) = \sup_{t<u} S(\vec{u} + (t - u_i)\vec{e}_i) \leq S(\vec{u} + (v_i - u_i)\vec{e}_i) \leq \hat{S}(\vec{v}).$$

Case 4 $\vec{u}, \vec{v} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$. We have

$$\hat{S}(\vec{u}) = \sup_{t<u_i} S(\vec{u}+(t-u_i)\vec{e}_i) \leq \sup_{t<u_i} S(\vec{v}+(t-v_i)\vec{e}_i) \leq \sup_{t<v_i} S(\vec{v}+(t-v_i)\vec{e}_i) = \hat{S}(\vec{v}).$$

Now, S is non-decreasing.
Suppose $S(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$. ($1 \in X_j$ for all $j \neq i$)
We will show that $\hat{S}(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in \hat{X}_i$. Let $u \in \hat{X}_i$.
If $u_i \in X_i$, then we are done. Consider in case $u_i \in \hat{X}_i \backslash X_i$. Thus, we obtain that $\sup\{t \in X_i : t < u_i\} = u^-_{X_i} = u_i$. It implies that

$$\hat{S}(\vec{1} - (1 - u_i)\vec{e}_i) = \sup_{t<u_i} S(\vec{1} - (1 - u_i)\vec{e}_i) = \sup_{t<u_i} t = u^-_{X_i} = u_i. \qquad \blacksquare$$

From Lemma 4.7, we can extend X_j to $\hat{X}_j$ for all $j = 1, 2, \ldots, k$ by the same method.

Lemma 4.8. *Let $S : \prod_{j=1}^{k} X_j \to \mathbb{I}$ be non-decreasing where $X_j \subseteq \mathbb{I}$ for all $j = 1, 2, \ldots, k$. For each j, define $\hat{X}_j = \{u \in \mathbb{I} : u^+_{X_j} = u\}$. For fixed $i \in \{1, 2, \ldots, k\}$, define*

$$\hat{S}(\vec{u}) = \begin{cases} S(\vec{u}) & \text{if } \vec{u} \in \prod_{j=1}^{k} X_j, \\ \inf_{t>u_i} S(\vec{u} + (t - u_i)\vec{e}_i) & \text{if } \vec{u} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j. \end{cases}$$

Then $\hat{S} : \hat{X}_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j \to \mathbb{I}$ is also non-decreasing. If $S(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$, then $\hat{S}(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in \hat{X}_i$ also.

Proof. Let $\vec{u}, \vec{v} \in \hat{X}_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$ with $\vec{u} \leq \vec{v}$. It can be shown that $X_j \subseteq \hat{X}_j$ for all j. To prove $\hat{S}$ is non-decreasing on its domain, we can divide the situation into four cases.
Case 1 $\vec{u}, \vec{v} \in \prod_{j=1}^{k} X_j$. It is obvious.
Case 2 $\vec{u} \in \prod_{j=1}^{k} X_j$ and $\vec{v} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$. Then

$$\hat{S}(\vec{u}) = S(\vec{u}) \leq \inf_{t\geq u_i} S(\vec{u}+(t-u_i)\vec{e}_i) \leq \inf_{t>v_i} S(\vec{u}+(t-u_i)\vec{e}_i) = \hat{S}(\vec{u}+(v-u_i)\vec{e}_i) \leq \hat{S}(\vec{v}).$$

Case 3 $\vec{u} \in \hat{X}_i \backslash X_i \times \prod_{\substack{j=1 \\ j\neq i}}^{k} X_j$ and $\vec{v} \in \prod_{j=1}^{k} X_j$. We have

$$\hat{S}(\vec{u}) = \inf_{t>u_i} S(\vec{u} + (t - u_i)\vec{e}_i) \leq S(\vec{u} + (v_i - u_i)\vec{e}_i) \leq S(\vec{v}) = \hat{S}(\vec{v}).$$

Case 4 $\vec{u},\vec{v} \in \hat{X}_i\backslash X_i \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j$. We have

$$\hat{S}(\vec{u}) = \inf_{t>u_i} S(\vec{u}+(t-u_i)\vec{e}_i) \le \inf_{t>v_i} S(\vec{u}+(v_i-u_i)\vec{e}_i) \le \inf_{t>v_i} S(\vec{v}+(t-v_i)\vec{e}_i) = \hat{S}(\vec{v}).$$

Similarly to Lemma 4.7, we can prove that $\hat{S}(\vec{1}-(1-u_i)\vec{e}_i) = u_i$. ∎

Lemma 4.9. *Let $S : \prod_{j=1}^{k} X_j \to \mathbb{I}$ be non-decreasing where $X_j \subseteq \mathbb{I}$ for all $j = 1,2,\ldots,k$. Then S can be extended to a non-decreasing function $\hat{S}$ on $\prod_{j=1}^{k} \bar{X}_j$. Moreover, if $S(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$, then $\hat{S}(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in \bar{X}_i$ also.*

Proof. Let $S : \prod_{j=1}^{k} X_j \to \mathbb{I}$ be non-decreasing where $X_j \subseteq \mathbb{I}$ for all $j = 1,2,\ldots,k$.
Fixed $i \in \{1,2,\ldots,k\}$, suppose $S(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$.
By Lemma 4.7, we can extend S to S' on $\prod_{j=1}^{k} X'_j$ where $X'_j = \{u \in \mathbb{I} : u^-_{X_j} = u\}$.
Moreover, $S'(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$.
By Lemma 4.8, S' can be extended to a non-decreasing function S'' on $\prod_{j=1}^{k} X''_j$ where $X''_j = \{u \in \mathbb{I} : u^+_{X'_j} = u\}$. Furthermore, $S''(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$.
By Proposition 4.2, we obtain that $\bar{X}_j \subseteq X''_j$ for all j.
Thus, we can restrict the domain of S'' into $\prod_{j=1}^{k} \bar{X}_j$. Hence, we get a non-decreasing function $\hat{S} : \prod_{j=1}^{k} \bar{X}_j \to \mathbb{I}$. In addition, $\hat{S}(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in \bar{X}_i$. ∎

Lemma 4.10. *Let X_j be a closed subset of $\mathbb{I}$ for all $j = 1,2,\ldots,k$. Let $S : \prod_{j=1}^{k} X_j \to \mathbb{I}$ be non-decreasing. For fixed $i \in \{1,2,\ldots,k\}$, define*

$$\hat{S}(\vec{u}) = \begin{cases} S(\vec{u}) & \text{if } \vec{u} \in \prod_{j=1}^{k} X_j, \\ \frac{1}{u^+_{X_i}-u^-_{X_i}}\Big[(u^+_{X_i}-u_i)S(\vec{u}+(u^-_{X_i}-u_i)\vec{e}_i) & \\ +(u_i-u^-_{X_i})S(\vec{u}+(u^+_{X_i}-u_i)\vec{e}_i)\Big] & \text{if } \vec{u} \in \mathbb{I}\backslash X_i \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j \end{cases}$$

*where $u^-_{X_i} = \sup\{t \in X_i : t < u_i\}$ and $u^+_{X_i} = \inf\{t \in X_i : t > u_i\}$.
Then $\hat{S} : \mathbb{I} \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j \to \mathbb{I}$ is also non-decreasing. Moreover, if $S(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$, then $\hat{S}(\vec{1}-(1-u_i)\vec{e}_i) = u_i$ for all $u_i \in \mathbb{I}$ also.*

Proof. Let $\vec{u},\vec{v} \in \mathbb{I} \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j$ with $\vec{u} \le \vec{v}$. To prove $\hat{S}$ is non-decreasing on its domain, we divide the situation into four cases.
Case 1 $\vec{u},\vec{v} \in \prod_{j=1}^{k} X_j$. It is obvious.
Case 2 $\vec{u} \in \prod_{j=1}^{k} X_j$ and $\vec{v} \in \mathbb{I}\backslash X_i \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j$. Then

$$\begin{aligned}\hat{S}(\vec{u}) &= S(\vec{u}) \\ &\le S(\vec{u}+(v^-_{X_i}-u_i)\vec{e}_i) \\ &\le S(\vec{v}+(v^-_{X_i}-v_i)\vec{e}_i) \\ &\le \left(\frac{v^+_{X_i}-v}{v^+_{X_i}-v^-_{X_i}}\right) S(\vec{u}+(v^-_{X_i}-v_i)\vec{e}_i) + \left(\frac{v-v^-_{X_i}}{v^+_{X_i}-v^-_{X_i}}\right) S(\vec{u}+(v^+_{X_i}-v_i)\vec{e}_i) = \hat{S}(\vec{v}).\end{aligned}$$

Case 3 $\vec{u} \in \mathbb{I}\backslash X_i \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j$ and $\vec{v} \in \prod_{j=1}^{k} X_j$. Then

$$\begin{aligned}
\hat{S}(\vec{u}) &= \left(\frac{u^{+}_{X_i} - u_i}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{-}_{X_i} - u_i)\vec{e}_i) + \left(\frac{u_i - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&\leq S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&\leq S(\vec{u} + (v_i - u_i)\vec{e}_i)\\
&\leq \hat{S}(\vec{v}).
\end{aligned}$$

Case 4 $\vec{u}, \vec{v} \in \mathbb{I}\backslash X_i \times \prod_{\substack{j=1\\ j\neq i}}^{k} X_j$.

Case 4.1 There is $a \in X_i$ such that $u_i \leq a \leq v_i$. Thus, $u^{+}_{X_i} \leq v^{-}_{X_i}$. It implies that

$$\begin{aligned}
\hat{S}(\vec{u}) &= \left(\frac{u^{+}_{X_i} - u_i}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{-}_{X_i} - u_i)\vec{e}_i) + \left(\frac{u_i - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&\leq \left(\frac{u^{+}_{X_i} - u_i}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i) + \left(\frac{u_i - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&= \left(\frac{u^{+}_{X_i} - u_i + u_i - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&= \left(\frac{u^{+}_{X_i} - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&= S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&\leq S(\vec{u} + (v^{-}_{X_i} - u_i)\vec{e}_i)\\
&\leq \left(\frac{v^{+}_{X_i} - v}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{u} + (v^{-}_{X_i} - u_i)\vec{e}_i) + \left(\frac{v - v^{-}_{X_i}}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{u} + (v^{+}_{X_i} - u_i)\vec{e}_i)\\
&\leq \hat{S}(\vec{v}).
\end{aligned}$$

Case 4.2 For each $a \in X_i$, either $a \leq u_i \leq v_i$ or $u_i \leq v_i \leq a$. Thus, $u^{-}_{X_i} = v^{-}_{X_i}$ and $u^{+}_{X_i} = v^{+}_{X_i}$. For convenience, for each $u \in X_i$, we put $S(\vec{u} + (u - u_i)\vec{e}_i)$ instead of $S(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_k)$.
Then

$$\begin{aligned}
\hat{S}(\vec{v}) - \hat{S}(\vec{u}) &= \left(\frac{v^{+}_{X_i} - v_i}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{v} + (v^{-}_{X_i} - v_i)\vec{e}_i) + \left(\frac{v_i - v^{-}_{X_i}}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{v} + (v^{+}_{X_i} - v_i)\vec{e}_i)\\
&\quad - \left(\frac{u^{+}_{X_i} - u_i}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{-}_{X_i} - u_i)\vec{e}_i) - \left(\frac{u_i - u^{-}_{X_i}}{u^{+}_{X_i} - u^{-}_{X_i}}\right) S(\vec{u} + (u^{+}_{X_i} - u_i)\vec{e}_i)\\
&= \left(\frac{v^{+}_{X_i} - v_i}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{v} + (v^{-}_{X_i} - v_i)\vec{e}_i) + \left(\frac{v_i - v^{-}_{X_i}}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{v} + (v^{+}_{X_i} - v_i)\vec{e}_i)\\
&\quad - \left(\frac{v^{+}_{X_i} - u_i}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{u} + (v^{-}_{X_i} - u_i)\vec{e}_i) - \left(\frac{u_i - v^{-}_{X_i}}{v^{+}_{X_i} - v^{-}_{X_i}}\right) S(\vec{u} + (v^{+}_{X_i} - u_i)\vec{e}_i)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{v^+_{X_i} - v_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) - \left(\frac{v^+_{X_i} - u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{u} + (v^-_{X_i} - u_i)\vec{e}_i) \\
&+ \left(\frac{v_i - v^-_{X_i}}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) - \left(\frac{u_i - v^-_{X_i}}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{u} + (v^+_{X_i} - u_i)\vec{e}_i) \\
&\geq \left(\frac{v^+_{X_i} - v_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) - \left(\frac{v^+_{X_i} - u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) \\
&+ \left(\frac{v_i - v^-_{X_i}}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) - \left(\frac{u_i - v^-_{X_i}}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) \\
&= \left(\frac{v^+_{X_i} - v_i - v^+_{X_i} + u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) \\
&+ \left(\frac{v_i - v^-_{X_i} - u_i + v^-_{X_i}}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) \\
&= \left(\frac{u_i - v_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) + \left(\frac{v_i - u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) \\
&= \left(\frac{v_i - u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) - \left(\frac{v_i - u_i}{v^+_{X_i} - v^-_{X_i}}\right) S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i) \\
&= \left(\frac{v_i - u_i}{v^+_{X_i} - v^-_{X_i}}\right) \left[S(\vec{v} + (v^+_{X_i} - v_i)\vec{e}_i) - S(\vec{v} + (v^-_{X_i} - v_i)\vec{e}_i)\right] \\
&\geq 0.
\end{aligned}
$$

Therefore, $\hat{S}$ is non-decreasing. Suppose $S(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in X_i$. We will prove that $\hat{S}(\vec{1} - (1 - u_i)\vec{e}_i) = u_i$ for all $u_i \in \mathbb{I}$. Let $u_i \in \mathbb{I}$. If $u_i \in X_i$, then we are done. In the case of $u_i \in \mathbb{I} \backslash X_i$, consider

$$
\begin{aligned}
\hat{S}(\vec{1} - (1 - u_i)\vec{e}_i) &= \left(\frac{u^+_{X_i} - u_i}{u^+_{X_i} - u^-_{X_i}}\right) S(\vec{1} - (1 - u^-_{X_i})\vec{e}_i) \\
&+ \left(\frac{u_i - u^-_{X_i}}{u^+_{X_i} - u^-_{X_i}}\right) S(\vec{1} - (1 - u^+_{X_i})\vec{e}_i) \\
&= \left(\frac{u^+_{X_i} - u_i}{u^+_{X_i} - u^-_{X_i}}\right) u^-_{X_i} + \left(\frac{u_i - u^-_{X_i}}{u^+_{X_i} - u^-_{X_i}}\right) u^+_{X_i} \\
&= \left(\frac{u^+_{X_i} u^-_{X_i} - u_i u^-_{X_i}}{u^+_{X_i} - u^-_{X_i}}\right) + \left(\frac{u_i u^+_{X_i} - u^-_{X_i} u^+_{X_i}}{u^+_{X_i} - u^-_{X_i}}\right) \\
&= \frac{u_i(u^+_{X_i} - u^-_{X_i})}{u^+_{X_i} - u^-_{X_i}} = u_i.
\end{aligned}
$$

■

From Lemma 4.10, we can extend X_j to $\mathbb{I}$ for all $j = 1, 2, \ldots, k$ by the same method.

Theorem 4.11. *Let S be non-decreasing such that $S(\vec{1}-(1-u_i)\vec{e_i}) = u_i$ where $u_i \in A_i$ and $A_i \subseteq \mathbb{I}$ for all $i = 1, 2, \ldots, k$. Then S can be extended to a semi-copula.*

Proof. Direct proof by using Lemma 4.9 and 4.10. ∎

From previous results, we are now able to show the proof of Theorem 3.3.

Proof of Theorem 3.3. Fixed $i \in \{1, 2, \ldots, k\}$. Since $1 \in Range(F_j)$ for all j, we get $F_j F_j^* = 1 = F_j(1)$ by Lemma 4.3. Let $u \in \mathbb{I}$. By assumption,

$$A(F_1^*(1), \ldots, F_{i-1}^*(1), u, F_{i+1}^*(1), \ldots, F_k^*(1)) = A(\vec{1} - (1 - F_i^*(1))\vec{e_i}) = F_i(u).$$

By Lemma 4.5, we have $A(\sum_{j=1}^k F_j^*(1)\vec{e_j} + (F_i^*(u_i) - F_i^*(1))\vec{e_i}) = u_i$ for all $u_i \in Range(F_i)$. Define $\hat{S} : \prod_{j=1}^k A_j \to \mathbb{I}$ by $\hat{S}(u_1, \ldots, u_k) = A(F_1^*(u_1), \ldots, F_k^*(u_k))$ for all $u_j \in Range(F_j)$. By Lemma 4.6, we get $\hat{S}$ is non-decreasing with $\hat{S}(\vec{1} - (1-u)\vec{e_j}) = u_j$ for all $u_j \in Range(F_j)$. By Theorem 4.11, $\hat{S}$ can be extended to a semi-copula S.
Furthermore, by Lemma 4.3, we get $F_j F_j^* F_j(u_j) = F_j(u_j)$ for all j. Therefore,

$$\begin{aligned} S(F_1(u_1), \ldots, F_k(u_k)) &= A(F_1 F_1^*(u_1), \ldots, F_k F_k^*(u_k)) \\ &= A(u_1, F_2 F_2^*(u_2), \ldots, F_k F_k^*(u_k)) \\ &= A(u_1, u_2, F_3 F_3^*(u_3), \ldots, F_k F_k^*(u_k)) \\ &\ \vdots \\ &= A(u_1, \ldots, u_k). \end{aligned}$$ ∎

5. Conclusions

In this work, we resemble Sklars construction of multivariate distribution functions by replacing copulas with semi-copulas. We find necessary and sufficient conditions of aggregation functions that can be constructed by composing between semi-copulas and tuples of non-decreasing univariate functions. Furthermore, we prove that almost all aggregation functions can be constructed by this way. This construction method can be used to construct all strictly increasing aggregation functions. Also, all aggregation functions can be approximated by aggregation functions in our form.

During the revision process, we also found that one of our results (Theorem 3.3) is similar to [Scarsini [25],Theorem 9] (see also [26]) where the latter is proved for distribution functions of supermodular fuzzy measures instead of general aggregation functions. One interesting question is whether an aggregation function satisfied (3.2) can be used to construct a supermodular fuzzy measure. Hopefully, this can be answered in the future work.

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