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# Kannan Contraction Maps on the Space of Null Variable Exponent Second-Order Quantum Backward Difference Sequences of Soft Functions and Its Pre-Quasi Ideal

**Awad A. Bakery**

*University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
*e-mail : aabhassan@uj.edu.sa; awad_bakery@yahoo.com*

**Abstract** In this article, we develop and study the space of null variable exponent second-order quantum backward difference sequences of soft functions, which are critical extensions to the concept of modular spaces. The mappings have been idealized through the use of extended $s-$soft functions and this soft function sequence space. The topological and geometric features of this new space are described, as well as the ideal mappings that correspond to them. We establish the existence of a Kannan contraction mapping fixed point acting on this space and its associated pre-quasi ideal. It's fascinating that we give various numerical experiments to show our findings. Additionally, several practical applications of the existence of solutions to nonlinear difference equations involving soft functions are discussed.

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## 1. Introduction

Let $\mathcal{N}$ be the set of non-negative integers. Yaying et al. [45], defined quantum second-order backward difference operator, $\nabla_p^2$, where $v_a = 0$ for $a < 0$, and $\nabla_p^2 v_a = v_a - (1+p)v_{a-1} + pv_{a-2}$, for all $p \in (0,1)$ and $a \in \mathcal{N}$. Recall that the operator $\nabla_p^2$ becomes $\nabla^2$ if $p \to 1^-$, which defined and studied in [15]. They proved that the spaces $c_0(\nabla_p^2)$ and $c(\nabla_p^2)$ are Banach spaces linearly isomorphic to $c_0$ and $c$, respectively, and obtained their Schauder bases and $\alpha-$, $\beta-$ and $\gamma-$duals. They investigated the spectrum, the point spectrum, the continuous spectrum, and the residual spectrum of the operator $\nabla_p^2$ over the Banach space $c_0$ of null sequences. Evidently $c_0 \subsetneq c_0(\nabla_p^2) \subsetneq c_0(\nabla^2)$. For strict

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inclusion. We have $(1,1,\cdots)\notin c_0$ and $(1,1,\cdots)\in c_0(\nabla_p^2)$, and $(0,1,2,\cdots)\notin c_0(\nabla_p^2)$ and $(0,1,2,\cdots)\in c_0(\nabla^2)$.
In functional analysis, the ideal theory of mappings is highly regarded. The ideals of closed mappings are certain to play a significant role in the Banach lattice principle. The following theories use mappings' ideal: fixed point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations. $s-$numbers is a critical method. Pietsch [33–36] explored in depth and examined the theory of $s$-numbers of linear bounded mappings between Banach spaces. He presented and explained some topological and geometric structures of the quasi ideals of $\ell_p$ type mappings. Then, Constantin [8], generalized the class of $\ell_p$ type mappings to the class of $ces_p$ type mappings. Makarov and Faried [26] proved some inclusion relations of $\ell_p$ type mappings. As a generalization of $\ell_p$ type mappings, Stolz mappings and mappings ideal were investigated by Tita [42, 43]. In [25], Maji and Srivastava offered the class $A_p^{(s)}$ of $s$-type $ces_p$ mappings using $s$-number sequence and Cesàro sequence spaces and they introduced a new class$A_{p,q}^{(s)}$ of $s$-type $ces(p,q)$ mappings by weighted $ces_p$ with $1<p<\infty$. In [21], the class of $s$-type $Z(u,v;\ell_p)$ mappings was defined and some of their properties were explained. Pre-quasi mappings ideals are more extensive than quasi mappings ideals, according to Faried and Bakery [16]. Bakery and Abou Elmatty [**?** ] investigated the necessary conditions on any $s-$type sequence space to form an operator's ideal. They showed that the $s-$type Nakano generalized difference sequence space $X$ fails to generate an operator's ideal. They investigated the sufficient conditions on $X$ to be pre-modular Banach special space of sequences. The constructed pre-quasi-operator ideal becomes a small, simple, and closed Banach space and has eigenvalues identical to its $s-$numbers. Finally, they introduced necessary and sufficient conditions on $X$, explaining some topological and geometrical structures of the multiplication operator defined on $X$. The study of variable exponent Lebesgue spaces was accelerated by the mathematical explanation of the hydrodynamics of non-Newtonian fluids. (see [38, 40]). Electrorheological fluids are used in various sectors, including military science, civil engineering, and orthopedics. Guo and Zhu [19] examined a class of stochastic VolterraLevin equations with Poisson jumps. Mao et al. [27], concerned with neutral stochastic functional differential equations driven by pure jumps (NSFDEwPJs). They showed the existence and uniqueness of the solution to NSFDEwPJs. The coefficients of the latter satisfy the local Lipschitz condition and establish the $p$-th exponential estimations and almost surely asymptotic estimations of the solution for NSFDEwJs. Yang and Zhu [44], concerned with a class of stochastic neutral functional differential equations of Sobolev-type with Poisson jumps. Since the publication of the Banach fixed point theorem [5], there have been numerous developments in the field of mathematics. While contractions have fixed-point actions, Kannan [20] illustrated a non-continuous mapping. In Reference [18], a single attempt was made to explain Kannan operators in modular vector spaces. The mathematics underpinnings of fuzzy set theory, pioneered by Zadeh [46] in 1965 and have made significant progress, are well understood in fuzzy theory. The fuzzy theory applies to a wide variety of real-world challenges. For instance, various researchers established the possibility theory, including Dubois and Prade [14] and Nahmias [29]. The contribution of probability theory, fuzzy set theory, and rough sets to the study of uncertainty are critical. Yet, these theories have some limitations as well as advantages. The theory of soft sets, developed by Molodtsov [28], was introduced as a new mathematical strategy for dealing with uncertainties to overcome these characteristics. Soft sets have been widely used in various disciplines and

technologies. In particular, Maji et al. [23, 24] studied several operations on soft sets and applied their findings to decision-making problems in the literature. Several writers, including Chen [6], Pei and Miao [32], Zou and Xiao [47], and Kong et al. [22], have discovered significant characteristics of soft sets. Soft semirings, soft ideals, and idealistic soft semirings were all investigated by Feng et al. [17]. Das and Samanta developed the ideas of a soft real number and a soft real set in [10] and discussed the characteristics of each concept. These principles served as the foundation for their investigation into the concept of "soft metrics" in [11] (see [9, 12] for a more in-depth examination). Based on the idea of soft elements of soft metric spaces, Abbas et al. [1] developed the concept of soft contraction mapping, which they named "soft contraction mapping". They focused on fixed points of soft contraction maps and obtained, among other things, a soft Banach contraction principle due to their efforts. In their paper, Abbas et al. [2] demonstrated that every complete soft metric induces an equivalent complete usual metric. They obtained in a direct way soft metric versions of various significant fixed point theorems for metric spaces, such as the Banach contraction principle, Kannan and Meir-Keeler fixed point theorems, and Caristi-theorem, Kirk's, among other things. In [7], Chen and Lin presented an extension of the Meir and Keeler fixed point theorem to soft metric spaces, which was previously published. Many researchers working on sequence spaces and summability theory were involved in introducing fuzzy sequence spaces and studying their many characteristics. When it comes to fuzzy numbers, Nuray and Savaş [31] defined and explored the Nakano sequences of fuzzy numbers, $\ell^F(\tau)$ equipped with a definite function. Numerous fixed point theorems are effective when applied to a particular space because they either increase the size of the self-mapping acting on it or the space itself. We develop and examine the space of null variable exponent second-order quantum backward difference sequences of soft functions in this paper, which are critical extensions to the concept of modular spaces. The mappings have been idealized through extended $s-$soft functions and this soft function sequence space. The topological and geometric features of this new space are described, as well as the ideal mappings that correspond to them. We establish the existence of a Kannan contraction mapping fixed point functioning in this space and its associated pre-quasi ideal. It's fascinating that we give various numerical experiments to show our findings. Additionally, several practical applications of the existence of solutions to nonlinear difference equations involving soft functions are discussed.

## 2. Definitions and Preliminaries:

**Definition 2.1.** [28] Let $U$ be an initial universe set and $E$ be a set of parameters. A pair $(F, E)$ is called a soft set (over $U$) if and only if F is a mapping of $E$ into the set of all subsets of the set $U$.

If $\mathfrak{R}$ is the set of real numbers. We denote the collection of all nonempty bounded subsets of $\mathfrak{R}$ by $\mathfrak{B}(\mathfrak{R})$ and $E$ is the set of parameters.

**Definition 2.2.** [10] A soft real set denoted by $(\widetilde{f}, A)$, or simply by $\widetilde{f}$, is a mapping $\widetilde{f} : A \to \mathfrak{B}(\mathfrak{R})$. If $\widetilde{f}$ is a single-valued mapping on $A \subset E$ taking values in $\mathfrak{R}$, then $\widetilde{f}$ is called a soft element of $\mathfrak{R}$ or a soft real number. If $\widetilde{f}$ is a single-valued mapping on $A \subset E$ taking values in the set $\mathfrak{R}^+$ of nonnegative real numbers, then $\widetilde{f}$ is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers (corresponding to A) by $\mathfrak{R}(A)^*$. A constant soft real number $\widetilde{c}$ is a soft real number such that for each $a \in A$, we have $\widetilde{c}(a) = c$, where $c$ is some real number.

**Definition 2.3.** [13] For two soft real numbers $\widetilde{f}$ ,$\widetilde{g}$, we say that

- **(a):** $\widetilde{f}\widetilde{\leq}\widetilde{g}$ if $\widetilde{f}(a)\widetilde{\leq}\widetilde{g}(a)$, for all $a \in A$,
- **(b):** $\widetilde{f}\widetilde{\geq}\widetilde{g}$ if $\widetilde{f}(a)\widetilde{\geq}\widetilde{g}(a)$, for all $a \in A$,
- **(c):** $\widetilde{f}\widetilde{<}\widetilde{g}$ if $\widetilde{f}(a)\widetilde{<}\widetilde{g}(a)$, for all $a \in A$, and
- **(d):** $\widetilde{f}\widetilde{>}\widetilde{g}$ if $\widetilde{f}(a)\widetilde{>}\widetilde{g}(a)$, for all $a \in A$.

Note that the relation $\widetilde{\leq}$ is a partial order on $\mathfrak{R}(A)$. The additive identity and multiplicative identity in $\mathfrak{R}(A)$ are denoted by $\widetilde{0}$ and $\widetilde{1}$, respectively.
The arithmetic operations on $\mathfrak{R}(A)$ are defined as follows:

$$(\widetilde{f} \oplus \widetilde{g})(\lambda) = \left\{\widetilde{f}(\lambda) + \widetilde{g}(\lambda) : \lambda \in A\right\},$$
$$(\widetilde{f} \ominus \widetilde{g})(\lambda) = \left\{\widetilde{f}(\lambda) - \widetilde{g}(\lambda) : \lambda \in A\right\},$$
$$(\widetilde{f} \otimes \widetilde{g})(\lambda) = \left\{\widetilde{f}(\lambda)\widetilde{g}(\lambda) : \lambda \in A\right\},$$
$$\left(\frac{f}{g}\right)(\lambda) = \left\{\frac{\widetilde{f}(\lambda)}{\widetilde{g}(\lambda)} : \lambda \in A \text{ and } 0 \notin \widetilde{g}(\lambda)\right\}.$$

The absolute value $|\widetilde{f}|$ of $\widetilde{f} \in \mathfrak{R}(A)$ is defined by

$$|\widetilde{f}|(\lambda) = \left\{|\widetilde{f}(\lambda)| : \lambda \in A\right\}.$$

Let $d : \mathfrak{R}(A) \times \mathfrak{R}(A) \to \mathfrak{R}(A)^*$, where $d(\widetilde{f}, \widetilde{g}) = |\widetilde{f} - \widetilde{g}|$ for all $\widetilde{f}, \widetilde{g} \in \mathfrak{R}(A)$. Assume $m_d : \mathfrak{R}(A) \times \mathfrak{R}(A) \to \mathfrak{R}^+$ is defined by $m_d(\widetilde{f}, \widetilde{g}) = \max\limits_{\lambda \in A} d(\widetilde{f}, \widetilde{g})(\lambda)$.
Note that:

1. $(\mathfrak{R}(A), m_d)$ is a complete metric space.
2. $m_d(\widetilde{f} + \widetilde{k}, \widetilde{g} + \widetilde{k}) = m_d(\widetilde{f}, \widetilde{g})$ for all $\widetilde{f}, \widetilde{g}, \widetilde{k} \in \mathfrak{R}(A)$.
3. $m_d(\widetilde{f} + \widetilde{k}, \widetilde{g} + \widetilde{l}) \leq m_d(\widetilde{f}, \widetilde{g}) + m_d(\widetilde{k}, \widetilde{l})$.
4. $m_d(\xi\widetilde{f}, \xi\widetilde{g}) = |\xi| m_d(f, g)$, for all $\xi \in \mathfrak{R}$.

**Definition 2.4.** A sequence $\widetilde{f} = (\widetilde{f}_j)$ of soft real numbers is said to be

**(a):** bounded if the set $\{\widetilde{f}_j : \ j \in \mathcal{N}\}$ of soft real numbers is bounded i.e., if a sequence $(\widetilde{f}_j)$ is bounded, then there are two soft real numbers $\widetilde{g}, \widetilde{l}$ such that $\widetilde{g}\widetilde{\leq}\widetilde{f}_j\widetilde{\leq}\widetilde{l}$,

**(b):** convergent to a soft real number $\widetilde{f}_0$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $m_d(\widetilde{f}_j, \widetilde{f}_0) < \varepsilon$, for all $j \geq j_0$.

By $c_0$, $\ell_\infty$ and $\ell_r$, we denote the space of null, bounded and $r$-absolutely summable sequences of real numbers. We indicate the space of all bounded, finite rank linear mappings from an infinite dimensional Banach space $\Omega$ into an infinite dimensional Banach space $\Lambda$ by $\mathcal{L}(\Omega, \Lambda)$, and $\mathfrak{F}(\Omega, \Lambda)$ and when $\Omega = \Lambda$, we inscribe $\mathcal{L}(\Omega)$ and $\mathfrak{F}(\Omega)$. The space of approximable and compact bounded linear mappings from $\Omega$ into $\Lambda$ will be denoted by $\Upsilon(\Omega, \Lambda)$ and $\mathcal{L}_c(\Omega, \Lambda)$, and if $\Omega = \Lambda$, we mark $\Upsilon(\Omega)$ and $\mathcal{L}_c(\Omega)$, respectively.

**Definition 2.5.** [37] An $s$-number function is a mapping $s : \mathcal{L}(\Omega, \Lambda) \to \mathfrak{R}^{+\mathcal{N}}$ that gives all $V \in \mathcal{L}(\Omega, \Lambda)$ a $(s_d(V))_{d=0}^{\infty}$ satisfies the following conditions:

**(a):** $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq ... \geq 0$, for every $V \in \mathcal{L}(\Omega, \Lambda)$,

**(b):** $s_{l+d-1}(V_1+V_2) \leq s_l(V_1)+s_d(V_2)$, for every $V_1, V_2 \in \mathcal{L}(\Omega, \Lambda)$ and $l$, $d \in \mathcal{N}$,

**(c):** $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for every $W \in \mathcal{L}(\Omega_0, \Omega)$, $Y \in \mathcal{L}(\Omega, \Lambda)$ and $V \in \mathcal{L}(\Lambda, \Lambda_0)$, where $\Omega_0$ and $\Lambda_0$ are arbitrary Banach spaces,

**(d):** assume $V \in \mathcal{L}(\Omega, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$,

**(e):** if $rank(V) \leq d$, then $s_d(V) = 0$, for all $V \in \mathcal{L}(\Omega, \Lambda)$,

**(f):** $s_{l\geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where $I_a$ indicates the unit mapping on the $a$-dimensional Hilbert space $\ell_2^a$.

We give here some examples of $s$-numbers:

**(1):** The $q$-th Kolmogorov number, denoted by $d_q(X)$, is marked by $d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\|\leq 1} \inf_{g\in J} \|Xf - g\|$.

**(2):** The $q$-th approximation number, indicated by $\alpha_q(X)$, is marked by $\alpha_q(X) = \inf\{ \|X - Y\| : Y \in \mathcal{L}(\Omega, \Lambda) \text{ and } \operatorname{rank}(Y) \leq q\}$.

**Definition 2.6.** [36] Let $\mathcal{L}$ be the class of all bounded linear operators within any two arbitrary Banach spaces. A sub class $\mathcal{U}$ of $\mathcal{L}$ is said to be a mappings ideal, if every $\mathcal{U}(\Omega, \Lambda) = \mathcal{U} \cap \mathcal{L}(\Omega, \Lambda)$ satisfies the following setups:

**(i):** $I_\Gamma \in \mathcal{U}$, where $\Gamma$ indicates Banach space of one dimension.

**(ii):** The space $\mathcal{U}(\Omega, \Lambda)$ is linear over $\mathfrak{R}$.

**(iii):** If $W \in \mathcal{L}(\Omega_0, \Omega)$, $X \in \mathcal{U}(\Omega, \Lambda)$ and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$, then $YXW \in \mathcal{U}(\Omega_0, \Lambda_0)$.

**Definition 2.7.** [16] A function $H \in [0, \infty)^{\mathcal{U}}$ is said to be a pre-quasi norm on the ideal $\mathcal{U}$ if the following conditions hold:

**(1):** Assume $V \in \mathcal{U}(\Omega, \Lambda)$, $H(V) \geq 0$ and $H(V) = 0$, if and only if, $V = 0$,

**(2):** one has $Q \geq 1$ with $H(\alpha V) \leq D|\alpha| H(V)$, for all $V \in \mathcal{U}(\Omega, \Lambda)$ and $\alpha \in \mathfrak{R}$,

**(3):** there are $P \geq 1$ such that $H(V_1+V_2) \leq P[H(V_1)+H(V_2)]$, for all $V_1, V_2 \in \mathcal{U}(\Omega, \Lambda)$,

**(4):** there are $\sigma \geq 1$ so that if $V \in \mathcal{L}(\Omega_0, \Omega)$, $X \in \mathcal{U}(\Omega, \Lambda)$ and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$ then $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$.

**Theorem 2.8** ([16]). *$H$ is a pre-quasi norm on the ideal $\mathcal{U}$, whenever $H$ is a quasi norm on the ideal $\mathcal{U}$.*

**Lemma 2.9** ([3]). *If $\tau_a > 0$ and $v_a, t_a \in \mathfrak{R}$, for all $a \in \mathcal{N}$, then $|v_a+t_a|^{\tau_a} \leq 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = \max\{1, \sup_a \tau_a\}$.*

## 3. Some Characteristics of $c_0^S(\nabla_p^2, \tau)$

We have presented in this section sufficient setups of the space of null variable exponent second-order quantum backward difference sequences of soft functions, $c_0^S(\nabla_p^2, \tau)$, equipped with the definite function $h$ to be pre-quasi Banach (certain space of sequences of soft reals, or in short (csss)). We have investigated some algebraic and topological properties like completeness, solidness, symmetry, convergence-free, etc. The Fatou property of various pre-quasi norms $h$ on $c_0^S(\nabla_p^2, \tau)$ has been studied.

If $\omega(S)$ is the space of all sequence spaces of soft reals. Assume $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, where

$\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences. The space of null variable exponent second-order quantum backward difference sequences of soft functions is defined as:
$c_0^S(\nabla_p^2,\tau)=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\mu\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}=0\text{ , for some }\mu>0\Big\}.$

**Theorem 3.1.** *If $(\tau_a)\in\ell_\infty$, then*

$$c_0^S(\nabla_p^2,\tau)=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\mu\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}=0\text{ , }\textit{for any }\mu>0\Big\}.$$

*Proof.*

$$\begin{aligned}c_0^S(\nabla_p^2,\tau)&=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\mu\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}=0\text{ , for some }\mu>0\Big\}\\&=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\inf_a|\mu|^{\frac{\tau_a}{K}}\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}\le0\text{ , for some }\mu>0\Big\}\\&=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}=0\Big\}\\&=\Big\{\widetilde{v}=(\widetilde{v_a})\in\omega(S):\lim_{a\to\infty}\Big[m_d\Big(\big|\nabla_p^2|\mu\widetilde{v_a}|\big|,\widetilde{0}\Big)\Big]^{\frac{\tau_a}{K}}=0\text{ , for any }\mu>0\Big\}.\end{aligned}$$

∎

Clearly, if $(\tau_a)\in\ell_\infty$, then

$$c_0^S(\tau)\subsetneqq c_0^S(\nabla_p^2,\tau)\subsetneqq c_0^S(\nabla^2,\tau).$$

For the strict inclusion. We have $(\widetilde{1},\widetilde{1},\widetilde{1},\cdots)\notin c_0^S$ and $(\widetilde{1},\widetilde{1},\widetilde{1},\cdots)\in c_0^S(\nabla_p^2,\tau)$, and $(\widetilde{0},\widetilde{1},\widetilde{2},\cdots)\notin c_0^S(\nabla_p^2,\tau)$ and $(\widetilde{0},\widetilde{1},\widetilde{2},\cdots)\in c_0^S(\nabla^2,\tau)$.
For $X=(X_k)$, a given sequence $S(X)$ denotes the set of all permutation of the elements of $(X_k)$, that is $S(X)=\big\{(X_{\pi(k)})\big\}$.

**Definition 3.2.** **(1):** A sequence space of soft numbers $\mathbf{U}$ is said to be symmetric, if $S(X)\in\mathbf{U}$, for all $X\in\mathbf{U}$.
**(2):** A sequence space of soft numbers $\mathbf{U}$ is said to be convergence free if $(Y_k)\in\mathbf{U}$ whenever $(X_k)\in\mathbf{U}$ and $X_k=\widetilde{0}$ implies $Y_k=\widetilde{0}$.

**Theorem 3.3.** *If $(\tau_a)\in\ell_\infty$, then the space $\big(c_0^S(\nabla_p^2,\tau)\big)_h$ is not symmetric.*

*Proof.* Consider $(X_k)=(\overbrace{\widetilde{1},...,\widetilde{1}}^{3\text{ times}},\overbrace{-\widetilde{1},...,-\widetilde{1}}^{3\text{ times}},\overbrace{\widetilde{1},...,\widetilde{1}}^{6\text{ times}},\overbrace{-\widetilde{1},...,-\widetilde{1}}^{6\text{ times}},\overbrace{\widetilde{1},...,\widetilde{1}}^{9\text{ times}},\overbrace{-\widetilde{1},...,-\widetilde{1}}^{9\text{ times}},...)$. Then $(X_k)\in\big(c_0^S(\nabla_p^2,\tau)\big)_h$. Now if $(Y_k)$ is the rearrangement of $(X_k)$ defined by $(Y_k)=(\widetilde{1},-\widetilde{1},\widetilde{1},-\widetilde{1},\cdots)$. Then $(Y_k)\notin\big(c_0^S(\nabla_p^2,\tau)\big)_h$. Therefore, the space $\big(c_0^S(\nabla_p^2,\tau)\big)_h$ is not symmetric. ∎

**Theorem 3.4.** *If $(\tau_a)\in\ell_\infty$, then the space $\big(c_0^S(\nabla_p^2,\tau)\big)_h$ is not convergence free.*

*Proof.* Consider the sequence $(X_k)=(\widetilde{1},\widetilde{1},\cdots)$. Then $(X_k)\in\big(c_0^S(\nabla_p^2,\tau)\big)_h$. Again if $(Y_k)=(\widetilde{k})$. Clearly, $(Y_k)\notin\big(c_0^S(\nabla_p^2,\tau)\big)_h$. Hence the space $\big(c_0^S(\nabla_p^2,\tau)\big)_h$ is not convergence free. ∎

Let us mark the space of all functions $h:\mathbf{U}\longrightarrow[0,\infty)$ by $[0,\infty)^{\mathbf{U}}$.

**Definition 3.5.** [30] If $\mathbf{U}$ is a vector space of soft reals. A function $h \in [0,\infty]^{\mathbf{U}}$ is said to be modular if the following conditions hold:

**(a):** Assume $\widetilde{Y} \in \mathbf{U}$, $\widetilde{Y} = \widetilde{\vartheta} \Leftrightarrow h(\widetilde{Y}) = 0$ with $h(\widetilde{Y}) \geq 0$, where $\widetilde{\vartheta} = (\widetilde{0}, \widetilde{0}, \widetilde{0}, \ldots)$,
**(b):** $h(\eta\widetilde{Z}) = h(\widetilde{Z})$ verifies, for every $\widetilde{Z} \in \mathbf{U}$ and $|\eta| = 1$,
**(c):** the inequality $h(\alpha\widetilde{Y} + (1-\alpha)\widetilde{Z}) \leq h(\widetilde{Y}) + h(\widetilde{Z})$ satisfies, for every $\widetilde{Y}, \widetilde{Z} \in \mathbf{U}$ and $\alpha \in [0,1]$.

**Definition 3.6.** The linear space $\mathbf{U}$ is called a certain space of sequences of soft reals (csss), when

**(1):** $\{\widetilde{b}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\widetilde{b}_q = \{\widetilde{0}, \widetilde{0}, ..., \widetilde{1}, \widetilde{0}, \widetilde{0}, ...\}$, for $\widetilde{1}$ marks at the $q^{th}$ place,
**(2):** $\mathbf{U}$ is solid i.e., if $\widetilde{Y} = (\widetilde{Y_q}) \in \omega(S)$, $\widetilde{Z} = (\widetilde{Z_q}) \in \mathbf{U}$ and $|\widetilde{Y_q}| \widetilde{\leq} |\widetilde{Z_q}|$, for all $q \in \mathcal{N}$, one has $\widetilde{Y} \in \mathbf{U}$,
**(3):** $(\widetilde{Y_{[\frac{q}{2}]}})_{q=0}^{\infty} \in \mathbf{U}$, where $[\frac{q}{2}]$ indicates the integral part of $\frac{q}{2}$, assume $(\widetilde{Y_q})_{q=0}^{\infty} \in \mathbf{U}$.

**Definition 3.7.** A subclass $\mathbf{U}_h$ of $\mathbf{U}$ is said to be a pre-modular (csss), if there is $h \in [0,\infty)^{\mathbf{U}}$ satisfies the following conditions:

**(i):** Suppose $\widetilde{Y} \in \mathbf{U}$, $\widetilde{Y} = \widetilde{\vartheta} \Leftrightarrow h(\widetilde{Y}) = 0$ with $h(\widetilde{Y}) \geq 0$,
**(ii):** we have $Q \geq 1$, the inequality $h(\alpha\widetilde{Y}) \leq Q|\alpha| h(\widetilde{Y})$ satisfies, for all $\widetilde{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$,
**(iii):** one has $P \geq 1$, the inequality $h(\widetilde{Y} + \widetilde{Z}) \leq P(h(\widetilde{Y}) + h(\widetilde{Z}))$ satisfies, for all $\widetilde{Y}, \widetilde{Z} \in \mathbf{U}$,
**(iv):** when $|\widetilde{Y_q}| \widetilde{\leq} |\widetilde{Z_q}|$, for all $q \in \mathcal{N}$, we have $h((\widetilde{Y_q})) \leq h((\widetilde{Z_q}))$,
**(v):** the inequality, $h((\widetilde{Y_q})) \leq h((\widetilde{Y_{[\frac{q}{2}]}})) \leq P_0 h((\widetilde{Y_q}))$ verifies, for some $P_0 \geq 1$,
**(vi):** assume $\mathbb{E}$ is the space of finite sequences of soft real numbers, one has the closure of $\mathbb{E} = \mathbf{U}_h$,
**(vii):** we have $\sigma > 0$ with $h(\widetilde{\alpha}, \widetilde{0}, \widetilde{0}, \widetilde{0}, ...) \geq \sigma|\alpha| h(\widetilde{1}, \widetilde{0}, \widetilde{0}, \widetilde{0}, ...)$, where $\widetilde{\alpha}(a) = \alpha$, for every $a \in A$.

Note that the notion of pre-modular vector spaces is more general than modular vector spaces. Some examples of pre-modular vector spaces but not modular vector spaces.

**Example 3.8.** The function $h(\widetilde{Z}) = \sup_q \left[m_d\left(\left|\nabla_p^2 |\widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\frac{4q+1}{q+4}}$ on the vector space $c_0^S\left(\nabla_p^2, \left(\frac{4q+1}{q+4}\right)\right)$. As for every $\widetilde{Z}, \widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{4q+1}{q+4}\right)\right)$, one has

$$h\left(\frac{\widetilde{Z} + \widetilde{Y}}{2}\right) = \sup_q \left[m_d\left(\left|\nabla_p^2 \left|\frac{\widetilde{Z_q} + \widetilde{Y_q}}{2}\right|\right|, \widetilde{0}\right)\right]^{\frac{4q+1}{q+4}} \leq \frac{8}{\sqrt[4]{2}}(h(\widetilde{Z}) + h(\widetilde{Y})).$$

**Example 3.9.** The function $h(\widetilde{Z}) = \sup_q \left[m_d\left(\left|\nabla_p^2 |\widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\frac{5q+2}{q+1}}$ on the vector space $c_0^S\left(\nabla_p^2, \left(\frac{5q+2}{q+1}\right)\right)$. As for every $\widetilde{Z}, \widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{5q+2}{q+1}\right)\right)$, one has

$$h\left(\frac{\widetilde{Z} + \widetilde{Y}}{2}\right) = \sup_q \left[m_d\left(\left|\nabla_p^2 \left|\frac{\widetilde{Z_q} + \widetilde{Y_q}}{2}\right|\right|, \widetilde{0}\right)\right]^{\frac{5q+2}{q+1}} \leq 4(h(\widetilde{Z}) + h(\widetilde{Y})).$$

Some examples of pre-modular vector spaces and modular vector spaces.

**Example 3.10.** The function $h(\widetilde{Z}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\frac{q+1}{3q+4}}$ on the vector space $c_0^S\left(\nabla_p^2, \left(\frac{q+1}{3q+4}\right)\right)$. As for every $\widetilde{Z}, \widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{q+1}{3q+4}\right)\right)$, one has

$$h\left(\frac{\widetilde{Z}+\widetilde{Y}}{2}\right) = \sup_q \left[m_d\left(\left|\nabla_p^2\left|\frac{\widetilde{Z_q}+\widetilde{Y_q}}{2}\right|\right|, \widetilde{0}\right)\right]^{\frac{q+1}{3q+4}} \leq \frac{1}{\sqrt[4]{2}}(h(\widetilde{Z}) + h(\widetilde{Y})).$$

**Example 3.11.** The function $h(\widetilde{Y}) = \inf\left\{\alpha > 0 : \sup_q \left[m_d\left(\left|\nabla_p^2|\frac{\widetilde{Y_q}}{\alpha}|\right|, \widetilde{0}\right)\right]^{\frac{2q+3}{q+2}} \leq 1\right\}$ is a pre-modular (modular) on the vector space $c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$.

**Definition 3.12.** If $\mathbf{U}$ is a (csss). The function $h \in [0,\infty)^{\mathbf{U}}$ is said to be a pre-quasi norm on $\mathbf{U}$, if it verifies the following settings:

**(i):** Suppose $\widetilde{Y} \in \mathbf{U}$, $\widetilde{Y} = \widetilde{\vartheta} \Leftrightarrow h(\widetilde{Y}) = 0$ with $h(\widetilde{Y}) \geq 0$,

**(ii):** we have $Q \geq 1$, the inequality $h(\alpha\widetilde{Y}) \leq Q|\alpha|h(\widetilde{Y})$ satisfies, for all $\widetilde{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$,

**(iii):** one has $P \geq 1$, the inequality $h(\widetilde{Y} + \widetilde{Z}) \leq P(h(\widetilde{Y}) + h(\widetilde{Z}))$ satisfies, for all $\widetilde{Y}, \widetilde{Z} \in \mathbf{U}$.

**Theorem 3.13.** *Suppose* $\mathbf{U}$ *is a pre-modular (csss), then it is pre-quasi normed (csss).*

**Theorem 3.14.** $\mathbf{U}$ *is a pre-quasi normed (csss), if it is quasi-normed (csss).*

**Definition 3.15.** **(a):** The function $h$ on $c_0^S(\nabla_p^2, \tau)$ is called $h$-convex, when

$$h(\alpha\widetilde{Y} + (1-\alpha)\widetilde{Z}) \leq \alpha h(\widetilde{Y}) + (1-\alpha)h(\widetilde{Z}),$$

for all $\alpha \in [0,1]$ and $\widetilde{Y}, \widetilde{Z} \in c_0^S(\nabla_p^2, \tau)$.

**(b):** $\{\widetilde{Y_q}\}_{q\in\mathcal{N}} \subseteq \left(c_0^S(\nabla_p^2, \tau)\right)_h$ is $h$-convergent to $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, if and only if, $\lim_{q\to\infty} h(\widetilde{Y_q} - \widetilde{Y}) = 0$. If the $h$-limit exists, then it is unique.

**(c):** $\{\widetilde{Y_q}\}_{q\in\mathcal{N}} \subseteq \left(c_0^S(\nabla_p^2, \tau)\right)_h$ is $h$-Cauchy, if $\lim_{q,r\to\infty} h(\widetilde{Y_q} - \widetilde{Y_r}) = 0$.

**(d):** $\Gamma \subset \left(c_0^S(\nabla_p^2, \tau)\right)_h$ is $h$-closed, if for every $h$-converges $\{\widetilde{Y_q}\}_{a\in\mathcal{N}} \subset \Gamma$ to $\widetilde{Y}$, one has $\widetilde{Y} \in \Gamma$.

**(e):** $\Gamma \subset \left(c_0^S(\nabla_p^2, \tau)\right)_h$ is $h$-bounded, assume $\delta_h(\Gamma) = \sup\left\{h(\widetilde{Y} - \widetilde{Z}) : \widetilde{Y}, \widetilde{Z} \in \Gamma\right\} < \infty$.

**(f):** The $h$-ball of radius $\varepsilon \geq 0$ and center $\widetilde{Y}$, for all $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, is denoted by:

$$\mathbf{B}_h(\widetilde{Y}, \varepsilon) = \left\{\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h : h(\widetilde{Y} - \widetilde{Z}) \leq \varepsilon\right\}.$$

**(g):** A pre-quasi norm $h$ on $c_0^S(\nabla_p^2, \tau)$ verifies the Fatou property, if for all sequence $\{\widetilde{Z^{(q)}}\} \subseteq \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\lim_{q\to\infty} h(\widetilde{Z^{(q)}} - \widetilde{Z}) = 0$ and every $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, we have $h(\widetilde{Y} - \widetilde{Z}) \leq \sup_r \inf_{q\geq r} h(\widetilde{Y} - \widetilde{Z^{(q)}})$.

We will denote the space of all increasing sequences of real numbers by **I**.

**Theorem 3.16.** $\left(c_0^S(\nabla_p^2,\tau)\right)_h$, *where* $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\left|\nabla_p^2\right|\widetilde{Y_q}\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, *for every* $\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$, *is a pre-modular (csss), if the following conditions are satisfied:*

**a.:** $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ *with* $\tau_0>0$,
**b.:** $\nabla_p^2$ *is an absolute non-decreasing, i.e., if* $|\widetilde{Z_i}|\widetilde{\leq}|\widetilde{Y_i}|$ *for all* $i\in\mathbb{N}$, *then* $\left|\nabla_p^2|\widetilde{Z_i}|\right|\widetilde{\leq}\left|\nabla_p^2|\widetilde{Y_i}|\right|$.

*Proof.* (i) Clearly, $h(\widetilde{Y})\geq 0$ and $h(\widetilde{Y})=0\Leftrightarrow\widetilde{Y}=\widetilde{\vartheta}$.
(1-i) Assume $\widetilde{Y},\widetilde{Z}\in c_0^S(\nabla_p^2,\tau)$. We have

$$\begin{aligned}h(\widetilde{Y}+\widetilde{Z})&=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}+\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&\leq\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}+\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&=h(\widetilde{Y})+h(\widetilde{Z})<\infty,\end{aligned}$$

then $\widetilde{Y}+\widetilde{Z}\in c_0^S(\nabla_p^2,\tau)$.
(iii) There are $P\geq 1$ with $h(\widetilde{Y}+\widetilde{Z})\leq P(h(\widetilde{Y})+h(\widetilde{Z}))$, for every $\widetilde{Y},\widetilde{Z}\in c_0^S(\nabla_p^2,\tau)$.
(1-ii) If $\alpha\in\mathfrak{R}$ and $\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$, one has

$$\begin{aligned}h(\alpha\widetilde{Y})&=\sup_q\left[m_d\left(\left|\nabla_p^2|\alpha\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&\leq\sup_q|\alpha|^{\frac{\tau_q}{K}}\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&\leq Q|\alpha|h(\widetilde{Y})<\infty.\end{aligned}$$

Since $\alpha\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$, hence from parts (1-i) and (1-ii), we have $c_0^S(\nabla_p^2,\tau)$ is linear. Also $\widetilde{b}_p\in c_0^S(\nabla_p^2,\tau)$, for every $p\in\mathcal{N}$, as $h(\widetilde{b}_p)=\sup_q\left[m_d\left(\left|\nabla_p^2|(\widetilde{b}_p)_q|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}=1$.
(ii) One has $Q=\max\left\{1,\sup_q|\alpha|^{\frac{\tau_q}{K}-1}\right\}\geq 1$ with $h(\alpha\widetilde{Y})\leq Q|\alpha|h(\widetilde{Y})$, for every $\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$ and $\alpha\in\mathfrak{R}$.
(2) If $|\widetilde{Y_q}|\leq|\widetilde{Z_q}|$, for every $q\in\mathcal{N}$ and $\widetilde{Z}\in c_0^S(\nabla_p^2,\tau)$. We obtain

$$h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\leq\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}=h(\widetilde{Z})<\infty,$$

then $\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$.
(iv) Evidently, from (2).

(3) Assume $(\widetilde{Y_q}) \in c_0^S(\nabla_p^2, \tau)$, one can see

$$\begin{aligned} h\Big((\widetilde{Y_{[\frac{q}{2}]}})\Big) &= \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_{[\frac{q}{2}]}}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\le \max\left\{\sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_{2q}}{K}}, \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_{2q+1}}{K}}\right\} \\ &\le \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} = h\Big((\widetilde{Y_q})\Big), \end{aligned}$$

then $(\widetilde{Y_{[\frac{q}{2}]}}) \in c_0^S(\nabla_p^2, \tau)$. (v) From (3), one has $P_0 = 1$.
(vi) Clearly, the closure of $\mathbb{E} = c_0^S(\nabla_p^2, \tau)$.
(vii) One gets $0 < \sigma \le \sup_q |\alpha|^{\frac{\tau_q}{K}-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with

$$h(\widetilde{\alpha}, \widetilde{0}, \widetilde{0}, \widetilde{0}, ...) \ge \sigma|\alpha| h(\widetilde{1}, \widetilde{0}, \widetilde{0}, \widetilde{0}, ...).$$

∎

**Theorem 3.17.** *If the conditions of Theorem 3.16 are satisfied, then* $\Big(c_0^S(\nabla_p^2, \tau)\Big)_h$ *is a pre-quasi Banach (csss), where* $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, *for all* $Y \in c_0^S(\nabla_p^2, \tau)$.

*Proof.* According to Theorem 3.16 and Theorem 3.13, the space $\Big(c_0^S(\nabla_p^2, \tau)\Big)_h$ is a pre-quasi normed (csss). If $\widetilde{Y^l} = (\widetilde{Y_q^l})_{q=0}^\infty$ is a Cauchy sequence in $\Big(c_0^S(\nabla_p^2, \tau)\Big)_h$, hence for all $\varepsilon \in (0, 1)$, then $l_0 \in \mathcal{N}$ such that for every $l, m \ge l_0$, we have

$$h(\widetilde{Y^l} - \widetilde{Y^m}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q^l} - \widetilde{Y_q^m}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} < \varepsilon.$$

Therefore, $m_d\left(\left|\nabla_p^2|\widetilde{Y_q^l} - \widetilde{Y_q^m}|\right|, \widetilde{0}\right) < \varepsilon$. Since $(\mathfrak{R}(A), m_d)$ is a complete metric space. So $(\widetilde{Y_q^m})$ is a Cauchy sequence in $\mathfrak{R}(A)$, for fixed $q \in \mathcal{N}$. This gives $\lim_{m\to\infty} \widetilde{Y_q^m} \cong \widetilde{Y_q^0}$, for fixed $q \in \mathcal{N}$. Then $h(\widetilde{Y^l} - \widetilde{Y^0}) < \varepsilon$, for all $l \ge l_0$. As $h(\widetilde{Y^0}) = h(\widetilde{Y^0} - \widetilde{Y^l} + \widetilde{Y^l}) \le h(\widetilde{Y^l} - \widetilde{Y^0}) + h(\widetilde{Y^l}) < \infty$. Then $\widetilde{Y^0} \in c_0^S(\nabla_p^2, \tau)$. ∎

**Theorem 3.18.** *The function* $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$ *satisfies the Fatou property, when the conditions of Theorem 3.16 are satisfied.*

*Proof.* Let $\{\widetilde{Z^r}\} \subseteq \Big(c_0^S(\nabla_p^2, \tau)\Big)_h$ such that $\lim_{r\to\infty} h(\widetilde{Z^r} - \widetilde{Z}) = 0$. Since $\Big(c_0^S(\nabla_p^2, \tau)\Big)_h$ is a pre-quasi closed space, we have $\widetilde{Z} \in \Big(c_0^S(\nabla_p^2, \tau)\Big)_h$. For every $\widetilde{Y} \in \Big(c_0^S(\nabla_p^2, \tau)\Big)_h$, then

$$\begin{aligned} h(\widetilde{Y} - \widetilde{Z}) &= \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q} - \widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\le \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q} - \widetilde{Z_q^r}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q^r} - \widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\le \sup_m \inf_{r\ge m} h(\widetilde{Y} - \widetilde{Z^r}). \end{aligned}$$

∎

**Theorem 3.19.** *The function* $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\tau_q}$ *does not satisfy the Fatou property, for all* $\widetilde{Y}\in c_0^S(\nabla_p^2,\tau)$*, if the conditions of Theorem 3.16 are satisfied with* $\tau_0>1$.

*Proof.* Assume $\{\widetilde{Z^r}\}\subseteq\left(c_0^S(\nabla_p^2,\tau)\right)_h$ such that $\lim_{r\to\infty}h(\widetilde{Z^r}-\widetilde{Z})=0$. As $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ is a pre-quasi closed space, we have $\widetilde{Z}\in\left(c_0^S(\nabla_p^2,\tau)\right)_h$. For all $\widetilde{Z}\in\left(c_0^S(\nabla_p^2,\tau)\right)_h$, then

$$
\begin{aligned}
h(\widetilde{Y}-\widetilde{Z})&=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}-\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\tau_q}\\
&\le 2^{\sup_q\tau_q-1}\left(\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}-\widetilde{Z_q^r}|\right|,\widetilde{0}\right)\right]^{\tau_q}+\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q^r}-\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\tau_q}\right)\\
&\le 2^{\sup_q\tau_q-1}\sup_m\inf_{r\ge m}h(\widetilde{Y}-\widetilde{Z^r}).
\end{aligned}
$$

■

**Example 3.20.** For $(\tau_q)\in[1,\infty)^{\mathcal{N}}$, the function

$$h(\widetilde{Y})=\inf\left\{\alpha>0:\sup_q\left[m_d\left(\left|\nabla_p^2\left|\frac{\widetilde{Y_q}}{\alpha}\right|\right|,\widetilde{0}\right)\right]^{\tau_q}\le 1\right\}\text{ is a norm on } c_0^S(\nabla_p^2,\tau).$$

**Example 3.21.** The function $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{3q+2}{q+3}}$ is a pre-quasi norm (not a norm) on $c_0^S\left(\nabla_p^2,(\frac{3q+2}{q+3})_{q=0}^\infty\right)$.

**Example 3.22.** The function $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{3q+2}{q+1}}$ is a pre-quasi norm (not a quasi norm) on $c_0^S\left(\nabla_p^2,(\frac{3q+2}{q+1})_{q=0}^\infty\right)$.

## 4. Structure of Mappings' Ideal

The structure of the mappings' ideal by $\left(c_0^S(\nabla_p^2,\tau)\right)_h$, where

$$h(\widetilde{g})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{g_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}},$$

for all $\widetilde{g}\in c_0^S(\nabla_p^2,\tau)$, and extended $s-$soft functions have been explained. We study enough setups on $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ such that the class $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is complete. We investigate conditions setups (not necessary) on $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ such that the closure of $\mathfrak{F}=\widetilde{\mathcal{A}^\alpha}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$. This gives a negative answer of Rhoades [39] open problem about the linearity of $s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ spaces. We explain enough setups on $\left(c_0^S(\nabla_p^2,\tau)\right)_h$

such that $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is strictly contained for different powers and backward generalized differences, the class $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is simple, and the space of every bounded linear mappings which sequence of eigenvalues in $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ equals $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$.

**Remark 4.1.**

$$\widetilde{\mathcal{A}}_{\mathbf{U}} := \left\{\widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)\right\}, \text{ where } \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda) := \left\{V \in \mathcal{L}(\Omega,\Lambda) : ((\widetilde{s_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\},$$
$$\widetilde{\mathcal{A}^{\alpha}}_{\mathbf{U}} := \left\{\widetilde{\mathcal{A}^{\alpha}}_{\mathbf{U}}(\Omega,\Lambda)\right\}, \text{ where } \widetilde{\mathcal{A}^{\alpha}}_{\mathbf{U}}(\Omega,\Lambda) := \left\{V \in \mathcal{L}(\Omega,\Lambda) : ((\widetilde{\alpha_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\},$$
$$\widetilde{\mathcal{A}^{d}}_{\mathbf{U}} := \left\{\widetilde{\mathcal{A}^{d}}_{\mathbf{U}}(\Omega,\Lambda)\right\}, \text{ where } \widetilde{\mathcal{A}^{d}}_{\mathbf{U}}(\Omega,\Lambda) := \left\{V \in \mathcal{L}(\Omega,\Lambda) : ((\widetilde{d_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\}.$$

**Theorem 4.2.** *If $\mathbf{U}$ is a (csss), then $\widetilde{\mathcal{A}}_{\mathbf{U}}$ is a mappings' ideal.*

*Proof.* (i) Suppose $V \in \mathfrak{F}(\Omega,\Lambda)$ and $rank(V) = n$, for every $n \in \mathcal{N}$, since $\widetilde{b_i} \in \mathbf{U}$, for every $i \in \mathcal{N}$, and $\mathbf{U}$ is a linear space, then
$(\widetilde{s_i(V)})_{i=0}^{\infty} = (\widetilde{s_0(V)}, \widetilde{s_1(V)}, ..., \widetilde{s_{n-1}(V)}, \widetilde{0}, \widetilde{0}, \widetilde{0}, ...) = \sum_{i=0}^{n-1} \widetilde{s_i(V)}\widetilde{b_i} \in \mathbf{U}$; for that $V \in \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)$ then $\mathfrak{F}(\Omega,\Lambda) \subseteq \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)$.
(ii) If $V_1, V_2 \in \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$ so by Definition 3.6 condition (3) one has $(\widetilde{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$ and $(\widetilde{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$, as $i \geq 2[\frac{i}{2}]$, by the definition of $s-$numbers and $\widetilde{s_i(V)}$ is decreasing, we have
$\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{2[\frac{i}{2}]}(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{[\frac{i}{2}]}(\beta_1 V_1)} + \widetilde{s_{[\frac{i}{2}]}(\beta_2 V_2)} = |\beta_1|\widetilde{s_{[\frac{i}{2}]}(V_1)} + |\beta_2|\widetilde{s_{[\frac{i}{2}]}(V_2)}$
for all $i \in \mathcal{N}$. By Definition 3.6 part (2) and $\mathbf{U}$ is a linear space, we get $(\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^{\infty} \in \mathbf{U}$, hence $\beta_1 V_1 + \beta_2 V_2 \in \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)$.
(iii) Assume $P \in \mathcal{L}(\Omega_0,\Omega)$, $T \in \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega,\Lambda)$ and $R \in \mathcal{L}(\Lambda,\Lambda_0)$, then
$(\widetilde{s_i(T)})_{i=0}^{\infty} \in \mathbf{U}$ and since $\widetilde{s_i(RTP)} \leq \|R\|\widetilde{s_i(T)}\|P\|$, from Definition 3.6 parts (1) and (2), then
$(\widetilde{s_i(RTP)})_{i=0}^{\infty} \in \mathbf{U}$, then $RTP \in \widetilde{\mathcal{A}}_{\mathbf{U}}(\Omega_0,\Lambda_0)$. ∎

In view of Theorem 3.16 and Theorem 4.2, one has the following theorem.

**Theorem 4.3.** *If the conditions of Theorem 3.16 are satisfied, then $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is a mappings' ideal.*

**Theorem 4.4.** *If the conditions of Theorem 3.16 are satisfied, then the function $H$ is a pre-quasi norm on $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$, with $H(\widetilde{Z}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(Z)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, for every $\widetilde{Z} \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$.*

*Proof.* **(1):** Suppose $Z \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$, $H(Z) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(Z)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$ $\geq 0$ and $H(Z) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(Z)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} = 0$, if and only if, $\widetilde{s_q(Z)} = \widetilde{0}$, for all $q \in \mathcal{N}$, if and only if, $Z = 0$,

**(2):** one has $Q \geq 1$ with $H(\alpha X) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(\alpha X)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \leq Q|\alpha|H(X)$, for all $X \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ and $\alpha \in \mathfrak{R}$,

**(3):** for $X_1, X_2 \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$, we have

$$\begin{aligned} H(X_1+X_2) = & \sup_q \left[m_d\left(\left|\nabla_p^2|s_q(\widetilde{X_1+X_2})|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ & \leq \left(h(\widetilde{s_{[\frac{q}{2}]}(X_1)})_{q=0}^{\infty} + h(\widetilde{s_{[\frac{q}{2}]}(X_2)})_{q=0}^{\infty}\right) \\ & \leq \left(h(\widetilde{s_q(X_1)})_{q=0}^{\infty} + h(\widetilde{s_q(X_2)})_{q=0}^{\infty}\right), \end{aligned}$$

**(4):** there are $\varrho \geq 1$, if $X \in \mathcal{L}(\Omega_0,\Omega)$, $Y \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ and $Z \in \mathcal{L}(\Lambda,\Lambda_0)$, then

$$H(ZYX) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(ZYX)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \leq h(\,\|X\|\,\|Z\|\widetilde{s_q(Y)})_{q=0}^{\infty} \leq \varrho\,\|X\|H(Y)\,\|Z\|.$$

∎

In the next theorems, we will use the notation $\left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}, H\right)$, where $H(V) = h\left((\widetilde{s_q(V)})_{q=0}^{\infty}\right)$, for all $V \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$.

**Theorem 4.5.** *Assume the conditions of Theorem 3.16 are satisfied, then* $\left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}, H\right)$ *is a pre-quasi Banach mappings ideal.*

*Proof.* Let $(V_a)_{a\in\mathcal{N}}$ be a Cauchy sequence in $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$. Since $\mathcal{L}(\Omega,\Lambda) \supseteq S_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$, then

$$H(V_r - V_a) = \sup_q \left[m_d\left(\left|\nabla_p^2|s_q(\widetilde{V_r - V_a})|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \geq h\left(s_0(\widetilde{V_r - V_a}), \widetilde{0}, \widetilde{0}, \widetilde{0}, ...\right) \geq \|V_r - V_a\|^{\frac{\tau_0}{K}},$$

this implies $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathcal{L}(\Omega,\Lambda)$. Since $\mathcal{L}(\Omega,\Lambda)$ is a Banach space, one has $V \in \mathcal{L}(\Omega,\Lambda)$ such that $\lim_{a\to\infty} \|V_a - V\| = 0$ and as $(\widetilde{s_q(V_a)})_{q=0}^{\infty} \in \left(c_0^S(\nabla_p^2,\tau)\right)_h$, for every $a \in \mathcal{N}$ and $\left(c_0^S(\nabla_p^2, \tau)\right)_h$ is a pre-modular (csss). Then we have

$$\begin{aligned} H(V) &= h\Big(\widetilde{(s_q(V))_{q=0}^{\infty}}\Big) \leq h\Big(\widetilde{(s_{[\frac{q}{2}]}(V-V_a)\Big)_{q=0}^{\infty}}) + h\Big(\widetilde{(s_{[\frac{q}{2}]}(V_a)_{q=0}^{\infty})}\Big) \\ &\leq h\Big(\widetilde{(\,\|V_a - V\,\|1)_{q=0}^{\infty}}\Big) + h\Big(\widetilde{(s_q(V_a))_{q=0}^{\infty}}\Big) < \varepsilon, \end{aligned}$$

hence one has $\widetilde{(s_q(V))_{q=0}^{\infty}} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, then $V \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$. ∎

**Definition 4.6.** A pre-quasi norm $H$ on the ideal $\widetilde{\mathcal{A}}_{\mathbf{U}_h}$ satisfies the Fatou property if for all $\{T_q\}_{q\in\mathcal{N}} \subseteq \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega, \Lambda)$ such that $\lim_{q\to\infty} H(T_q - T) = 0$ and $M \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega, \Lambda)$, then

$$H(M - T) \leq \sup_q \inf_{j\geq q} H(M - T_j).$$

**Theorem 4.7.** *If the conditions of Theorem 3.16 are satisfied, then* $\left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}, H\right)$ *does not satisfy the Fatou property.*

*Proof.* Let $\{T_q\}_{q\in\mathcal{N}} \subseteq \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$ with $\lim_{q\to\infty} H(T_q - T) = 0$. Since $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is a pre-quasi closed ideal, then $T \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$, hence for all $M \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$, we have

$$\begin{aligned} H(M - T) &= \sup_q \left[m_d\left(\left|\nabla_p^2 s_q(\widetilde{M-T})\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\leq \sup_q \left[m_d\left(\left|\nabla_p^2 s_{[\frac{q}{2}]}(\widetilde{M-T_i})\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[m_d\left(\left|\nabla_p^2 s_{[\frac{q}{2}]}(\widetilde{T_i-T})\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\leq \sup_r \inf_{i\geq r} \sup_q \left[m_d\left(\left|\nabla_p^2 s_q(\widetilde{M-T_i})\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}. \end{aligned}$$

∎

**Theorem 4.8.** $\widetilde{\mathcal{A}^{\alpha}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda) =$ *the closure of* $\mathfrak{F}(\Omega, \Lambda)$, *if the conditions of Theorem 3.16 are satisfied. But the converse is not necessarily true.*

*Proof.* As $\widetilde{b}_x \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, for all $x \in \mathcal{N}$ and $\left(c_0^S(\nabla_p^2, \tau)\right)_h$ is a linear space. If $Z \in \mathfrak{F}(\Omega, \Lambda)$, one has $\widetilde{(\alpha_x(Z))_{x=0}^{\infty}} \in E$. Then the closure of $\mathfrak{F}(\Omega, \Lambda) \subseteq \widetilde{\mathcal{A}^{\alpha}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$. Suppose $Z \in \widetilde{\mathcal{A}^{\alpha}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$, one has $\widetilde{(\alpha_x(Z))_{x=0}^{\infty}} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$. Since $h\widetilde{(\alpha_x(Z))_{x=0}^{\infty}} < \infty$, if $\rho \in (0, 1)$, one has $x_0 \in \mathcal{N} - \{0\}$ so that $h(\widetilde{(\alpha_x(Z))_{x=x_0}^{\infty}}) < \frac{\rho}{4}$. As $\widetilde{(\alpha_x(Z))_{x=0}^{\infty}}$ is decreasing, one gets

$$\begin{aligned} &\sup_{x=x_0+1}^{2x_0} \left[m_d\left(\left|\nabla_p^2 \widetilde{\alpha_{2x_0}(Z)}\right|, \widetilde{0}\right)\right]^{\frac{\tau_x}{K}} \leq \sup_{x=x_0+1}^{2x_0} \left[m_d\left(\left|\nabla_p^2 \widetilde{\alpha_x(Z)}\right|, \widetilde{0}\right)\right]^{\frac{\tau_x}{K}} \\ &\leq \sup_{x=x_0}^{\infty} \left[m_d\left(\left|\nabla_p^2 \widetilde{\alpha_x(Z)}\right|, \widetilde{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}. \end{aligned} \tag{4.1}$$

Then one has $Y \in \mathfrak{F}_{2x_0}(\Omega,\Lambda)$ such that $\mathrm{rank}(Y) \le 2x_0$ and

$$\sup_{x=2x_0+1}^{3x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} \le \sup_{x=x_0+1}^{2x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}, \tag{4.2}$$

as $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, take

$$\sup_{x=0}^{x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}. \tag{4.3}$$

According to inequalities (4.1)-(4.3), then

$$\begin{aligned}
d(Z,Y) &= \sup_{x=0}^{\infty}\left[m_d\left(\left|\nabla_p^2\alpha_x\widetilde{(Z-Y)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}}\\
&\le \sup_{x=0}^{3x_0-1}\left[m_d\left(\left|\nabla_p^2\alpha_x\widetilde{(Z-Y)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=3x_0}^{\infty}\left[m_d\left(\left|\nabla_p^2\alpha_x\widetilde{(Z-Y)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}}\\
&\le \sup_{x=0}^{3x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=x_0}^{\infty}\left[m_d\left(\left|\nabla_p^2\alpha_{x+2x_0}\widetilde{(Z-Y)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_{x+2x_0}}{K}}\\
&\le \sup_{x=0}^{3x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=x_0}^{\infty}\left[m_d\left(\left|\nabla_p^2\widetilde{\alpha_x(Z)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}}\\
&\le 3\sup_{x=0}^{x_0}\left[m_d\left(\left|\nabla_p^2\|\widetilde{Z-Y}\|\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=x_0}^{\infty}\left[m_d\left(\left|\nabla_p^2\widetilde{\alpha_x(Z)}\right|,\widetilde{0}\right)\right]^{\frac{\tau_x}{K}} < \rho.
\end{aligned}$$

This implies $\widetilde{\mathcal{A}}^{\alpha}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda) \subseteq$ the closure of $\mathfrak{F}(\Omega,\Lambda)$. Contrarily, one has a counter-example as $I_3 \in \widetilde{\mathcal{A}}^{\alpha}_{\left(c_0^S(\nabla_p^2,(0,0,1,1,\cdots))\right)}(\Omega,\Lambda)$, but $\tau_0 > 0$ is not satisfied. ∎

**Theorem 4.9.** *Assume the conditions of Theorem 3.16 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$, for every $x \in \mathcal{N}$, then*

$$\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda) \subsetneqq \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda) \subsetneqq \mathcal{L}(\Omega,\Lambda).$$

*Proof.* Suppose $Z \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)})\right)_h}(\Omega,\Lambda)$, then $(\widetilde{s_x(Z)}) \in \left(c_0^S(\nabla_q^2,(\tau_x^{(1)})\right)_h$. We have

$$\lim_{x\to\infty}\left[m_d\left(\left|\nabla^2\widetilde{s_x(Z)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(2)}} = \lim_{x\to\infty}\left[m_d\left(\left|\nabla_q^2\widetilde{s_x(Z)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(1)}} = 0,$$

then $Z \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda)$. Next, if we take $(\widetilde{s_x(Z)})_{x=0}^{\infty} = (\widetilde{0},\widetilde{1},\widetilde{2},\cdots)$, one has $Z \in \mathcal{L}(\Omega,\Lambda)$ so that

$$\lim_{x\to\infty}\left[m_d\left(\left|\nabla_q^2\widetilde{s_x(Z)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(1)}} \ne 0,$$

and

$$\lim_{x\to\infty}\left[m_d\left(\left|\nabla^2\widetilde{s_x(Z)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(2)}} = 0.$$

Therefore, $Z \notin \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)$ and $Z \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda)$.
Evidently, $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda) \subset \mathcal{L}(\Omega,\Lambda)$. After, if we choose $(\widetilde{s_x(Z)})_{x=0}^{\infty}$ so that $\left(\nabla^2\widetilde{s_x(Z)}\right) = (\widetilde{1},\widetilde{1},\cdots)$. One has $Z \in \mathcal{L}(\Omega,\Lambda)$ such that $Z \notin \widetilde{\mathcal{A}}_{\left(\chi_0^F(\nabla_q^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda)$. ■

**Lemma 4.10** ([36]). *Suppose $B \in \mathcal{L}(\Omega,\Lambda)$ and $B \notin \Upsilon(\Omega,\Lambda)$, then $D \in \mathcal{L}(\Omega)$ and $M \in \mathcal{L}(\Lambda)$ with $MBDe_b = e_b$, with $b \in \mathcal{N}$.*

**Theorem 4.11** ([36]). *In general, one has*

$$\mathfrak{F}(\Omega) \subsetneqq \Upsilon(\Omega) \subsetneqq \mathcal{L}_c(\Omega) \subsetneqq \mathcal{L}(\Omega).$$

**Theorem 4.12.** *If the conditions of Theorem 3.16 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, then*

$$\mathcal{L}\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$$
$$= \Upsilon\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big).$$

*Proof.* Let $X \in \mathcal{L}\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$ and
$X \notin \Upsilon\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$. In view of Lemma 4.10, one has $Y \in \mathcal{L}\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda)\Big)$ and $Z \in \mathcal{L}\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$ so that $ZXYI_b = I_b$, then with $b \in \mathcal{N}$, we have

$$\begin{aligned}
\|I_b\|_{\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)} &= \sup_x \left[m_d\left(\left|\nabla_q^2\widetilde{s_x(I_b)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(1)}} \\
&\le \|ZXY\|\|I_b\|_{\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda)} \\
&\le \sup_x \left[m_d\left(\left|\nabla^2\widetilde{s_x(I_b)}\right|,\widetilde{0}\right)\right]^{\tau_x^{(2)}}.
\end{aligned}$$

Which contradicts Theorem 4.9. As $X \in \Upsilon\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$.
■

**Corollary 4.13.** *Suppose the conditions of Theorem 3.16 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$, for every $x \in \mathcal{N}$, then*

$$\mathcal{L}\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big)$$
$$= \mathcal{L}_c\Big(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla^2,(\tau_x^{(2)}))\right)_h}(\Omega,\Lambda), \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,(\tau_x^{(1)}))\right)_h}(\Omega,\Lambda)\Big).$$

*Proof.* Obviously, since $\Upsilon \subset \mathcal{L}_c$. ■

**Definition 4.14.** [36] A Banach space $\Omega$ called simple if there is only one non-trivial closed ideal in $\mathcal{L}(\Omega)$.

**Theorem 4.15.** *Assume the conditions of Theorem 3.16 are verified, then* $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ *is simple.*

*Proof.* Let $X \in \mathcal{L}_c(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda))$ and $X \notin \Upsilon(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda))$. From Lemma 4.10, there exist $Y, Z \in \mathcal{L}(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda))$ with $ZXYI_b = I_b$. Which implies $I_{\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)} \in \mathcal{L}_c(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda))$. Then

$$\mathcal{L}(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)) = \mathcal{L}_c(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)),$$

then $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is a simple Banach space. ■

**Remark 4.16.**

$$\left(\widetilde{\mathcal{A}}_{\mathbf{U}}\right)^{\lambda} := \left\{\left(\widetilde{\mathcal{A}}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda);\ \Omega \text{ and } \Lambda \text{ are Banach Spaces}\right\}, \text{ where}$$

$$\left(\widetilde{\mathcal{A}}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda) = \Big\{X \in \mathcal{L}(\Omega,\Lambda) : ((\lambda_x(X))_{x=0}^{\infty} \in \mathbf{U} \text{ and } \|X - m_d(\lambda_x(X),\widetilde{0})\| \text{ is not invertible, with } x \in \mathcal{N}\Big\}.$$

**Theorem 4.17.** *If the conditions of Theorem 3.16 are satisfied, then*

$$\left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}\right)^{\lambda}(\Omega,\Lambda) = \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda).$$

*Proof.* Let $X \in \left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}\right)^{\lambda}(\Omega,\Lambda)$, then $(\lambda_x(X))_{x=0}^{\infty} \in \left(c_0^S(\nabla_p^2,\tau)\right)_h$ and $\|X - m_d(\lambda_x(X),\widetilde{0})I\| = 0$, for all $x \in \mathcal{N}$. Therefore, $\lim_{q\to\infty}\left[m_d\left(\left|\nabla_p^2|\lambda_q(X)|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}} = 0$. One has $X = m_d(\lambda_x(X),\widetilde{0})I$, for every $x \in \mathcal{N}$, so

$$m_d(\widetilde{s_x(X)},\widetilde{0}) = m_d(\widetilde{s_x(m_d(\lambda_x(X),\widetilde{0})I)},\widetilde{0}) = m_d(\lambda_x(X),\widetilde{0}),$$

for every $x \in \mathcal{N}$. Hence $(\widetilde{s_x(X)})_{x=0}^{\infty} \in \left(c_0^S(\nabla_p^2,\tau)\right)_h$, then $X \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$. After, assume $X \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$. Hence $(\widetilde{s_x(X)})_{x=0}^{\infty} \in \left(c_0^S(\nabla_p^2,\tau)\right)_h$. We have

$$\lim_{q\to\infty}\left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(X)}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}} = 0.$$

As $\nabla_p^2$ is continuous, then $\lim_{x\to\infty} m_d(\widetilde{s_x(X)},\widetilde{0}) = 0$. Suppose $\|X - m_d(\widetilde{s_x(X)},\widetilde{0})I\|^{-1}$ exists, with $x \in \mathcal{N}$. Hence $\|X - m_d(\widetilde{s_x(X)},\widetilde{0})I\|^{-1}$ exists and bounded, for every $x \in \mathcal{N}$.

As $\lim_{x\to\infty} \|X - m_d(\widetilde{s_x(X)}, \widetilde{0})I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $\left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}, H\right)$ is a pre-quasi Mappings ideal, one gets

$$I = XX^{-1} \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda) \Rightarrow (\widetilde{s_x(I)})_{x=0}^{\infty} \in c_0^S(\nabla_p^2,\tau) \Rightarrow \lim_{x\to\infty} m_d(\widetilde{s_x(I)}, \widetilde{0}) = 0.$$

We have a contradiction, since $\lim_{x\to\infty} m_d(\widetilde{s_x(I)}, \widetilde{0}) = 1$. Then $\|X - m_d(\widetilde{s_x(X)}, \widetilde{0})I\| = 0$, with $x \in \mathcal{N}$. Which proves that $X \in \left(\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}\right)^{\lambda}(\Omega,\Lambda)$. ∎

**Theorem 4.18.** *For $s-$ type $\mathbf{U}_h := \Big\{f = (\widetilde{s_r(X)}) \in \omega(S) :\ X \in \mathcal{L}(\Omega,\Lambda)$ and $h(f) < \infty\Big\}$. If $\widetilde{\mathcal{A}}_{\mathbf{U}_h}$ is a mappings' ideal, then the following conditions are verified:*

1. *$E \subset s-$ type $\mathbf{U}_h$.*
2. *Assume $\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$ and $\left(\widetilde{s_r(X_2)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$, then $\left(s_r(\widetilde{X_1+X_2})\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$.*
3. *If $\lambda \in \mathfrak{R}$ and $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$, then $|\lambda|\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$.*
4. *The sequence space $\mathbf{U}_h$ is solid. i.e., if $\left(\widetilde{s_r(Y)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$ and $\widetilde{s_r(X)} \le \widetilde{s_r(Y)}$, for all $r \in \mathcal{N}$ and $X, Y \in \mathcal{L}(\Omega,\Lambda)$, then $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$.*

*Proof.* If $\widetilde{\mathcal{A}}_{\mathbf{U}_h}$ is a mappings ideal.

**(i):** We have $\mathfrak{F}(\Omega,\Lambda) \subset \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$. Hence for all $X \in \mathfrak{F}(\Omega,\Lambda)$, we have $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in E$. This gives $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$. Hence $E \subset s-$ type $\mathbf{U}_h$.

**(ii):** The space $\widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$ is linear over $\mathfrak{R}$. Hence for each $\lambda \in \mathfrak{R}$ and $X_1, X_2 \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$, we have $X_1 + X_2 \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$ and $\lambda X_1 \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$. This implies

$$\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty} \in s-\text{type } \mathbf{U}_h \text{ and } \left(\widetilde{s_r(X_2)}\right)_{r=0}^{\infty} \in s-\text{type } \mathbf{U}_h$$
$$\Rightarrow \left(s_r(\widetilde{X_1+X_2})\right)_{r=0}^{\infty} \in s-\text{type } \mathbf{U}_h$$

and

$$\lambda \in \mathfrak{R} \text{ and } \left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty} \in s-\text{type } \mathbf{U}_h \Rightarrow |\lambda|\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty} \in s-\text{type } \mathbf{U}_h.$$

**(iii):** If $A \in \mathcal{L}(\Omega_0,\Omega)$, $B \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$ and $D \in \mathcal{L}(\Lambda,\Lambda_0)$, then $DBA \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega_0,\Lambda_0)$, where $\Omega_0$ and $\Lambda_0$ are arbitrary Banach spaces. Therefore, since $\left(\widetilde{s_r(B)}\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$, then $\left(s_r(\widetilde{DBA})\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$.
Since $s_r(\widetilde{DBA}) \widetilde{\le} \|D\|\widetilde{s_r(B)}\,\|A\|$. By using condition 3, if $\left(\|D\|\,\|A\|\widetilde{s_r(B)}\right)_{r=0}^{\infty} \in \mathbf{U}_h$, we have $\left(s_r(\widetilde{DBA})\right)_{r=0}^{\infty} \in s-$ type $\mathbf{U}_h$. This means $s-$ type $\mathbf{U}_h$ is solid.

∎

In view of Theorem 4.3 and Theorem 4.18, we conclude the following properties of the $s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ space.

**Theorem 4.19.** *If $s-$type $\left(c_0^S(\nabla_p^2,\tau)\right)_h := \Big\{f = (\widetilde{s_r(X)}) \in \omega(S):\ X \in \mathcal{L}(\Omega,\Lambda)$ and $h(f) < \infty\Big\}$,
then the following conditions are verified:*

1. *$E \subset s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$.*
2. *Assume $\left(\widetilde{s_r(X_1)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ and $\left(\widetilde{s_r(X_2)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$, then $\left(s_r(\widetilde{X_1+X_2})\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$.*
3. *If $\lambda \in \mathfrak{R}$ and $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$, then $|\lambda|\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$.*
4. *The sequence space $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ is solid. i.e., if $\left(\widetilde{s_r(Y)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ and $\widetilde{s_r(X)}\widetilde{\leq}\widetilde{s_r(Y)}$, for all $r \in \mathcal{N}$ and $X, Y \in \mathcal{L}(\Omega,\Lambda)$, then $\left(\widetilde{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\left(c_0^S(\nabla_p^2,\tau)\right)_h$.*

**Theorem 4.20.** *The space $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_q^2,\tau)\right)_h}$ is not mappings' ideal, if the conditions (a) and (c) of Theorem 3.16 are satisfied*

*Proof.* If we choose $m = 1$, $n = 1$, $w_k = \widetilde{1}$, $v_k = w_k$ for $k = 3s$ or $v_k = \widetilde{0}$, otherwise, for all $s, k \in \mathcal{N}$. We have $|v_k|\widetilde{\leq}|w_k|$, for all $k \in \mathcal{N}$, $w \in \left(c_0^S(\nabla_p^2,\tau)\right)_h$ and $v \notin \left(c_0^S(\nabla_p^2,\tau)\right)_h$. Hence the space $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ is not solid. ∎

## 5. Kannan Contraction Mapping on $c_0^S(\nabla_p^2,\tau)$

In this section, we look at how to configure $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ with different $h$ so that there is only one fixed point of Kannan contraction mapping. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. Interestingly, several numerical experiments are presented to illustrate our results.

**Definition 5.1.** An operator $V : \mathbf{U}_h \to \mathbf{U}_h$ is said to be a Kannan $h$-contraction, if one gets $\alpha \in [0, \frac{1}{2})$ with $h(V\widetilde{Y} - V\widetilde{Z}) \leq \alpha(h(V\widetilde{Y} - \widetilde{Y}) + h(V\widetilde{Z} - \widetilde{Z}))$, for all $\widetilde{Y}, \widetilde{Z} \in \mathbf{U}_h$. An element $\widetilde{Y} \in \mathbf{U}_h$ is called a fixed point of $V$, when $V(\widetilde{Y}) = \widetilde{Y}$.

**Theorem 5.2.** *If the conditions of Theorem 3.16 are satisfied, and $V : \left(c_0^S(\nabla_p^2,\tau)\right)_h \to \left(c_0^S(\nabla_p^2,\tau)\right)_h$ is Kannan $h$-contraction mapping, where $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, for all $\widetilde{Y} \in c_0^S(\nabla_p^2,\tau)$, then $V$ has a unique fixed point.*

*Proof.* If $\widetilde{Y} \in c_0^S(\nabla_p^2, \tau)$, one has $V^p\widetilde{Y} \in c_0^S(\nabla_p^2, \tau)$. As $V$ is a Kannan $h$-contraction mapping, one gets

$$h(V^{l+1}\widetilde{Y} - V^l\widetilde{Y}) \leq \alpha\left(h(V^{l+1}\widetilde{Y} - V^l\widetilde{Y}) + h(V^l\widetilde{Y} - V^{l-1}\widetilde{Y})\right) \Rightarrow$$

$$h(V^{l+1}\widetilde{Y} - V^l\widetilde{Y}) \leq \frac{\alpha}{1-\alpha}h(V^l\widetilde{Y} - V^{l-1}\widetilde{Y}) \leq \left(\frac{\alpha}{1-\alpha}\right)^2 h(V^{l-1}\widetilde{Y} - V^{l-2}\widetilde{Y}) \leq$$

$$\ldots \leq \left(\frac{\alpha}{1-\alpha}\right)^l h(V\widetilde{Y} - \widetilde{Y}).$$

So for all $l, m \in \mathcal{N}$ with $m > l$, one gets

$$h(V^l\widetilde{Y} - V^m\widetilde{Y}) \leq \alpha\left(h(V^l\widetilde{Y} - V^{l-1}\widetilde{Y}) + h(V^m\widetilde{Y} - V^{m-1}\widetilde{Y})\right)$$

$$\leq \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right) h(V\widetilde{Y} - \widetilde{Y}).$$

Then, $\{V^l\widetilde{Y}\}$ is a Cauchy sequence in $\left(c_0^S(\nabla_p^2, \tau)\right)_h$. As the space $\left(c_0^S(\nabla_p^2, \tau)\right)_h$ is pre-quasi Banach space. One has $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\lim_{l\to\infty} V^l\widetilde{Y} \widehat{=} \widetilde{Z}$. To prove that $V\widetilde{Z} \widehat{=} \widetilde{Z}$. Since $h$ has the Fatou property, one obtains

$$h(V\widetilde{Z} - \widetilde{Z}) \leq \sup_i \inf_{l\geq i} h(V^{l+1}\widetilde{Y} - V^l\widetilde{Y}) \leq \sup_i \inf_{l\geq i} \left(\frac{\alpha}{1-\alpha}\right)^l h(V\widetilde{Y} - \widetilde{Y}) = 0,$$

then $V\widetilde{Z} \widehat{=} \widetilde{Z}$. So $\widetilde{Z}$ is a fixed point of $V$. To show the uniqueness. Let $\widetilde{Y}, \widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ be two not equal fixed points of $V$. One has

$$h(\widetilde{Y} - \widetilde{Z}) \leq h(V\widetilde{Y} - V\widetilde{Z}) \leq \alpha\left(h(V\widetilde{Y} - \widetilde{Y}) + h(V\widetilde{Z} - \widetilde{Z})\right) = 0.$$

So, $\widetilde{Y} \widehat{=} \widetilde{Z}$. ∎

**Corollary 5.3.** *If the conditions of Theorem 3.16 are satisfied, and* $V : \left(c_0^S(\nabla_p^2, \tau)\right)_h \to \left(c_0^S(\nabla_p^2, \tau)\right)_h$ *is Kannan $h$-contraction mapping, where* $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, *for all* $\widetilde{Y} \in c_0^S(\nabla_p^2, \tau)$, *one has $V$ has unique fixed point $\widetilde{Z}$ so that* $h(V^l\widetilde{Y} - \widetilde{Z}) \leq \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(V\widetilde{Y} - \widetilde{Y})$.

*Proof.* In view of Theorem 5.2, one has a unique fixed point $\widetilde{Z}$ of $V$. So

$$h(V^l\widetilde{Y} - \widetilde{Z}) = h(V^l\widetilde{Y} - V\widetilde{Z}) \leq \alpha\left(h(V^l\widetilde{Y} - V^{l-1}\widetilde{Y}) + h(V\widetilde{Z} - \widetilde{Z})\right)$$

$$= \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(V\widetilde{Y} - \widetilde{Y}).$$

∎

**Example 5.4.** Assume $V:\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h\to\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$, where $h(\widetilde{g})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{g}_q|\right|,\widetilde{0}\right)\right]^{\frac{2q+3}{2q+4}}$, for every $\widetilde{g}\in c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$ and

$$V(\widetilde{g})=\begin{cases}\frac{\widetilde{g}}{4}, & h(\widetilde{g})\in[0,1),\\ \frac{\widetilde{g}}{5}, & h(\widetilde{g})\in[1,\infty).\end{cases}$$

As for each $\widetilde{g}_1,\widetilde{g}_2\in\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{g}_1),h(\widetilde{g}_2)\in[0,1)$, one has

$$\begin{aligned}h(V\widetilde{g}_1-V\widetilde{g}_2)&=h(\frac{\widetilde{g}_1}{4}-\frac{\widetilde{g}_2}{4})\le\frac{1}{\sqrt[4]{27}}\Big(h(\frac{3\widetilde{g}_1}{4})+h(\frac{3\widetilde{g}_2}{4})\Big)\\&=\frac{1}{\sqrt[4]{27}}\Big(h(V\widetilde{g}_1-\widetilde{g}_1)+h(V\widetilde{g}_2-\widetilde{g}_2)\Big).\end{aligned}$$

For all $\widetilde{g}_1,\widetilde{g}_2\in\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{g}_1),h(\widetilde{g}_2)\in[1,\infty)$, one has

$$\begin{aligned}h(V\widetilde{g}_1-V\widetilde{g}_2)&=h(\frac{\widetilde{g}_1}{5}-\frac{\widetilde{g}_2}{5})\le\frac{1}{\sqrt[4]{64}}\Big(h(\frac{4\widetilde{g}_1}{5})+h(\frac{4\widetilde{g}_2}{5})\Big)\\&=\frac{1}{\sqrt[4]{64}}\Big(h(V\widetilde{g}_1-\widetilde{g}_1)+h(V\widetilde{g}_2-\widetilde{g}_2)\Big).\end{aligned}$$

For all $\widetilde{g}_1,\widetilde{g}_2\in\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{g}_1)\in[0,1)$ and $h(\widetilde{g}_2)\in[1,\infty)$, we get

$$\begin{aligned}h(V\widetilde{g}_1-V\widetilde{g}_2)&=h(\frac{\widetilde{g}_1}{4}-\frac{\widetilde{g}_2}{5})\le\frac{1}{\sqrt[4]{27}}h(\frac{3\widetilde{g}_1}{4})+\frac{1}{\sqrt[4]{64}}h(\frac{4\widetilde{g}_2}{5})\\&\le\frac{1}{\sqrt[4]{27}}\Big(h(\frac{3\widetilde{g}_1}{4})+h(\frac{4\widetilde{g}_2}{5})\Big)\\&=\frac{1}{\sqrt[4]{27}}\Big(h(V\widetilde{g}_1-\widetilde{g}_1)+h(V\widetilde{g}_2-\widetilde{g}_2)\Big).\end{aligned}$$

Hence $V$ is Kannan $h$-contraction. As $h$ satisfies the Fatou property. From Theorem 5.2, one has $V$ satisfies one fixed point $\widetilde{\vartheta}\in\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$.

**Definition 5.5.** Pick up $\mathbf{U}_h$ be a pre-quasi normed (csss), $V:\mathbf{U}_h\to\mathbf{U}_h$ and $\widetilde{Z}\in\mathbf{U}_h$. The operator $V$ is called $h$-sequentially continuous at $\widetilde{Z}$, if and only if, when $\lim_{q\to\infty}h(\widetilde{Y_q}-\widetilde{Z})=0$, then $\lim_{q\to\infty}h(V\widetilde{Y_q}-V\widetilde{Z})=0$.

**Example 5.6.** Suppose

$$V:\left(c_0^S\left(\nabla_p^2,\left(\frac{q+1}{2q+4}\right)\right)\right)_h\to\left(c_0^S\left(\nabla_p^2,\left(\frac{q+1}{2q+4}\right)\right)\right)_h,$$

where

$$h(\widetilde{Z})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\frac{4q+4}{2q+4}},$$

for every $\widetilde{Z} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{q+1}{2q+4}\right)\right)\right)_h$ and

$$V(\widetilde{Z}) = \begin{cases} \frac{1}{18}(\widetilde{b}_0 + \widetilde{Z}), & \widetilde{Z}_0(a) \in [0, \frac{1}{17}), \\ \frac{1}{17}\widetilde{b}_0, & \widetilde{Z}_0(a) = \frac{1}{17}, \\ \frac{1}{18}\widetilde{b}_0, & \widetilde{Z}_0(a) \in (\frac{1}{17}, 1]. \end{cases}$$

$V$ is clearly both $h$-sequentially continuous and discontinuous at $\frac{1}{17}\widetilde{b}_0 \in \left(c_0^S\left(\nabla_p^2, \left(\frac{q+1}{2q+4}\right)\right)\right)_h$.

**Example 5.7.** Assume $V$ is defined as in Example 5.4. Suppose $\{\widetilde{Z^{(n)}}\} \subseteq \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ is such that $\lim_{n\to\infty} h(\widetilde{Z^{(n)}} - \widetilde{Z^{(0)}}) = 0$, where $Z^{(0)} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(Z^{(0)}) = 1$. As the pre-quasi norm $h$ is continuous, we have

$$\lim_{n\to\infty} h(V\widetilde{Z^{(n)}} - V\widetilde{Z^{(0)}}) = \lim_{n\to\infty} h\Big(\frac{\widetilde{Z^{(n)}}}{4} - \frac{\widetilde{Z^{(0)}}}{5}\Big) = h\Big(\frac{\widetilde{Z^{(0)}}}{20}\Big) > 0.$$

Therefore, $V$ is not $h$-sequentially continuous at $\widetilde{Z^{(0)}}$.

**Theorem 5.8.** *If the conditions of Theorem 3.16 are satisfied with $\tau_0 > 1$, and $V : \left(c_0^S(\nabla_p^2, \tau)\right)_h \to \left(c_0^S(\nabla_p^2, \tau)\right)_h$, where $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\tau_q}$, for all $\widetilde{Y} \in c_0^S(\nabla_p^2, \tau)$. Suppose*

**(1):** *$V$ is Kannan $h$-contraction mapping,*
**(2):** *$V$ is $h$-sequentially continuous at $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$,*
**(3):** *there is $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\{V^l\widetilde{Y}\}$ has $\{V^{l_j}\widetilde{Y}\}$ converging to $\widetilde{Z}$.*

*Then $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ is the only fixed point of $V$.*

*Proof.* Assume $\widetilde{Z}$ is not a fixed point of $V$, one has $V\widetilde{Z} \neq \widetilde{Z}$. From parts (2) and (3), we get

$$\lim_{l_j\to\infty} h(V^{l_j}\widetilde{Y} - \widetilde{Z}) = 0 \text{ and } \lim_{l_j\to\infty} h(V^{l_j+1}\widetilde{Y} - V\widetilde{Z}) = 0.$$

As $V$ is Kannan $h$-contraction, one obtains

$$\begin{aligned} 0 < h(V\widetilde{Z} - \widetilde{Z}) &= h\left((V\widetilde{Z} - V^{l_j+1}\widetilde{Y}) + (V^{l_j}\widetilde{Y} - \widetilde{Z}) + (V^{l_j+1}\widetilde{Y} - V^{l_j}\widetilde{Y})\right) \\ &\leq 2^{2\sup_i \tau_i - 2} h\left(V^{l_j+1}\widetilde{Y} - V\widetilde{Z}\right) + 2^{2\sup_i \tau_i - 2} h\left(V^{l_j}\widetilde{Y} - \widetilde{Z}\right) \\ &+ 2^{\sup_i \tau_i - 1}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j-1} h(V\widetilde{Y} - \widetilde{Y}). \end{aligned}$$

As $l_j \to \infty$, one has a contradiction. Then $\widetilde{Z}$ is a fixed point of $V$. To show that the uniqueness. Let $\widetilde{Z}, \widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ be two not equal fixed points of $V$. One obtains

$$h(\widetilde{Z} - \widetilde{Y}) \leq h(V\widetilde{Z} - V\widetilde{Y}) \leq \alpha\left(h(V\widetilde{Z} - \widetilde{Z}) + h(V\widetilde{Y} - \widetilde{Y})\right) = 0.$$

Hence $\widetilde{Z} = \widetilde{Y}$. ∎

**Example 5.9.** Assume $V$ is defined as in Example 5.4. Let $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{2q+3}{q+2}}$, for all $\widetilde{Y} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$. Since for all $\widetilde{Y_1}, \widetilde{Y_2} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{Y_1}), h(\widetilde{Y_2}) \in [0,1)$, one gets

$$\begin{aligned} h(V\widetilde{Y_1} - V\widetilde{Y_2}) &= h(\frac{\widetilde{Y_1}}{4} - \frac{\widetilde{Y_2}}{4}) \le \frac{2}{\sqrt{27}}\Big(h(\frac{3\widetilde{Y_1}}{4}) + h(\frac{3\widetilde{Y_2}}{4})\Big) \\ &= \frac{2}{\sqrt{27}}\Big(h(V\widetilde{Y_1} - \widetilde{Y_1}) + h(V\widetilde{Y_2} - \widetilde{Y_2})\Big). \end{aligned}$$

For all $\widetilde{Y_1}, \widetilde{Y_2} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{Y_1}), h(\widetilde{Y_2}) \in [1,\infty)$, one gets

$$\begin{aligned} h(V\widetilde{Y_1} - V\widetilde{Y_2}) &= h(\frac{\widetilde{Y_1}}{5} - \frac{\widetilde{Y_2}}{5}) \le \frac{1}{4}\Big(h(\frac{4\widetilde{Y_1}}{5}) + h(\frac{4\widetilde{Y_2}}{5})\Big) \\ &= \frac{1}{4}\Big(h(V\widetilde{Y_1} - \widetilde{Y_1}) + h(V\widetilde{Y_2} - \widetilde{Y_2})\Big). \end{aligned}$$

For all $\widetilde{Y_1}, \widetilde{Y_2} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ with $h(\widetilde{Y_1}) \in [0,1)$ and $h(\widetilde{Y_2}) \in [1,\infty)$, one gets

$$\begin{aligned} h(V\widetilde{Y_1} - V\widetilde{Y_2}) &= h(\frac{\widetilde{Y_1}}{4} - \frac{\widetilde{Y_2}}{5}) \le \frac{2}{\sqrt{27}}h(\frac{3\widetilde{Y_1}}{4}) + \frac{1}{4}h(\frac{4\widetilde{Y_2}}{5}) \\ &\le \frac{2}{\sqrt{27}}\Big(h(\frac{3\widetilde{Y_1}}{4}) + h(\frac{4\widetilde{Y_2}}{5})\Big) \\ &= \frac{2}{\sqrt{27}}\Big(h(V\widetilde{Y_1} - \widetilde{Y_1}) + h(V\widetilde{Y_2} - \widetilde{Y_2})\Big). \end{aligned}$$

So $V$ is Kannan $h$-contraction and $V^p(\widetilde{Y}) = \begin{cases} \frac{\widetilde{Y}}{4^p}, & h(\widetilde{Y}) \in [0,1), \\ \frac{\widetilde{Y}}{5^p}, & h(\widetilde{Y}) \in [1,\infty). \end{cases}$ Obviously, $V$ is $h$-sequentially continuous at $\widetilde{\vartheta} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ and $\{V^p\widetilde{Y}\}$ satisfies $\{V^{l_j}\widetilde{Y}\}$ converges to $\widetilde{\vartheta}$. By Theorem 5.8, the point $\widetilde{\vartheta} \in \left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$ is the only fixed point of $V$.

**Definition 5.10.** An operator $V : \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$ is said to be a Kannan $H$-contraction, if one has $\alpha \in [0, \frac{1}{2})$ with $H(VT - VM) \le \alpha\Big(H(VT - T) + H(VM - M)\Big)$, for all $T, M \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$.

**Definition 5.11.** An operator $V : \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$ is said to be $H$-sequentially continuous at $M$, where $M \in \widetilde{\mathcal{A}}_{\mathbf{U}_h}(\Omega,\Lambda)$, if and only if, $\lim_{r\to\infty} H(T_r - M) = 0 \Rightarrow \lim_{r\to\infty} H(VT_r - VM) = 0$.

**Example 5.12.** If $V : \widetilde{\mathcal{A}}_{\left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h}(\Omega,\Lambda)$, where

$$H(T) = \sup_q \left[m_d\left(\left|\nabla_p^2 \widetilde{s_q(T)}\right|, \widetilde{0}\right)\right]^{\frac{2q+3}{2q+4}},$$

for every $T \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}(\Omega,\Lambda)$ and

$$V(T) = \begin{cases} \frac{T}{6}, & H(T) \in [0,1), \\ \frac{T}{7}, & H(T) \in [1,\infty). \end{cases}$$

Evidently, $V$ is $H$-sequentially continuous at the zero operator $\Theta \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$. Let $\{T^{(j)}\} \subseteq \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ be such that $\lim_{j\to\infty} H(T^{(j)} - T^{(0)}) = 0$, where $T^{(0)} \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ with $H(T^{(0)}) = 1$. Since the pre-quasi norm $H$ is continuous, one gets

$$\lim_{j\to\infty} H(VT^{(j)} - VT^{(0)}) = \lim_{j\to\infty} H\Big(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\Big) = H\Big(\frac{T^{(0)}}{42}\Big) > 0.$$

Therefore, $V$ is not $H$-sequentially continuous at $T^{(0)}$.

**Theorem 5.13.** *Suppose the conditions of Theorem 3.16 are satisfied and $V : \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$. Assume*

**(i):** *$V$ is Kannan $H$-contraction mapping,*

**(ii):** *$V$ is $H$-sequentially continuous at an element $M \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$,*

**(iii):** *there are $G \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ such that the sequence of iterates $\{V^r G\}$ has a $\{V^{r_m} G\}$ converging to $M$.*

*Then $M \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ is the unique fixed point of $V$.*

*Proof.* Let $M$ be not a fixed point of $V$, hence $VM \neq M$. By using parts (ii) and (iii), we get

$$\lim_{r_m\to\infty} H(V^{r_m}G - M) = 0 \text{ and } \lim_{r_m\to\infty} H(V^{r_m+1}G - VM) = 0.$$

Since $V$ is Kannan $H$-contraction, one obtains

$$\begin{aligned} 0 < H(VM - M) &= H\left((VM - V^{r_m+1}G) + (V^{r_m}G - M) + (V^{r_m+1}G - V^{r_m}G)\right) \\ &\leq 2H\left(V^{r_m+1}G - VM\right) + 4H\left(V^{r_m}G - M\right) + 4\alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_m-1} H(VG - G). \end{aligned}$$

As $r_m \to \infty$, there is a contradiction. Hence $M$ is a fixed point of $V$. To prove the uniqueness of the fixed point $M$. Suppose one has two not equal fixed points $M, J \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ of $V$. So, one gets

$H(M - J) \leq H(VM - VJ) \leq \alpha\Big(H(VM - M) + H(VJ - J)\Big) = 0$. Then, $M = J$. ■

**Example 5.14.** Given example 5.12. Since for all $T_1, T_2 \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ with $H(T_1), H(T_2) \in [0,1)$, we have

$$H(VT_1 - VT_2) = H(\frac{T_1}{6} - \frac{T_2}{6}) \leq \frac{1}{\sqrt[4]{125}}\Big(H(\frac{5T_1}{6}) + H(\frac{5T_2}{6})\Big)$$
$$= \frac{1}{\sqrt[4]{125}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

For all $T_1, T_2 \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ with $H(T_1), H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H(\frac{T_1}{7} - \frac{T_2}{7}) \leq \frac{1}{\sqrt[4]{216}}\Big(H(\frac{6T_1}{7}) + H(\frac{6T_2}{7})\Big)$$
$$= \frac{1}{\sqrt[4]{216}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

For all $T_1, T_2 \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ with $H(T_1) \in [0,1)$ and $H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H(\frac{T_1}{6} - \frac{T_2}{7}) \leq \frac{1}{\sqrt[4]{125}}H(\frac{5T_1}{6}) + \frac{1}{\sqrt[4]{216}}H(\frac{6T_2}{7})$$
$$\leq \frac{1}{\sqrt[4]{125}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

Hence $V$ is Kannan $H$-contraction and $V^r(T) = \begin{cases} \frac{T}{6^r}, & H(T) \in [0,1), \\ \frac{T}{7^r}, & H(T) \in [1,\infty). \end{cases}$

Obviously, $V$ is $H$-sequentially continuous at $\Theta \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2q+3}{q+2}))\right)_h}$ and $\{V^rT\}$ has a subsequence $\{V^{r_m}T\}$ converges to $\Theta$. By Theorem 5.13, $\Theta$ is the only fixed point of $G$.

## 6. Applications

This section introduces some successful applications to the existence of solutions of nonlinear difference equations of soft functions.

**Theorem 6.1.** *Consider the summable equations*

$$\widetilde{Y_q} = \widetilde{R_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Y_r}), \tag{6.1}$$

*which presented by Salimi et al.* [41], *and assume* $V : \left(c_0^S(\nabla_p^2,\tau)\right)_h \to \left(c_0^S(\nabla_p^2,\tau)\right)_h$,

*where the conditions of Theorem* 3.16 *are satisfied and* $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$,

*for every* $\widetilde{Y} \in c_0^S(\nabla_p^2,\tau)$, *defined by*

$$V(\widetilde{Y_q})_{q\in\mathcal{N}} = \left(\widetilde{R_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Y_r})\right)_{q\in\mathcal{N}}. \tag{6.2}$$

*The summable equation* (6.1) *has a unique solution in* $\left(c_0^S(\nabla_p^2,\tau)\right)_h$, *if* $D:\mathcal{N}^2\to\mathfrak{R}$, $m:\mathcal{N}\times\mathfrak{R}(A)\to\mathfrak{R}(A)$, $\widetilde{R}:\mathcal{N}\to\mathfrak{R}(A)$, $\widetilde{Z}:\mathcal{N}\to\mathfrak{R}(A)$, *there is* $\varepsilon$ *so that* $\sup_q\varepsilon^{\frac{\tau_q}{K}}\in[0,0.5)$, *and for all* $q\in\mathcal{N}$ *we have*

$$\begin{aligned}&\left|\sum_{r\in\mathcal{N}}D(q,r)(m(r,\widetilde{Y_r})-m(r,\widetilde{Z_r}))\right|\\&\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\widetilde{Y_r})\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\widetilde{Z_r})\right|\right].\end{aligned}$$

*Proof.* One has

$$\begin{aligned}h(V\widetilde{Y}-V\widetilde{Z})&=\sup_q\left[m_d\left(\left|\nabla_p^2|V\widetilde{Y_q}-V\widetilde{Z_q}|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&=\sup_q\left[m_d\left(\left|\nabla_p^2\left|\sum_{r\in\mathcal{N}}D(q,r)(m(r,\widetilde{Y_r})-m(r,\widetilde{Z_r}))\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&\leq\sup_q\varepsilon^{\frac{\tau_q}{K}}\sup_q\left[m_d\left(\left|\nabla_p^2\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\widetilde{Y_r})\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&\quad+\sup_q\varepsilon^{\frac{\tau_q}{K}}\sup_q\left[m_d\left(\left|\nabla_p^2\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\widetilde{Z_r})\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_q}{K}}\\&=\sup_q\varepsilon^{\frac{\tau_q}{K}}\left(h(V\widetilde{Y}-\widetilde{Y})+h(V\widetilde{Z}-\widetilde{Z})\right).\end{aligned}$$

By Theorem 5.2, one gets a unique solution of equation (6.1) in $\left(c_0^S(\nabla_p^2,\tau)\right)_h$. ■

**Example 6.2.** Suppose $\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$, where $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{2q+3}{2q+4}}$, for all $\widetilde{Y}\in c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$. Consider the summable equations

$$\widetilde{Y_q}=\widetilde{R_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t,\tag{6.3}$$

with $t>0$. Let $V:c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\to c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$ defined by

$$V(\widetilde{Y_q})=\left(\widetilde{R_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right).\tag{6.4}$$

Obviously

$$\begin{aligned}&\left|\sum_{r=0}^{\infty}(-1)^q\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\left((-1)^r-(-1)^r\right)\right|\\&\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Z_q}}{q^2+r^2+1}\right)^t\right|\right].\end{aligned}$$

By Theorem 6.1, the summable equations (6.5) have a unique solution in $c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$.

**Example 6.3.** Suppose $\left(c_0^S\left(\nabla_p^2,\left(\frac{q+3}{2q+4}\right)\right)\right)_h$, where $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{q+3}{2q+4}}$, for all $\widetilde{Y}\in c_0^S\left(\nabla_p^2,\left(\frac{q+3}{2q+4}\right)\right)$. Consider the summable equations

$$\widetilde{Y_q}=\widetilde{R_q}+\sum_{r=0}^{\infty}e^{q+r}\Big(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\Big)^t, \tag{6.5}$$

with $t>0$. Let $V:c_0^S\left(\nabla_p^2,\left(\frac{q+3}{2q+4}\right)\right)\to c_0^S\left(\nabla_p^2,\left(\frac{q+3}{2q+4}\right)\right)$ defined by

$$V(\widetilde{Y_q})=\Big(\widetilde{R_q}+\sum_{r=0}^{\infty}e^{q+r}\Big(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\Big)^t\Big). \tag{6.6}$$

Obviously

$$\left|\sum_{r=0}^{\infty}e^{q}\Big(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\Big)^t\Big(e^r-e^r\Big)\right|$$
$$\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}e^{q+r}\Big(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\Big)^t\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}e^{q+r}\Big(\frac{Z_q^5}{Z_q^3+\widetilde{Z_r}^2+\widetilde{1}}\Big)^t\right|\right].$$

By Theorem 6.1, the summable equations (6.5) have a unique solution in $c_0^S\left(\nabla_p^2,\left(\frac{q+3}{2q+4}\right)\right)$.

**Example 6.4.** Suppose $\left(c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\right)_h$, where $h(\widetilde{Y})=\sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|,\widetilde{0}\right)\right]^{\frac{2q+3}{2q+4}}$, for every $\widetilde{Y}\in c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$. Consider the non-linear difference equations:

$$\widetilde{Y_q}=\widetilde{R_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u}+\widetilde{l^2+1}}, \tag{6.7}$$

with $r,u>0$, $\widetilde{Y_{-2}}(a),\widetilde{Y_{-1}}(a)>0$, for all $a\in A$, and assume $V:c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)\to c_0^S\left(\nabla_p^2,\left(\frac{2q+3}{q+2}\right)\right)$, defined by

$$V(\widetilde{Y_q})_{q=0}^{\infty}=\Big(\widetilde{R_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u}+\widetilde{l^2+1}}\Big)_{q=0}^{\infty}. \tag{6.8}$$

Evidently

$$\left|\sum_{l=0}^{\infty}(-1)^{q}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u}+\widetilde{l^2+1}}\Big((-1)^l-(-1)^l\Big)\right|$$
$$\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u}+\widetilde{l^2+1}}\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Z_{q-2}^r}}{\widetilde{Z_{q-1}^u}+\widetilde{l^2+1}}\right|\right].$$

By Theorem 6.1, the non-linear difference equations (6.7) have a unique solution in $c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$.

**Theorem 6.5.** *Consider the summable equations (6.1), and assume* $V : \left(c_0^S(\nabla_p^2, \tau)\right)_h \to \left(c_0^S(\nabla_p^2, \tau)\right)_h$ *is defined by (6.2), where the conditions of Theorem 3.16 are satisfied with* $\tau_0 > 1$ *and* $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\tau_q}$, *for every* $\widetilde{Y} \in c_0^S(\nabla_p^2, \tau)$. *The summable equation (6.1) has a unique solution* $Z \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, *if the following conditions are satisfied:*

**(1):** *If* $D : \mathcal{N}^2 \to \mathfrak{R}$, $m : \mathcal{N} \times \mathfrak{R}(A) \to \mathfrak{R}(A)$, $\widetilde{R} : \mathcal{N} \to \mathfrak{R}(A)$, $\widetilde{Z} : \mathcal{N} \to \mathfrak{R}(A)$, *there is* $\varepsilon$ *so that* $2^{K-1}\sup_q \varepsilon^{\tau_q} \in [0, 0.5)$, *and for all* $q \in \mathcal{N}$ *we have*

$$\left|\sum_{r\in\mathcal{N}} D(q,r)(m(r,\widetilde{Y_r}) - m(r,\widetilde{Z_r}))\right|$$
$$\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q} - \widetilde{Y_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Y_r})\right| + \left|\widetilde{R_q} - \widetilde{Z_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Z_r})\right|\right],$$

**(2):** $V$ *is h-sequentially continuous at* $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$,

**(3):** *there is* $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ *with* $\{V^l\widetilde{Y}\}$ *has* $\{V^{l_j}\widetilde{Y}\}$ *converging to* $\widetilde{Z}$.

*Proof.* One has

$$h(V\widetilde{Y} - V\widetilde{Z}) = \sup_q \left[m_d\left(\left|\nabla_p^2|V\widetilde{Y_q} - V\widetilde{Z_q}|\right|, \widetilde{0}\right)\right]^{\tau_q}$$
$$= \sup_q \left[m_d\left(\left|\nabla_p^2\left|\sum_{r\in\mathcal{N}} D(q,r)(m(r,\widetilde{Y_r}) - m(r,\widetilde{Z_r}))\right|\right|, \widetilde{0}\right)\right]^{\tau_q}$$
$$\leq 2^{K-1}\sup_q \varepsilon^{\tau_q}\sup_q \left[m_d\left(\left|\nabla_p^2\left|\widetilde{R_q} - \widetilde{Y_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Y_r})\right|\right|, \widetilde{0}\right)\right]^{\tau_q}$$
$$+ 2^{K-1}\sup_q \varepsilon^{\tau_q}\sup_q \left[m_d\left(\left|\nabla_p^2\left|\widetilde{R_q} - \widetilde{Z_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\widetilde{Z_r})\right|\right|, \widetilde{0}\right)\right]^{\tau_q}$$
$$= 2^{K-1}\sup_q \varepsilon^{\tau_q}\left(h(V\widetilde{Y} - \widetilde{Y}) + h(V\widetilde{Z} - \widetilde{Z})\right).$$

By Theorem 5.8, one gets a unique solution $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ of equation (6.1). ∎

**Example 6.6.** Suppose $\left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$, where $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{2q+3}{q+2}}$, for all $\widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$. Consider the summable equations

$$\widetilde{Y_q} = \widetilde{R_q} + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2 + r^2 + 1}\right)^t, \tag{6.9}$$

with $t > 0$. Let $V : c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right) \to c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$ defined by

$$V(\widetilde{Y_q}) = \left(\widetilde{R_q} + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right). \tag{6.10}$$

Assume $V$ is $h$-sequentially continuous at $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, and there is $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\{V^l\widetilde{Y}\}$ has $\{V^{l_j}\widetilde{Y}\}$ converging to $\widetilde{Z}$. Obviously

$$\begin{aligned} &\left|\sum_{r=0}^{\infty}(-1)^q\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\left((-1)^r-(-1)^r\right)\right| \\ &\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Z_q}}{q^2+r^2+1}\right)^t\right|\right]. \end{aligned}$$

By Theorem 6.5, the summable equations (6.9) have a unique solution $\widetilde{Z} \in c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$.

**Example 6.7.** Suppose $\left(c_0^S\left(\nabla_p^2, \left(\frac{5q+3}{q+1}\right)\right)\right)_h$, where $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{5q+3}{q+1}}$, for all $\widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{5q+3}{q+1}\right)\right)$. Consider the summable equations

$$\widetilde{Y_q} = \widetilde{R_q} + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\right)^t, \tag{6.11}$$

with $t > 0$. Let $V : c_0^S\left(\nabla_p^2, \left(\frac{5q+3}{q+1}\right)\right) \to c_0^S\left(\nabla_p^2, \left(\frac{5q+3}{q+1}\right)\right)$ defined by

$$V(\widetilde{Y_q}) = \left(\widetilde{R_q} + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\right)^t\right). \tag{6.12}$$

Assume $V$ is $h$-sequentially continuous at $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, and there is $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\{V^l\widetilde{Y}\}$ has $\{V^{l_j}\widetilde{Y}\}$ converging to $\widetilde{Z}$. Obviously

$$\begin{aligned} &\left|\sum_{r=0}^{\infty} e^q\left(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\right)^t\left(e^r-e^r\right)\right| \\ &\widetilde{\leq}\ \varepsilon\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty} e^{q+r}\left(\frac{\widetilde{Y_q^5}}{\widetilde{Y_q^3}+\widetilde{Y_r}^2+\widetilde{1}}\right)^t\right|+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty} e^{q+r}\left(\frac{Z_q^5}{Z_q^3+\widetilde{Z_r}^2+\widetilde{1}}\right)^t\right|\right]. \end{aligned}$$

By Theorem 6.5, the summable equations (6.11) have a unique solution $\widetilde{Z} \in c_0^S\left(\nabla_p^2, \left(\frac{5q+3}{q+1}\right)\right)$.

**Example 6.8.** Suppose $\left(c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)\right)_h$, where $h(\widetilde{Y}) = \sup_q \left[m_d\left(\left|\nabla_p^2|\widetilde{Y_q}|\right|, \widetilde{0}\right)\right]^{\frac{2q+3}{q+2}}$, for every $\widetilde{Y} \in c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$. Consider the non-linear difference equations:

$$\widetilde{Y_q} = \widetilde{R_q} + \sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u}+\widetilde{l^2+1}}, \tag{6.13}$$

with $r, u > 0$, $\widetilde{Y_{-2}}(a), \widetilde{Y_{-1}}(a) > 0$, for all $a \in A$, and assume $V : c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right) \to c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$, defined by

$$V(\widetilde{Y_q})_{q=0}^{\infty} = \left(\widetilde{R_q} + \sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u} + \widetilde{l^2+1}}\right)_{q=0}^{\infty}. \tag{6.14}$$

Suppose $V$ is $h$-sequentially continuous at $\widetilde{Z} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$, and there is $\widetilde{Y} \in \left(c_0^S(\nabla_p^2, \tau)\right)_h$ with $\{V^l\widetilde{Y}\}$ has $\{V^{l_j}\widetilde{Y}\}$ converging to $\widetilde{Z}$. Evidently

$$\begin{aligned}
&\left|\sum_{l=0}^{\infty}(-1)^{q}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u} + \widetilde{l^2+1}}\left((-1)^l - (-1)^l\right)\right| \\
&\precsim \varepsilon\left[\left|\widetilde{R_q} - \widetilde{Y_q} + \sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^u} + \widetilde{l^2+1}}\right| + \left|\widetilde{R_q} - \widetilde{Z_q} + \sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Z_{q-2}^r}}{\widetilde{Z_{q-1}^u} + \widetilde{l^2+1}}\right|\right].
\end{aligned}$$

By Theorem 6.5, the non-linear difference equations (6.15) have a unique solution $\widetilde{Z} \in c_0^S\left(\nabla_p^2, \left(\frac{2q+3}{q+2}\right)\right)$.

In this part, we search for a solution to nonlinear matrix equations (6.7) at $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$, where $\Omega$ and $\Lambda$ are Banach spaces, the conditions of theorem 3.16 are satisfied, and
$\Psi(G) = \sup_q\left[m_d\left(\left|\nabla_p^2|\widetilde{s_q(G)}|\right|, \widetilde{0}\right)\right]^{\frac{\tau_q}{K}}$, for all $G \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$. Consider the summable equations

$$\widetilde{s_a(G)} = \widetilde{s_a(P)} + \sum_{m=0}^{\infty} A(a, m)f(m, \widetilde{s_m(G)}), \tag{6.15}$$

and suppose $W : \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda) \to \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$ is defined by

$$W(G) = \left(\widetilde{s_a(P)} + \sum_{m=0}^{\infty} A(a, m)f(m, \widetilde{s_m(G)})\right)I. \tag{6.16}$$

**Theorem 6.9.** *The summable equations (6.15) have one solution $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega, \Lambda)$, if the following conditions are satisfied:*

**(a):** *$A:\mathcal{N}^2\to\mathfrak{R}$, $f:\mathcal{N}\times\mathfrak{R}(A)\to\mathfrak{R}(A)$, $P\in\mathcal{L}(\Omega,\Lambda)$, $T\in\mathcal{L}(\Omega,\Lambda)$, and for every $a\in\mathcal{N}$, there is $\kappa$ so that $\sup_a\kappa^{\tau_a}\in[0,0.5)$, with*

$$\left|\sum_{m\in\mathcal{N}}A(a,m)\left(f(m,\widetilde{s_m(G)})-f(m,\widetilde{s_m(T)})\right)\right|$$
$$\widetilde{\leq}\kappa^K\left|\widetilde{s_a(P)}-\widetilde{s_a(G)}+\sum_{m\in\mathcal{N}}A(a,m)f(m,\widetilde{s_m(G)})\right|$$
$$+\kappa^K\left|\widetilde{s_a(P)}-\widetilde{s_a(T)}+\sum_{m\in\mathcal{N}}A(a,m)f(m,\widetilde{s_m(T)})\right|,$$

**(b):** *$W$ is $\Psi$-sequentially continuous at a point $D\in\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$,*

**(c):** *there is $B\in\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ so that the sequence of iterates $\{W^aB\}$ has a subsequence $\{W^{a_i}B\}$ converging to $D$.*

*Proof.* Suppose the settings are verified. Consider the mapping $W:\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)\to\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$ defined by (6.16). We have

$$\begin{aligned}\Psi(WG-WT)&=\sup_a\left[m_d\left(\left|\nabla_p^2\left|\sum_{m\in\mathcal{N}}A(a,m)\left(f(m,\widetilde{s_m(G)})-f(m,\widetilde{s_m(T)})\right)\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_a}{K}}\\
&\leq\sup_a\kappa^{\tau_a}\sup_a\left[m_d\left(\left|\nabla_p^2\left|\widetilde{s_a(P)}-\widetilde{s_a(G)}+\sum_{m\in\mathcal{N}}A(a,m)f(m,\widetilde{s_m(G)})\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_a}{K}}+\\
&\quad\sup_a\kappa^{\tau_a}\sup_a\left[m_d\left(\left|\nabla_p^2\left|\widetilde{s_a(T)}-\widetilde{s_a(G)}+\sum_{m\in\mathcal{N}}A(a,m)f(m,\widetilde{s_m(T)})\right|\right|,\widetilde{0}\right)\right]^{\frac{\tau_a}{K}}\\
&=\sup_a\kappa^{\tau_a}\left(\Psi(WG-G)+\Psi(WT-T)\right).\end{aligned}$$

In view of Theorem 5.13, one obtains a unique solution of equation (6.15) at $D\in\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}(\Omega,\Lambda)$. ■

**Example 6.10.** Assume the class $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^\infty)\right)_h}(\Omega,\Lambda)$, where

$\Psi(G)=\sup_a\left[m_d\left(\left|\nabla_p^2|\widetilde{s_a(G)}|\right|,\widetilde{0}\right)\right]^{\frac{a+1}{a+2}}$, for all $G\in\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^\infty)\right)_h}(\Omega,\Lambda)$.

Consider the non-linear difference equations:

$$\widetilde{s_a(G)}=e^{\widetilde{-(2a+3)}}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}|+\sin ma+\widetilde{1}},\tag{6.17}$$

where $a \geq 2$ and $r, q > 0$ and let $W : \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$ be defined as

$$W(G) = \left(\widetilde{e^{-(2a+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}| + \widetilde{\sin ma} + \widetilde{1}}\right) I. \qquad (6.18)$$

Suppose $W$ is $\Psi$-sequentially continuous at a point $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$, and there is $B \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to $D$. It is easy to see that

$$\begin{aligned}
&\left|\sum_{m=0}^{\infty} \frac{\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}| + \widetilde{\sin ma} + \widetilde{1}}\Big(\tan(2m+1) - \tan(2m+1)\Big)\right|^{\frac{a+1}{a+2}} \\
&\leq \frac{1}{5}\left|\widetilde{e^{-(2a+3)}} - \widetilde{s_a(G)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}| + \widetilde{\sin ma} + \widetilde{1}}\right|^{\frac{a+1}{a+2}} \\
&\quad + \frac{1}{5}\left|\widetilde{e^{-(2a+3)}} - \widetilde{s_a(T)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(T)}|}{\sinh^q|\widetilde{s_{a-1}(T)}| + \widetilde{\sin ma} + \widetilde{1}}\right|^{\frac{a+1}{a+2}}.
\end{aligned}$$

By Theorem 6.9, the non-linear difference equations (6.17) have one solution $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{a+1}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$.

**Example 6.11.** Assume the class $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$, where

$\Psi(G) = \sup_a \left[m_d\left(\left|\nabla_p^2|\widetilde{s_a(G)}|\right|, \widetilde{0}\right)\right]^{\frac{2a+3}{2a+4}}$, for all $G \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$.

Consider the non-linear difference equations (6.17) and let

$$W : \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda) \to \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$$

be defined as (6.18). Suppose $W$ is $\Psi$-sequentially continuous at a point $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$, and there is $B \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,(\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega,\Lambda)$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to $D$. It is easy to see that

$$\left|\sum_{m=0}^{\infty} \frac{\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}| + \widetilde{\sin ma} + \widetilde{1}}\Big(\tan(2m+1) - \tan(2m+1)\Big)\right|^{\frac{2a+3}{2a+4}}$$
$$\leq \frac{1}{25}\left|\widetilde{e^{-(2a+3)}} - \widetilde{s_a(G)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(G)}|}{\sinh^q|\widetilde{s_{a-1}(G)}| + \widetilde{\sin ma} + \widetilde{1}}\right|^{\frac{2a+3}{2a+4}}$$
$$+ \frac{1}{25}\left|\widetilde{e^{-(2a+3)}} - \widetilde{s_a(T)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1)\cosh(3m-a)\cos^r|\widetilde{s_{a-2}(T)}|}{\sinh^q|\widetilde{s_{a-1}(T)}| + \widetilde{\sin ma} + \widetilde{1}}\right|^{\frac{2a+3}{2a+4}}.$$

By Theorem 6.9, the non-linear difference equations (6.17) have one solution $D \in \widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2, (\frac{2a+3}{a+2})_{a=0}^{\infty})\right)_h}(\Omega, \Lambda)$.

## 7. Conclusion

In this paper, we have explained sufficient settings of the space $c_0^S(\nabla_p^2, \tau)$ equipped with the definite function $h$ to be pre-quasi Banach (csss). The Fatou property of various pre-quasi norms $h$ on $c_0^S(\nabla_p^2, \tau)$ has been investigated. The structure of the mappings ideal by this space and extended $s-$soft functions have been explained. We study enough setups on it such that the class $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is simple Banach and the closure of $\mathfrak{F} = \widetilde{\mathcal{A}^{\alpha}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$. We explain enough setups on it such that $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$ is strictly contained for different powers and backward generalized differences, and the space of every bounded linear mappings which sequence of eigenvalues in $\left(c_0^S(\nabla_p^2,\tau)\right)_h$ equals $\widetilde{\mathcal{A}}_{\left(c_0^S(\nabla_p^2,\tau)\right)_h}$. The existing results may be established under a wide range of flexible conditions. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of soft functions are introduced.

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