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A Fixed Point Theorem and Generalized Non-Archimedean Quasi-Ordered Metric Spaces and Its Applications

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Abstract In this paper, we introduce the concepts of quasi-orders which are generalization of partial orders and the concepts of a generalized non-Archimedean quasi-ordered metric space which are generalization of non-Archimedean metric spaces. And we investigate the generalized non-Archimedean quasi-ordered metric on the set of all mappings from a non-Archimedean normed space to a complete non-Archimedean normed space. Further, we prove a fixed point theorem for a contraction in d-complete metric spaces. As an application, we give the stability for some functional equations.

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1. Introduction and Preliminaries

Banachs contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Many kinds of generalizations of the above principle have been a heavily investigated branch of research. In particular, Diaz and Margolis [9] presented the following definition and fixed point theorem in a generalized complete metric space.

Definition 1.1. Let X be a nonempty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric* on X if d satisfies the following conditions:
(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.
In this case, (X, d) is called *a generalized metric space.*

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A sequence $\{x_n\}$ in a generalized metric space (X, d) is called *Cauchy* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and a generalized metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) is convergent.

Theorem 1.2. *Suppose that (X, d) is a generalized complete metric space and a function $T : X \longrightarrow X$ is a contraction, that is, T satisfies the following condition:*
(CI) There exists a constant L with $0 < L < 1$ such that, whenever $d(x, y) < \infty$,

$$d(Tx, Ty) \leq Ld(x, y). \tag{1.1}$$

Let $x_0 \in X$ and consider a sequence $\{T^n(x_0)\}$ of successive approximations with the initial element x_0. Then the following alternative holds:
either
(A) for all $n \geq 0$, one has $d(T^n(x_0), T^{n+1}(x_0)) = \infty$
or
(B) the sequence $\{T^n(x_0)\}$ is convergent to a fixed point of T in (X, d).

Further, Nieto and Rodriguez-Lopez [11] proved a fixed point theorem in partially ordered sets as follows.

Theorem 1.3. *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \longrightarrow X$ be a continuous and nondecreasing mapping such that there exists a constant $L \in [0, 1)$ with*

$$d(f(x), f(y)) \leq Ld(x, y) \tag{1.2}$$

for all $x, y \in X$. If there exists an $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Moreover, in [7], the following fixed point theorem for a partially ordered generalized complete metric space was proved.

Theorem 1.4. *Let $(X, \leq)$ be a partially ordered set. Suppose that (X, d) is a generalized complete metric space and a mapping $T : X \longrightarrow X$ is a continuous and nondecreasing mapping such that there exists a constant $L \in [0, 1)$ such that*

$$d(Tx, Ty) \leq Ld(x, y) \tag{1.3}$$

for all $x, y \in X$ with $x \geq y$. If there exists an x_0 in X with $x_0 \leq f(x_0)$, then the following alternative holds:
either
(A) for all $n \geq 0$, one has $d(T^n(x_0), T^{n+1}(x_0)) = \infty$
or
(B) the sequence $\{T^n(x_0)\}$ is convergent to a fixed point of T in (X, d).

The inequality in (D3) is called *the triangle inequality.* A metric d on X is said to be *non-Archimedean* if the triangle inequality is replaced with the following stronger inequality, called the ultrametric inequality :

(D4) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0, \infty)$ such that for any $r, s \in \mathbb{K}$, the following conditions hold: (i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, and (iii) $|r + s| \leq |r| + |s|$. A field $\mathbb{K}$ is called *a valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$, then the valuation $|\cdot|$ is called *a* *non-Archimedean valuation* and the field with a non-Archimedean valuation is called *a non-Archimedean field.* If $|\cdot|$ is a non-Archimedean valuation on $\mathbb{K}$, then clearly, $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called *a non-Archimedean norm* if it satisfies the following conditions:
(a) $\|x\| = 0$ if and only if $x = 0$,
(b) $\|rx\| = |r|\|x\|$, and
(c) the strong triangle inequality (ultrametric) holds, that is,

$$||x + y|| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and all $r \in \mathbb{K}$. If $\|\cdot\|$ is a non-Archimedean norm, then $(X, \|\cdot\|)$ is called *a non-Archimedean normed space.*

Let $(X, \|\cdot\|)$ be a non-Archimedean normed space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, x is called *the limit of the sequence* $\{x_n\}$ and one denotes it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ in $(X, \|\cdot\|)$. A sequence $\{x_n\}$ is said to be Cauchy in $(X, \|\cdot\|)$ if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p \in \mathbb{N}$. By the strong triangle inequality, we have

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| \mid m \leq j \leq n-1\} \quad (n > m)$$

and so the sequence $\{x_n\}$ is Cauchy in $(X, \|\cdot\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

In this paper, we introduce the concepts of quasi-orders on a set which are generalization of partial orders and the concepts of a generalized non-Archimedean quasi-ordered metric space which are generalization of non-Archimedean metric spaces. And we investigate the generalized non-Archimedean quasi-ordered metric on the set of all mappings from a non-Archimedean normed space to a complete non-Archimedean normed space. Further, we prove a fixed point theorem for a contraction in d-complete spaces. As an application, we give the stability for some functional equations.

Throughout in this paper, let $(X, \|\cdot\|)$ be a non-Archimedean normed space and $(Y, \|\cdot\|)$ a complete non-Archimedean normed space.

2. Fixed Point Theorem in d-Complete Metric Spaces

We start with the definition of quasi-orders which is a generalization of the concept of partial orders. A relation $\leq_X$ on a set X is called *a quasi-order on X* if $\leq_X$ satisfies reflexive and transitive. Let $\leq_X$ be a quasi-order on X. Then x and y are called *comparable*, denoted by $x \sim_X y$ or simply $x \sim y$, if $x \leq_X y$ or $y \leq_X x$.

Definition 2.1. Let X be a nonempty set and $\leq_X$ a quasi-order on X. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized non-Archimedean quasi-ordered metric on X* if d satisfies (D1), (D2), and the following inequality

(DO) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$

for all $x, y, z \in X$ with $x \sim_X z$ and $z \sim_X y$. In this case, $(X, d, \leq_X)$ is called *a generalized non-Archimedean quasi-ordered metric space.* And a generalized non-Archimedean quasi-ordered metric space $(X, d, \leq_X)$ is called *d-complete* if every nondecreasing Cauchy sequence in $(X, d, \leq_X)$ is convergent in (X, d).

Theorem 2.2. *Let $(X, d, \leq_X)$ be a d-complete metric space such that*
if $\{x_n\}$ is a nondecreasing sequence in $(X, \leq_X)$ and $x_n \to x$ in $(X, \|\cdot\|)$,
then $x_n \leq_X x$ for all $n \in \mathbb{N}$.
Let $T : X \longrightarrow X$ be a nondecreasing mapping such that there exists an $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y) \tag{2.1}$$

for all $x, y \in X$ with $x \sim y$. If there exist an x_0 in X with $x_0 \leq_X Tx_0$, then the following alternative holds:
either
(A) for all $n \geq 0$, one has $d(T^n x_0, T^{n+1} x_0) = \infty$
or
(B) the sequence $\{T^n x_0\}$ is convergent to a fixed point u of T in (X, d). Further, if $d(x_0, Tx_0) < \infty$, then

$$d(u, x_0) \leq Ld(x_0, Tx_0). \tag{2.2}$$

Proof. Suppose that there is an $l \in \mathbb{N} \cup \{0\}$ such that $d(T^l x_0, T^{l+1} x_o) < \infty$. Since T is nondecreasing and $x_0 \leq_X Tx_0$,

$$x_0 \leq_X Tx_0 \leq_X T^2 x_0 \leq_X \cdots \leq_X T^n x_0 \leq_X \cdots$$

and by (2.1), we have

$$d(T^n x_0, T^{n+1} x_0) \leq L^{n-l} d(T^l x_0, T^{l+1} x_0) \tag{2.3}$$

and

$$d(T^n x_0, T^{n+1} x_0) < \infty$$

for all positive integers n with $l \leq n$. Hence for $m > n \geq l$, by (DO) and (2.3), we have

$$\begin{aligned} &d(T^m x_0, T^n x_0) \\ &\leq \max\{d(T^i x_0, T^{i+1} x_0) \mid n \leq i \leq m-1\} \leq L^{n-l} d(T^l x_0, T^{l+1} x_0) \end{aligned} \tag{2.4}$$

and so the sequence $\{T^n x_0\}$ is a nondecreasing Cauchy sequence in $(X, d, \leq_X)$. Since $(X, d, \leq_X)$ is d-complete, there is a $u \in X$ such that $T^n x_o \to u$ in $(X, d, \leq_X)$.

Now, we will show that u is the fixed point of T. Let $\epsilon > 0$ be given. Since $T^n(x_o) \to u$, there is a $k \in \mathbb{N}$ such that

$$k \leq n \ \Rightarrow d(T^n x_0, u) \leq \epsilon$$

and since $T^n x_0 \leq_X u$, $T^{n+1} x_0 \leq_X Tu$ for all $n \in \mathbb{N}$. By (DO), we have

$$\begin{aligned} d(Tu, u) &\leq \max\{d(Tu, T^{k+1} x_0), d(T^{k+1} x_0, u)\} \\ &\leq \max\{Ld(u, T^k x_0), d(T^{k+1} x_0, u)\} \leq \epsilon \end{aligned}$$

and so $Tu = u$.

Moreover, if $d(x_0, Tx_0) < \infty$, then, by (2.4), we have (2.2). ∎

Define a relation $\leq_X$ on $(X, \|\cdot\|)$ by

$$x \leq_X y \ \ if \ \ \|x\| \leq \|y\|.$$

We can easily show the following proposition.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a non-Archimedean normed space. Then we have the following :*

(1) $\leq_X$ is a quasi-order on X.

(2) If $\{x_n\}$ is a nondecreasing sequence in $(X, \leq_X)$ and $x_n \to x$ in $(X, \|\cdot\|)$, then $x_n \leq_X x$ for all $n \in \mathbb{N}$.

Proof. (1) is trivial.

(2) Suppose that there is an $m \in \mathbb{N}$ such that $\epsilon = \|x_m\| - \|x\| > 0$. Since $x_n \to x$ in $(X, \|\cdot\|)$, there is a $k \in \mathbb{N}$ such that $m < k$ and $\|x_k - x\| < \epsilon$ and so $\|x_k\| < \|x_m\|$. Since $\{x_n\}$ is a nondecreasing sequence in $(X, \leq_X)$, $x_m \leq_X x_k$ which is a contradiction. ■

Let $S = \{g \mid g : X \longrightarrow Y\}$. Define a quasi-order $\leq_s$ on S by

$$g \leq_s h$$

if and only if

$$g(x) \leq_Y h(x), \ \forall x \in X.$$

Let $\phi : X^2 \longrightarrow [0, \infty)$ be a mapping. For any $g, l, h \in S$, let

$$m(g, l, h)(x) = \max\{\|g(x) - l(x)\|, \|l(x) - h(x)\|\}.$$

Define a mapping $d : S^2 \longrightarrow [0, \infty]$ by

$$d(g, h) = \inf\{c \in \mathbb{R}^+ \mid \textit{there is an } l \in S \textit{ such that } g \sim l,\ l \sim h, \textit{and} \\ m(g, l, h)(x) \leq c\phi(x, 0),\ \forall x \in X\}.$$

Lemma 2.4. *Let $g, h \in S$ with $g \sim h$. Then*

$$d(g, h) = \inf\{c \in \mathbb{R}^+ \mid \|g(x) - h(x)\| \leq c\phi(x, 0),\ \forall x \in X\}.$$

Proof. Let $c \in \mathbb{R}^+$ such that

$$\|g(x) - h(x)\| \leq c\phi(x, 0)$$

for all $x \in X$. Then $g \sim h$, $h \sim h$, and

$$\|g(x) - h(x)\| = \max\{\|g(x) - h(x)\|, \|h(x) - h(x)\|\} = m(g, h, h)(x) \leq c\phi(x, 0),$$

for all $x \in X$. Hence we have $d(g, h) \leq c$. For the converse, suppose that $r \in \mathbb{R}^+$ and $l \in S$ such that $g \sim l$, $l \sim h$, and

$$m(g, l, h)(x) \leq r\phi(x, 0)$$

for all $x \in X$. Since $\|g(x) - h(x)\| \leq m(g, l, h)(x) \leq r\phi(x, 0)$ for all $x \in X$,

$$\inf\{c \in \mathbb{R}^+ \mid \|g(x) - h(x)\| \leq c\phi(x, 0),\ \forall x \in X\} \leq r.$$

Hence one has the result. ■

Using Lemma 2.4, we have the following theorem:

Theorem 2.5. *$(S, d, \leq_s)$ is a d-complete metric space.*

Proof. Suppose that $d(g, h) = 0$ and $g \neq h$. Then there exists a $z \in X$ such that $g(z) \neq h(z)$. Let $\epsilon = \|g(z) - h(z)\|$. Then $\epsilon > 0$ and there exists a $\delta > 0$ such that $\delta\phi(z, 0) < \epsilon$. Since $d(g, h) = 0$, there exists a $c \in \mathbb{R}^+$ such that $c < \delta$ and

$$m(g, l, h)(x) \leq c\phi(x, 0)$$

for all $x \in X$ and some $l \in S$ with $g \sim l$ and $l \sim h$. Then

$$\|g(z) - h(z)\| \leq m(g,l,h)(z) \leq c\phi(z,0) < \epsilon$$

which is a contradiction. Hence (D1) holds and clearly, d satisfies (D2).

Let $g, l, h \in S$ such that $g \sim l$ and $l \sim h$. Since $g \sim l$ and $l \sim h$, by Lemma 2.4, we have

$$d(g,l) = \inf\{c \in \mathbb{R}^+ \mid \|g(x) - l(x)\| \leq c\phi(x,0),\ \forall x \in X\}$$

and

$$d(l,h) = \inf\{c \in \mathbb{R}^+ \mid \|l(x) - h(x)\| \leq c\phi(x,0),\ \forall x \in X\}.$$

Let $c_1, c_2 \in \mathbb{R}$ and $l_1, l_2 \in S$ such that $g \sim l_1$, $l_1 \sim l$, $l \sim l_2$, $l_2 \sim h$, and

$$m(g,l_1,l)(x) \leq c_1\phi(x,0),\ \ m(l,l_2,h)(x) \leq c_2\phi(x,0)$$

for all $x \in X$. Then we have

$$\begin{aligned}
&m(g,l,h)(x)\\
&= \max\{\|g(x) - l(x)\|, \|l(x) - h(x)\|\}\\
&\leq \max\{\|g(x) - l_1(x)\|, \|l_1(x) - l(x)\|, \|l(x) - l_2(x)\|, \|l_2(x) - h(x)\|\}\\
&\leq \max\{c_1, c_2\}\phi(x,0)
\end{aligned}$$

for all $x \in X$ and so d satisfies (DO). Hence d is a generalized non-Archimedean quasi-ordered metric on S.

Now, we claim that $(S, d, \leq_s)$ is d-complete. Let $\{h_n\}$ be a nondecreasing Cauchy sequence in $(S, d, \leq_s)$. For any $x \in X$, $\{h_n(x)\}$ is a nondecreasing Cauchy sequence in Y and since Y is complete, there exists $h(x)$ in Y such that $\lim_{n\to\infty} h_n(x) = h(x)$. By Proposition 2.3, $h_n(x) \leq_Y h(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Hence $h_n \leq_s h$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Then there is a $k \in \mathbb{N}$ such that for $n > m \geq k$,

$$d(h_n, h_m) < \epsilon. \tag{2.5}$$

Let $n > m \geq k$. Since $h_m \leq_s h_n$,

$$d(h_n, h_m) = \inf\{c \in \mathbb{R}^+ \mid \|h_n(x) - h_m(x)\| \leq c\phi(x,0),\ \forall x \in X\}$$

and by (2.5), we have $\|h_n(x) - h_m(x)\| \leq \epsilon\phi(x,0)$ for all $x \in X$. Since the mapping $t(y) = \|y - h_m(x)\|$ is continuous on Y, $\|h(x) - h_m(x)\| \leq \epsilon\phi(x,0)$ for all $x \in X$. Hence $d(h_m, h) \leq \epsilon$ for all $m \in \mathbb{N}$ with $m \geq k$, because $h_m \leq_s h$ and thus $h_n \to h$ in $(S, d, \leq_s)$.
■

3. Applications

In this section, we will prove the stability as an application of our fixed point theorem. We start with the following lemma:

Lemma 3.1. *Define a mapping $T : S \longrightarrow S$ by $Tg(x) = pg(rx)$ for some real numbers p, r. Then T is a nondecreasing mapping.*

By Theroem 2.5 and Lemma 3.1, we have the following theorem:

Theorem 3.2. *Let a, k be natural numbers and L be a positive real number such that $L < 1$ and*

$$\phi(ax, ay) \leq |a|^k L\phi(x, y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that

$$\|a^k f(x) - f(ax)\| \leq M\phi(x, 0), \ \ \|f(x)\| \leq \left\| \frac{1}{a^k} f(ax) \right\| \tag{3.2}$$

for all $x \in X$ and some positive real number M. Then there exists a unique mapping $F : X \longrightarrow Y$ such that $\lim_{n\to\infty} \frac{1}{a^{kn}} f(a^n x) = F(x)$ for all $x \in X$ and

$$\begin{aligned} &\left\| \frac{1}{a^{kn}} f(a^n x) \right\| \leq \|F(x)\|, \ \ F(ax) = a^k F(x), \\ &\|F(x) - f(x)\| \leq \frac{M}{|a|^k} \phi(x, 0) \end{aligned} \tag{3.3}$$

for all $x \in X$ and all $n \in \mathbb{N} \cup \{0\}$.

Proof. By Theorem 2.5, $(S, d, \leq_s)$ is a d-complete metric space. Define a mapping $T : S \longrightarrow S$ by $Tf(x) = \frac{1}{a^k} f(ax)$. By Lemma 3.1, T is a nondecreasing mapping and by (3.2), $f \leq_s Tf$. Let $f, g \in S$ with $f \leq_s g$. For any $c \in \mathbb{R}^+$ with

$$\|f(x) - g(x)\| \leq c\phi(x, 0)$$

for all $x \in X$, by (3.1), we have

$$\|Tf(x) - Tg(x)\| \leq \frac{c}{|a|^k} \phi(ax, 0) \leq cL\phi(x, 0) \tag{3.4}$$

for all $x \in X$. Since $Tf \leq_s Tg$, by Lemma 2.4, $d(Tf, Tg) \leq Ld(f, g)$ and by Theorem 2.2, there exists a unique mapping $F : X \longrightarrow Y$ with (3.3). ∎

In 1940, Ulam proposed the following stability problem ([14]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [8] gave a partial solution of Ulam′s problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [13] for linear mappings, to consider the stability problem with unbounded Cauchy differences. Rassias [13] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon(\geq 0)$, $p(< 1)$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [13] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. Rassias [12] investigated the following the cubic functional equation

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x),$$

and proved the generalized Hyers-Ulam stability for it. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians([3, 4, 5, 10]). Cádariu and Radu [2] applied the fixed point method to investigate the Jensen functional equation and presented a short and simple proof (different from the direct method initiated by Hyers in 1941 for the Hyers-Ulam stability of the Jensen functional equation).

Let D_1 be a functional operator on S defined by

$$D_1f(x,y)=f(ax+by)+f(ax-by)-2af(x)+c[f(x+y)+f(x-y)-2f(x)]$$

for a fixed non-zero rational number a and real numbers b, c such that $a^2 \neq b^2$. The we can easily show the following lemma.

Lemma 3.3. *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Then f is additive if and only if $D_1f(x,y) = 0$ for all $x, y \in X$.*

Using Theorem 3.2 and Lemma 3.3, we have the following theorem :

Theorem 3.4. *Let a be a natural number with $a \geq 2$ and $\phi : X^2 \longrightarrow [0,\infty)$ be a mapping such that*

$$\phi(ax,ay) \leq |a|L\phi(x,y) \tag{3.5}$$

for all $x, y \in X$ and some $L \in (0,1)$. Suppose that $f : X \longrightarrow Y$ is a mapping such that $f(0) = 0$,

$$\|f(x)\| \leq \left\|\frac{1}{a}f(ax)\right\|,$$

for all $x \in X$ and

$$\|D_1f(x,y)\| \leq \phi(x,y) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that $\lim_{n\to\infty} \frac{1}{a^n}f(a^nx) = A(x)$ for all $x \in X$ and

$$\left\|\frac{1}{a^n}f(a^nx)\right\| \leq \|A(x)\|,\ \ \|f(x)\| \leq \|A(x)\|,\ \ \|A(x)-f(x)\| \leq \frac{1}{|2a|}\phi(x,0) \tag{3.7}$$

for all $x \in X$ and $n \in \mathbb{N}\cup\{0\}$.

Proof. Letting $y = 0$ in (3.6), we have

$$\|af(x)-f(ax)\| \leq \frac{1}{|2|}\phi(x,0)$$

for all $x \in X$. By Theorem 3.2, there is a unique mapping $A : X \longrightarrow Y$ such that

$$\left\|\frac{1}{a^n}f(a^nx)\right\| \leq \|A(x)\|,\ \ A(ax) = aA(x),\ \ \|A(x)-f(x)\| \leq \frac{1}{|2a|}\phi(x,0)$$

for all $x \in X$ and all $n \in \mathbb{N}\cup\{0\}$. By (3.5) and (3.6), we have

$$\left\|\frac{1}{a^n}D_1f(a^nx,a^ny)\right\| \leq \frac{1}{|a|^n}\phi(a^nx,a^ny) \leq L^n\phi(x,y)$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we can show that $D_1 A(x, y) = 0$ for all $x, y \in X$ and hence A is an additive mapping.

Suppose that $G : X \longrightarrow Y$ is another additive mapping with (3.7). Then, by (3.7), we have

$$\begin{aligned}\|A(x) - G(x)\| &\leq \frac{1}{|a|^n} \max\{\|A(a^n x) - f(a^n x)\|, \|A(a^n x) - f(a^n x)\|\} \\ &\leq \frac{1}{|2a|} L^n \phi(x, 0)\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Since $0 < L < 1$, $A = G$. ■

Let D_2 and D_3 be functional operators on S defined by

$$D_2 f(x, y) = f(3x + y) + f(3x - y) - 18f(x) - 2f(y)$$

and

$$D_3 f(x, y) = f(3x + y) + f(3x - y) - 3f(x + y) - 3f(x - y) - 48f(x),$$

respectively. Then, clearly, for any mapping $f : X \longrightarrow Y$ with $f(0) = 0$, f is quadratic(cubic, resp.) if and only if $D_2 f(x, y) = 0 (D_3 f(x, y) = 0$, resp.) for all $x, y \in X$.

Similar to Theorem 3.4, we have the following remark:

Remark 3.5. Let $k = 2$ or $k = 3$ and $\phi : X^2 \longrightarrow [0, \infty)$ such that

$$\phi(3x, 3y) \leq |3|^k L\phi(x, y)$$

for all $x, y \in X$ and some $L \in (0, 1)$. Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(0) = 0$, $\|f(x)\| \leq \left\|\frac{1}{3^k} f(3x)\right\|$ for all $x \in X$, and

$$\|D_k f(x, y)\| \leq \phi(x, y) \tag{3.8}$$

for all $x, y \in X$. Further, suppose that (3.8) implies that

$$\|3^{kn} f(x) - f(3^n x)\| \leq M\phi(x, 0)$$

for all $x \in X$ and some positive real number M. Then there exists a unique mapping $F : X \longrightarrow Y$ such that $D_k F(x, y) = 0$ for all $x, y \in X$ and

$$\left\|\frac{1}{3^{kn}} f(3^n x)\right\| \leq \|F(x)\|,\ f \leq_s F,\ \|F(x) - f(x)\| \leq \frac{M}{|3|^k} \phi(x, 0)$$

for all $x \in X$.

Author Contributions

All authors contributed equally to the writing of this paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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