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New Treatment of the Perturbed Motions of a Rotating Symmetric Gyrostat about a Fixed Point

T. S. Amer

Abstract : In the present paper, we investigate the perturbed rotational motions of a symmetric gyrostat about a fixed point O, which are close to Lagrange's case. This gyrostat is acted upon by a central Newtonian force field arising from an attracting centre O_1 which is located on a downward fixed axis ($Z-$ axis); a third component of the gyrostatic moment vector $\underline{\ell}$ about the moving axis ($z-$ axis); restoring moment and perturbing moment vector $\underline{M}$. The moment k is introduced to express the rotation of the body under the action of uniform magnetic field of strength $\underline{H}$ and a point charge e located on the axis of symmetry. It is assumed that the angular velocity is large, its direction is close to the axis of dynamic symmetry of the body and that two projections of the perturbing moment vector onto the principal axes of inertia of the body are small as compared to the restoring moment k while the third one is of the same order as it. A small parameter is introduced in a special way and the averaging method is used to obtain the first order approximate solutions of the equations of motion. A theoretical description for this approach in the resonant and non-resonant cases is given. The graphical representations for these solutions are presented to describe the gyrostatic motion at any time.

Keywords : Euler's Equations, Rigid body, Averaging Method.
2000 Mathematics Subject Classification : 70E15, 70D05, 73V20.

1 Introduction

The considered problem had shed the interest of many researches. In [1], the perturbed motion of a heavy rigid body close to Lagrange's case was considered when the gyrostat is acted upon by a constant restoring moment which is generated by a force of constant magnitude and direction. The analytic solutions of the equations of motion were obtained using the averaging method [2]-[4] up to the first approximation. The same problem was investigated in [5] and [6] when the restoring moment is depending on the nutation angle θ. Averaged systems of the equations of motion are obtained in terms of the small parameter ε. The perturbed rotational motion of a symmetric gyrostat about a fixed point was studied in [7] when the third component of the gyrostatic moment vector $\underline{\ell}$; $(\ell_1 = \ell_2 = 0,\ \ell_3 \neq 0)$ is acted. In these works, the mentioned problem was studied under the action of perturbing moment vector $\underline{M}$ and some conditions that permits to introduce the small parameter ε. The scope of this paper, is to investigate the rotational motion of a symmetric gyrostat about a fixed point under the influence of Newtonian force field; the third component of the gyrostatic moment vector $\underline{\ell}$; some constant and linear dissipative moments acting in the same direction of the principal axes of the gyrostat; a variable restoring moment k which is the result of uniform electromagnetic field of strength $\underline{H}$ and a point charge e locating on the axis of symmetry. The equations of motion are studied under certain initial conditions which mean that the gyrostat rotates rapidly about the axis of dynamic symmetry. This velocity is very high and the magnitudes of the perturbing moments are less than or equal to the magnitude of the restoring moment k. These conditions allow us to introduce a small parameter ε, that causes the perturbed motion. The averaging method [2]-[4] is used to obtain the solutions of the equations of motion in the perturbed case up to the first approximation. A theoretical description for this approach in both resonant and non-resonant cases is given. The graphical representations of these solutions are performed to describe the motion of the gyrostat at any instant. The rigid body model has found a wide range of applications in various fields, for example the field of space research. This is because the body provides a convenient model for satellite, spacecraft and others.

2 Statement of the problem

Consider the motion of a dynamical symmetric gyrostat relative to a fixed point O, in response to a Newtonian force field arising from an attracting centre O_1 located at a distance from on the downward $Z-$ axis; a gyrostatic moment about $z-$ axis; a variable restoring moment k and perturbing moment vector $\underline{M}$. Two systems of coordinates are considered at the fixed point O; a fixed one $OXYZ$ and another rotating one $Oxyz$ which is fixed in the gyrostat and whose axes are directed along the principal axes of inertia of the gyrostat with origin O. Let us assume that

$$x_G = y_G = 0, \quad z_G = \ell^*, \quad \ell_1 = \ell_2 = 0, \quad \ell_3 \neq 0, \quad A = B \neq C,$$

where x_G, y_G and z_G are the coordinates of the centre of mass of the body (gyrostat); A, B and C are the principal moments of inertia and ℓ_1, ℓ_2 and ℓ_3 are the components of the gyrostatic moment vector. Then the equations of motion are

[[7] and [8]]

$$
\begin{aligned}
A\dot{p} + (C-A)\ q\ r + q\ \ell_3 &= k\ \sin\theta\ \cos\varphi + n\,(C-A)\ \gamma_2\ \gamma_3 + M_1, \\
A\dot{q} + (A-C)\ q\ r - p\ \ell_3 &= -k\ \sin\theta\ \sin\varphi + n\,(A-C)\ \gamma_2\ \gamma_3 + M_2, \\
C\dot{r} &= M_{3,} \\
\dot{\theta} &= p\ (\cos\varphi) - q\ (\sin\varphi)\ , \\
\dot{\varphi} &= r - (p\sin\varphi + q\cos\varphi)\cot\theta, \\
\dot{\psi} &= (p\sin\varphi + q\cos\varphi)\csc\theta,
\end{aligned}
\tag{2.1}
$$

$$
n = 3\lambda/R^3, \qquad M_i = M_i(p,q,r,\psi,\theta,\varphi,t) \qquad (i = 1,\ 2,\ 3).
$$

Dynamic equations (2.1) are written in projections onto the principal axes of inertia of the body, passing through point O . Here, (p,q,r) and M_i $(i = 1,\ 2,\ 3)$ are the projections of the angular velocity and the perturbing moment vectors of the body onto the principal axes of inertia; λ is the coefficient of the attracting centre O_1; R is the distance from O to O_1; and θ, φ and ψ are the Eulerian angles such that θ is the nutation angle, φ is the self-rotations angle and ψ is the precession angle.

Assume that the perturbing moments M_i $(i = 1,\ 2,\ 3)$ are 2π- periodic functions of the Euler's angles. In the case of a heavy solid, $k = mg\ell^*$ where m is the mass of the gyrostat, g is the acceleration due to gravity and ℓ^* is the distance from the centre of mass of the gyrostat to the fixed point O . In this work, we consider the following initial conditions

$$
p^2 + q^2 \ll r^2, \quad Cr^2 \gg k, \quad |M_i| \ll k \quad (i = 1,2)\,, \quad M_3 \approx k. \tag{2.2}
$$

These conditions mean that the direction of the angular velocity of the body is close to the axis of the dynamic symmetry; the angular velocity is large, so that the kinetic energy of the gyrostat is much greater than the potential energy resulting from the restoring moment; and two projections of the perturbing moment vector onto the principal axes of inertia of the body are small as compared with the restoring moment k, while the third one is of the same order of magnitude as it. Inequalities (2.2) allow us to introduce a small parameter $\varepsilon \ll 1$ and to set

$$
\begin{aligned}
p &= \varepsilon P, \qquad q = \varepsilon Q, \qquad k = \varepsilon K, \qquad n = \varepsilon N, \\
M_i &= \varepsilon^2 M_i^*\,(P,Q,r,\psi,\theta,\varphi) \qquad (i = 1,\ 2), \\
M_3 &= \varepsilon M_3^*\,(P,Q,r,\psi,\theta,\varphi)\,.
\end{aligned}
\tag{2.3}
$$

The new variables P and Q as well as the variables and constants r, ψ, θ, φ, K, N, A, C, M_i^* $(i = 1,\ 2,\ 3)$ are assumed to be bounded quantities of order unity as $\varepsilon \longrightarrow 0$.

The scope of this work is to investigate the asymptotic behavior of system (2.1) for the small parameter ε, if conditions (2.2) and (2.3) are satisfied. This will be performed by employing the averaging method [2]-[4] on the time interval of order ε^{-1}.

3 Variable restoring moment

In this section, we investigate our problem in presence of uniform electromagnetic field of strength $\underline{H}$ (along Z- axis) and a point charge e located on the axis of symmetry when the restoring moment depends on two components of the angular velocity vector and further, on the Euler's angles θ and φ. Thus, this gyrostat rotates under the influence of Newtonian force field, the gyrostatic moment about z- axis and the Lorentz force $e\,(\underline{V}\times\underline{H})$ [9] in which $\underline{V}=\left(\underline{\omega}\times\underline{\ell}'\right)$, $\underline{\ell}'\equiv(0,\ 0,\ \ell')$, where $\underline{\omega}$ is the angular velocity vector of that gyrostat and ℓ' is the distance of the position of the point charge e from O. Taking into account inequalities (2.2), the restoring moment K can be written in the form

$$\begin{aligned} K &= mg\ell^{*}+N(C-A)\cos\theta+eH\ell'^{2}\cos\theta[p^{2}+q^{2} \\ &\quad +\tan^{2}\theta\,(q\cos\varphi+p\sin\varphi)^{2}]^{1/2}. \end{aligned} \tag{3.1}$$

Making use of (2.1), (2.3) and (3.1), we obtain

$$\begin{aligned} A\dot{P}+(C-A)\ Q\ r+Q\ \ell_{3} &= [K+N\,(C-A)\cos\theta]\sin\theta\cos\varphi+\varepsilon M_{1}^{*}, \\ A\dot{Q}+(A-C)\ P\ r-P\ \ell_{3} &= -\,[K+N\,(C-A)\cos\theta]\sin\theta\sin\varphi+\varepsilon M_{2}^{*}, \\ C\dot{r} &= \varepsilon M_{3}^{*}, \\ \dot{\theta} &= \varepsilon\,(P\cos\varphi-Q\sin\varphi)\,, \\ \dot{\varphi} &= r-\varepsilon(P\sin\varphi+Q\cos\varphi)\cot\theta, \\ \dot{\psi} &= \varepsilon(P\sin\varphi+Q\cos\varphi)\csc\theta. \end{aligned} \tag{3.2}$$

The zero approximate solution for last four equations of system (3.2), gives

$$r=r_{0},\qquad \theta=\theta_{0},\qquad \varphi=r_{0}t+\varphi_{0},\qquad \psi=\psi_{0}, \tag{3.3}$$

where r_0, θ_0, φ_0 and ψ_0 are constants and are equal to the initial values of the corresponding ones. Substituting (3.3) into the first two equations of system (3.2) for $\varepsilon=0$, and integrating the resultant system of linear equations for P and Q to obtain

$$\begin{aligned} P &= a\cos\gamma_{0}+b\sin\gamma_{0}+E_{0}\sin\theta_{0}\sin\left(r_{0}t+\varphi_{0}\right), \\ Q &= a\sin\gamma_{0}-b\cos\gamma_{0}+E_{0}\sin\theta_{0}\cos\left(r_{0}t+\varphi_{0}\right), \\ a &= P_{0}-E_{0}\sin\theta_{0}\sin\varphi_{0},\qquad b=-Q_{0}+E_{0}\sin\theta_{0}\cos\varphi_{0}, \\ \gamma_{0} &= y_{0}t,\qquad y_{0}=n_{0}+A^{-1}\ell_{3}\neq 0, \\ n_{0} &= (C-A)\,A^{-1}r_{0},\qquad E_{0}=z_{0}\frac{[k_{0}+N\,(C-A)\cos\theta_{0}]}{(y_{0}^{2}-r_{0}^{2})}, \\ z_{0} &= (n_{0}-r_{0})\,A^{-1}+A^{-2}\ell_{3},\qquad k_{0}=K_{0},\qquad \left|\frac{y_{0}}{r_{0}}\right|\leq 1. \end{aligned} \tag{3.4}$$

Here P_0 and Q_0 are the initial values of the variables P and Q introduced in accordance with (2.3), while the variable γ_0 has the meaning of the oscillation phase of the generating system. System (3.2) is essentially non-linear, and therefore we introduce an additional variable γ , defined as

$$\frac{d\gamma}{dt} = y, \qquad \gamma(0) = 0. \tag{3.5}$$

For $\varepsilon = 0$, we have $\gamma = \gamma_0 = y_0 t$, in accordance with (3.4). Equations (3.3) and (3.4) define the general solution of (3.2) and (3.5) when $\varepsilon = 0$. Eliminating the constants with allowance of (3.3), the first two equations in (3.4) can be rewritten in equivalent form as

$$\begin{aligned} P &= a\cos\gamma + b\sin\gamma + E\sin\theta\sin\varphi, \\ Q &= a\sin\gamma - b\cos\gamma + E\sin\theta\cos\varphi, \end{aligned} \tag{3.6}$$

which can be solved for a and b, to get

$$\begin{aligned} a &= P\cos\gamma + Q\sin\gamma - E\sin\theta\sin\left(\gamma+\varphi\right), \\ b &= P\sin\gamma - Q\cos\gamma + E\sin\theta\cos\left(\gamma+\varphi\right). \end{aligned} \tag{3.7}$$

Let us consider system (3.2) for $\varepsilon \neq 0$ and expressions (3.6) and (3.7) as change of variable formulas (containing the variable γ), which specify the conversion from variables P and Q to the Van der Pol variables a and b, and vice versa [2]. Using these formulas in system (3.2) and (3.5), we convert from the variables P, Q, r, ψ, θ, φ, γ to the new ones a, b, r, ψ, θ, α, γ where

$$\alpha = \gamma + \varphi. \tag{3.8}$$

After some manipulation, we have the following system

$$\begin{aligned} \dot{a} = \; & \varepsilon A^{-1}\left[M_1^0\cos\gamma + M_2^0\sin\gamma\right] + \varepsilon E b\{2N(C-A)[K+N(C-A) \\ & \times\cos\theta]^{-1}\sin^2\theta\sin^2\alpha - \cos\theta\} + \varepsilon E^2(ACz)^{-1}\{2[y(C-A) \\ & -rA][K+N(C-A)\cos\theta]^{-1} - (CA^{-1}-2)\}M_3^0\sin\theta\sin\alpha \\ & -\frac{1}{2E}\varepsilon e H\ell'^2 b\sin^2\alpha\cos^2\theta[K+N(C-A)\cos\theta]^{-1}\{(EA)^{-1} \\ & \times[K+N(C-A)\cos\theta]^{-1}\{(2a^2-b^2)\cos^2\alpha - a^2\sin^2\alpha \\ & +[2E^2-(a^2+b^2)\cot^2\theta]\} + (y+r)\frac{\tan^2\theta}{\sin\theta}[((2a^2-b^2) \\ & \times\sin^2\theta + 4a^2 + b^2)\cos^2\alpha + (2E\tan^2\theta - a^2 - b^2)\cos^2\theta \\ & -a^2(2+\sin^2\theta)\sin^2\alpha]\}, \end{aligned}$$

$$
\begin{aligned}
\dot{b} &= \varepsilon A^{-1}\left[M_1^0\sin\gamma - M_2^0\cos\gamma\right] - \varepsilon E a\{2N(C-A)[K+N(C-A) \\
&\quad \times\cos\theta]^{-1}\sin^2\theta\cos^2\alpha - \cos\theta\} + \varepsilon E^2(ACz)^{-1}\{2[y(C-A) \\
&\quad -rA][K+N(C-A)\cos\theta]^{-1} - (CA^{-1}-2)\}M_3^0\sin\theta\cos\alpha \\
&\quad +\frac{1}{2E}\varepsilon e H\ell'^2 a\cos^2\alpha\cos^2\theta[K+N(C-A)\cos\theta]^{-1}\{(EA)^{-1} \\
&\quad \times[K+N(C-A)\cos\theta]^{-1}\{(2a^2-b^2)\sin^2\alpha - b^2\cos^2\alpha \\
&\quad +[2E^2-(a^2+b^2)\cot^2\theta]\} + (y+r)\frac{\tan^2\theta}{\sin\theta}[((2b^2-a^2) \\
&\quad \times\sin^2\theta + 4b^2 + a^2)\sin^2\alpha + (2E\tan^2\theta - a^2 - b^2)\cos^2\theta \\
&\quad -b^2(2+\sin^2\theta)\cos^2\alpha]\}, \\
\dot{r} &= \varepsilon C^{-1}M_3^0, \\
\dot{\psi} &= \varepsilon[(a\sin\alpha - b\cos\alpha)\csc\theta + E], \\
\dot{\theta} &= \varepsilon(a\cos\alpha + b\sin\alpha), \\
\dot{\varphi} &= r - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta, \\
\dot{\alpha} &= A^{-1}(Cr+\ell_3) - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta, \\
\dot{\gamma} &= A^{-1}[(C-A)r+\ell_3].
\end{aligned} \tag{3.9}
$$

Here, M_i^0 denote functions obtained from M_i^* as a result of substitution (3.6)-(3.8), i.e.,

$$
M_i^0\,(a,b,r,\psi,\theta,\alpha,\gamma,t) = M_i^*\,(P,Q,r,\psi,\theta,\varphi,t)\,; \qquad (i=1,\ 2,\ 3). \tag{3.10}
$$

We note that, the change from the two variables P and Q to the three variables a, b and γ is due to the sake of convenience; for $\varepsilon = 0$,the system for P and Q has the form of a linear system while substitution (3.6) is non-singular for all a and b. Let us consider a vector-valued function x whose components are provided by the slow variables a, b, r, ψ and θ of system (3.9). Thus, this system can be written as

$$
\begin{aligned}
\dot{x} &= \varepsilon X(x,\alpha,\gamma,t), \\
\dot{\alpha} &= A^{-1}(Cr+\ell_3) + \varepsilon Y(x,\alpha), \\
\dot{\gamma} &= A^{-1}[(C-A)r+\ell_3], \\
x(0) &= x_0, \qquad \alpha(0)=\alpha_0, \qquad \gamma(0)=0.
\end{aligned} \tag{3.11}
$$

Here the vector-valued function X and the scalar function Y are defined by the right-hand sides of (3.9) whose initial values can be obtained in accordance with equations (3.3) to (3.5) and (3.8).

Let us consider the system (3.9) or (3.11) from the stand point of employing the averaging method of [2], [3] and [10]. System (3.9) contains the slow variables

$a,\ b,\ r,\ \psi,\ \theta$ and fast variables namely the phases $\alpha,\ \gamma$ and the time t , with γ appearing only in the first three equations of (3.9). This system is essentially non-linear and it is extremely difficult to employ the averaging method directly [11]. For simplicity, we will assume that the perturbing moments M_i^* are independent of t. Since M_i^* $(i = 1,\ 2,\ 3)$ are 2π- periodic in φ, in accordance with substitutions (3.6)-(3.8) we have that functions M_i^0 from (3.10) will be 2π- periodic functions of α and γ. Then system (3.11) contains two rotating phases α and γ, and the corresponding frequencies $A^{-1}(Cr + \ell_3)$ and $A^{-1}[(C - A)r + \ell_3]$ are variables. In the averaging system (3.9) or (3.11), two cases should be distinguished [11]; the non-resonant case, when the frequencies $A^{-1}(Cr + \ell_3)$ and $A^{-1}[(C - A)r + \ell_3]$ are non-commensurable and the resonant case, when these frequencies are commensurable. A very important feature of system (3.11) is the fact that the ratio $A^{-1}[(C - A)r + \ell_3]/A^{-1}(Cr + \ell_3) = 1 - [Ar/(Cr + \ell_3)]$, of the frequencies is , and the resonant case occurs for

$$(Cr + \ell_3)\,/Ar\, = i/j \leq 2, \tag{3.12}$$

where i and j are relatively prime natural numbers while in the non-resonant case $(Cr + \ell_3)/Ar$ is an irrational number.

As a result of (3.12), averaging of the non-linear system (3.11), in which X is independent of t, is equivalent to averaging of a quasilinear system with constant frequencies. This can be achieved by introducing the independent variable γ. In the non-resonant case $(Cr + \ell_3)/Ar \neq i/j$, we obtain the first approximation averaged system by independent averaging of the right-hand sides of the system (3.9) with respect to the fast variables α and γ. As a result, we obtain the following equations for the slow variables

$$\begin{aligned}
\dot{a} &= \varepsilon A^{-1}\mu_1 - \varepsilon E b\cos\theta + \varepsilon E b N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1}\sin^2\theta\sin^2\alpha + \varepsilon E^2(Cz)^{-1}\{2[y(CA^{-1}-1) \\
&\quad -r][K + N(C-A)\cos\theta]^{-1} - A^{-1}(CA^{-1}-2)\} \\
&\quad \times\sin\theta\ \mu_4 + \mu_1^k, \\
\dot{b} &= \varepsilon A^{-1}\mu_2 + \varepsilon E a\cos\theta - \varepsilon E a N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1}\sin^2\theta\cos^2\alpha - \varepsilon E^2(Cz)^{-1}\{2[y(CA^{-1}-1) \\
&\quad -r][K + N(C-A)\cos\theta]^{-1} - A^{-1}(CA^{-1}-2)\} \\
&\quad \times\sin\theta\ \mu_5 + \mu_2^k, \\
\dot{r} &= \varepsilon C^{-1}\mu_3, \quad \dot{\psi} = \varepsilon E, \quad \dot{\theta} = 0, \quad \dot{\varphi} = r - \varepsilon E\cos\theta,
\end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
\mu_1\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[M_1^0\cos\gamma + M_2^0\sin\gamma\right]d\alpha d\gamma,\\
\mu_2\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[M_1^0\sin\gamma - M_2^0\cos\gamma\right]d\alpha d\gamma,\\
\mu_3\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0 d\alpha d\gamma,\\
\mu_4\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0\sin\alpha d\alpha d\gamma,\\
\mu_5\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M_3^0\cos\alpha d\alpha d\gamma,\\
\mu_1^k\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} K_1 d\alpha d\gamma,\\
\mu_2^k\left(a,b,r,\psi,\theta\right) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} K_2 d\alpha d\gamma,
\end{aligned}
$$

$$
\begin{aligned}
K_1 = {} & \varepsilon EbN(C-A)[K+N(C-A)\cos\theta]^{-1}\sin^2\theta\sin^2\alpha\\
& -\frac{\varepsilon}{2E^2}A^{-1}eH\ell'^2 b\sin\theta\cos^2\theta\sin^2\alpha[(2E^2-\frac{a^2+b^2}{\tan^2\theta})\\
& +(2a^2-b^2)\cos^2\alpha - a^2\sin^2\alpha] - \frac{\varepsilon}{E}eH\ell'^2 b(y+r)\\
& \times[K+N(C-A)\cos\theta]^{-1}\sin^2\alpha\{[a^2(1+\sin\theta)-b^2]\\
& \times\cos^2\alpha - a^2\sin\theta\sin^2\alpha + \frac{\sin^3\theta}{2}[(2E^2-\frac{a^2+b^2}{\tan^2\theta})\\
& +(2a^2-b^2)\cos^2\alpha - a^2\sin^2\alpha]\},
\end{aligned}
$$

$$
\begin{aligned}
K_2 = {} & -\varepsilon EaN(C-A)[K+N(C-A)\cos\theta]^{-1}\sin^2\theta\cos^2\alpha\\
& +\frac{\varepsilon}{2E^2}A^{-1}eH\ell'^2 a\sin\theta\cos^2\theta\cos^2\alpha[(2E^2-\frac{a^2+b^2}{\tan^2\theta})\\
& +(2b^2-a^2)\sin^2\alpha - b^2\cos^2\alpha] + \frac{\varepsilon}{E}eH\ell'^2 a(y+r)\\
& \times[K+N(C-A)\cos\theta]^{-1}\cos^2\alpha\{[b^2(1+\sin\theta)-a^2]\\
& \times\sin^2\alpha - b^2\sin\theta\cos^2\alpha + \frac{\sin^3\theta}{2}[(2E^2-\frac{a^2+b^2}{\tan^2\theta})\\
& +(2b^2-a^2)\sin^2\alpha - b^2\cos^2\alpha]\}.
\end{aligned}
$$

Solving the averaged system (3.13) for perturbing moments of specific form, we can determine the motion of the gyrostat in the non-resonant case with an error of order ε on an interval of time variation of order ε^{-1}. The last equation in the system (3.13) can be integrated; it yields $\theta = \text{constant} = \theta_0$. The above system is equivalent to a two-frequency system with constant frequencies, since both frequencies are proportional to the axial component r of the angular velocity vector. Therefore, the applicability of the averaging method can be shown in the same way as for a quasi-linear system.

In the resonant case (3.12), system (3.11) is a single frequency system. We replace α by a new slow variable that comprises a linear combination of the phases with integer coefficients

$$\lambda = \alpha - i\gamma(i-j)^{-1}, \quad i/j \neq 1, \quad i/j \leq 2; \quad i.j > 0. \tag{3.14}$$

System (3.11), assumes the form of a standard system with rotating phase

$$\begin{aligned}
\dot{x} &= \varepsilon X(x, i\gamma(i-j)^{-1} + \lambda, \gamma), \\
\dot{\lambda} &= \varepsilon Y(x, i\gamma(i-j)^{-1} + \lambda), \\
\dot{\gamma} &= A^{-1}[(C-A)r + \ell_3],
\end{aligned} \tag{3.15}$$

where its right hand sides being $(2\,|i-j|\,\pi$- periodic in γ. We set up the first approximation by averaging the right-hand sides of (3.15) with respect to the above period of variation of the argument γ. As a result, we obtain the following system of relations for the slow variables

$$\begin{aligned}
\dot{a} &= \varepsilon A^{-1}\mu_1^* - \varepsilon E b\cos\theta + \varepsilon E b N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1}\sin^2\theta\sin^2\alpha + \varepsilon E^2(Cz)^{-1}\{2[y(CA^{-1}-1) \\
&\quad -r][K + N(C-A)\cos\theta]^{-1} - A^{-1}(CA^{-1}-2)\} \\
&\quad \times \sin\theta\ \mu_4^* + \mu_1^{*k}, \\
\dot{b} &= \varepsilon A^{-1}\mu_2^* + \varepsilon E a\cos\theta - \varepsilon E a N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1}\sin^2\theta\cos^2\alpha - \varepsilon E^2(Cz)^{-1}\{2[y(CA^{-1}-1) \\
&\quad -r][K + N(C-A)\cos\theta]^{-1} - A^{-1}(CA^{-1}-2)\} \\
&\quad \times \sin\theta\ \mu_5^* + \mu_2^{*k}, \\
\dot{r} &= \varepsilon C^{-1}\mu_3^*, \qquad \dot{\psi} = \varepsilon E, \qquad \dot{\theta} = 0, \\
\dot{\varphi} &= r - \varepsilon E\cos\theta, \qquad \dot{\lambda} = -\varepsilon E\cos\theta,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
\mu_1^*(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\left[M_1^0\cos\gamma+M_2^0\sin\gamma\right]d\gamma,\\
\mu_2^*(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\left[M_1^0\sin\gamma-M_2^0\cos\gamma\right]d\gamma,\\
\mu_3^*(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0 d\gamma,\\
\mu_4^*(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0\sin\alpha d\gamma,\\
\mu_5^*(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}M_3^0\cos\alpha d\gamma,\\
\mu_1^{*k}(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}K_1 d\gamma,\\
\mu_2^{*k}(a,b,r,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}K_2 d\gamma,
\end{aligned}$$

It is assumed that, the variable α in the integrands is to be replaced by λ in accordance with (3.14). Solving the previous averaged system (3.16) for perturbing moments of a particular form, we can determine the motion of the body in the resonant case with an error of order ε on a time interval of order ε^{-1}.

4 Linear dissipative perturbed moments

Let us consider the perturbed motion of the gyrostat analogous to that of the Lagrange's case with allowance for the moments acting on our gyrostat from the external medium and the shape of the gyrostat. We assume that the perturbing moments having linear dissipative form as

$$M_1=-\varepsilon I_1 p,\quad M_2=-\varepsilon I_1 q,\quad M_3=-\varepsilon I_3 r;\quad I_1,I_3>0, \tag{4.1}$$

where I_1 and I_3 are constant proportionality factors that depend on the properties of the medium and the shape of the gyrostat. For the case of constant small moment along the axis of symmetry, the perturbed moments M_i $(i=1,\ 2,\ 3)$ take the form

$$M_1=M_2=0,\qquad M_3=\varepsilon M_3^*=const. \tag{4.2}$$

Each of equations (4.1) and (4.2), has been studied separately in [1] and [6] when $K\equiv K(\theta)$ only. Now, we consider the motion of the gyrostat acted upon by the sum of the two cases together. So, we write

$$M_1=-\varepsilon I_1 p,\quad M_2=-\varepsilon I_1 q,\quad M_3=-\varepsilon I_3 r+\varepsilon M_3^*;\quad I_1,I_3>0, \tag{4.3}$$

Substituting from (2.3) into (4.3), we get

$$M_1 = -\varepsilon^2 I_1 P, \quad M_2 = -\varepsilon^2 I_1 Q, \quad M_3 = -\varepsilon I_3 r + \varepsilon M_3^*; \quad I_1, I_3 > 0, \tag{4.4}$$

In accordance with the previous section, for the non-resonant case, we change over to new slow variables a, b, r, ψ and θ, so that the averaged system (3.13) has the forms

$$\begin{aligned}
\dot{a} &= -\varepsilon a I_1 A^{-1} - \varepsilon b \cos\theta + \varepsilon E b N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1} \sin^2\theta - \frac{\varepsilon}{16E^2} e H \ell'^2 b \sin\theta \{A^{-1} \cos^2\theta [8E^2 \\
&\quad -(a^2+b^2)(4\cot^2\theta + 1)] + E(y+r)[K + N(C-A) \\
&\quad \cos\theta]^{-1}[8E^2 \sin^2\theta - 3(a^2+b^2)(\cos^2\theta + 1)]\}, \\
\dot{b} &= -\varepsilon b I_1 A^{-1} + \varepsilon a \cos\theta - \varepsilon E a N(C-A)[K + N(C-A) \\
&\quad \times \cos\theta]^{-1} \sin^2\theta + \frac{\varepsilon}{16E^2} e H \ell'^2 a \sin\theta \{A^{-1} \cos^2\theta [8E^2 \\
&\quad -(a^2+b^2)(4\cot^2\theta + 1)] + E(y+r)[K + N(C-A) \\
&\quad \cos\theta]^{-1}[8E^2 \sin^2\theta - 3(a^2+b^2)(\cos^2\theta + 1)]\}, \\
\dot{r} &= -\varepsilon C^{-1}(I_3 r - M_3^*), \quad \dot{\psi} = \varepsilon E, \quad \dot{\theta} = 0, \quad \dot{\varphi} = r - \varepsilon E \cos\theta.
\end{aligned} \tag{4.5}$$

Integrating the third equation in (4.5), we obtain:

$$r = (r_0 - I_3^{-1} M_3^*) e^{-\varepsilon I_3 C^{-1} t} + I_3^{-1} M_3^*. \tag{4.6}$$

Making use of (4.5) and (4.6), $\dot{\psi}$, $\dot{\varphi}$ can be integrated to have

$$\begin{aligned}
\psi &= \psi_0 - \frac{C[K + N(C-A)\cos\theta_0]}{(\ell_3 I_3 + C M_3^*)} \\
&\quad \times \ln\{\frac{I_3(C r_0 + \ell_3) e^{-\varepsilon I_3 C^{-1} t}}{C(r_0 I_3 - M_3^*) e^{-\varepsilon I_3 C^{-1} t} + \ell_3 I_3 + C M_3^*}\}, \\
\varphi &= \varphi_0 + \frac{C(M_3^* - r_0 I_3)}{\varepsilon I_3^2} (e^{-\varepsilon I_3 C^{-1} t} - 1) + I_3^{-1} M_3^* t \\
&\quad + \frac{\varepsilon \cos\theta_0 [K + N(C-A)\cos\theta_0]}{(\ell_3 I_3 + C M_3^*)} \\
&\quad \times \ln\{\frac{I_3(C r_0 + \ell_3) e^{-\varepsilon I_3 C^{-1} t}}{C(r_0 I_3 - M_3^*) e^{-\varepsilon I_3 C^{-1} t} + \ell_3 I_3 + C M_3^*}\}.
\end{aligned} \tag{4.7}$$

In addition, as can be seen from (4.5), the nutation angle maintains constant value i.e.,

$$\theta = \theta_0. \tag{4.8}$$

Here r_0, ψ_0, φ_0 and θ_0 were obtained in (3.3). Making use of (4.6), (4.8) and the first two equations of (4.5), one obtains

$$\begin{aligned} a &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \cos\eta + Q_0 \sin\eta - E_0 \sin\theta_0 \sin(\eta+\varphi_0)], \\ b &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \sin\eta - Q_0 \cos\eta + E_0 \sin\theta_0 \cos(\eta+\varphi_0)], \\ \eta &= E_0 C I_3^{-1} \cos\theta_0 (e^{-\varepsilon I_3 C^{-1} t} - 1). \end{aligned} \tag{4.9}$$

As a results of substitution of the expressions for a and b from (4.9), and for r from (4.6) into expressions (3.6) and (2.3) for P, Q, p, q, we obtain

$$\begin{aligned} P &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \cos(\gamma-\eta) - Q_0 \sin(\gamma-\eta) + E_0 \sin\theta_0 \\ &\quad \times \sin(\gamma-\eta-\varphi_0)] + E_0 \sin\theta_0 \sin\varphi_0, \\ Q &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \sin(\gamma-\eta) + Q_0 \cos(\gamma-\eta) - E_0 \sin\theta_0 \\ &\quad \times \cos(\gamma-\eta-\varphi_0)] + E_0 \sin\theta_0 \cos\varphi_0, \\ \gamma &= A^{-1}\{(C-A)[(r_0 - I_3^{-1} M_3^*) e^{-\varepsilon I_3 C^{-1} t} + I_3^{-1} M_3^*] + \ell_3\} t, \\ p_0 &= \varepsilon P_0, \qquad q_0 = \varepsilon Q_0, \qquad k = \varepsilon K_0, \qquad n = \varepsilon N. \end{aligned} \tag{4.10}$$

Now we have developed the solution of the first approximation system for the slow variables in the case of dissipative moment. If the resonance relation (3.12) is satisfied, the averaging should be performed in accordance with the scheme (3.16). In this case, all the integrals μ_i^* are equal to the corresponding integrals μ_i of (3.13). Therefore, the resonance in effect does not appear and the resultant solution is suitable for describing the motion for any ratio $(Cr + \ell_3)/Ar \neq 1$. One can see that the axial rotational velocity r decreases monotonically in an exponential fashion in accordance with (4.6). It is obvious from (4.10) that the terms representing the projections P and Q resulting from the initial values P_0 and Q_0, attenuate exponentially. At the same time, the projections P and Q contain exponential increasing terms that are proportional to the restoring moment and the third component of gyrostatic moment ℓ_3 We conclude also from (4.7) and (4.8) that the precession angle ψ and the self-rotations angle φ are functionally dependent on the time t and behave in logarithmic manner while the nutation angle θ remains constant during the motion. For zero order approximation of ε, we note that

$$\dot{\psi} = 0, \qquad \dot{\varphi} = r_0, \qquad \dot{\theta} = 0. \tag{4.11}$$

The case of rotation with fast spin r_0 about the symmetry axis is then obtained.

5 The graphical representations

This section is given to discuss the obtained results through the graphical representations. For the concerned problem, the following data are used

$$\begin{aligned}
A &= B = 6\ kg.mm^2, \quad C = 10\ kg.mm^2, \quad r_0 = 2\ mm,\\
M &= 300\ kg, \quad m_3^* = 2.5, \quad H = 20, \quad \ell^* = 25\ mm,\\
\ell' &= 13\ mm, \quad I_1 = 0.2, \quad I_3 = 0.6, \quad \varepsilon = 0.0001,\\
\ell &= (100,\ 300,\ 500)\ kg.mm^2\ s^{-1}, \quad \theta_0 = (\pi/6,\ \pi/4,\ \pi/3),\\
T &= 12.566371
\end{aligned}$$

Consider p and q to denote the obtained solutions for the considered problem. The graphical representations of these solutions via t are represented in figures (5.1)-(5.4) and (5.5)-(5.8) respectively. The phase plane diagrams describing the stability for the solutions p and q are given in figures (5.9)-(5.12) and (5.13)-(5.16).

It is obvious that, when ℓ increases the frequency numbers increase while the amplitude of the waves decreases, see figures (5.1)-(5.4) and (5.5)-(5.8). On other hand, when e increases, the amplitude of the waves increases and the numbers of waves remain unchanged, for example see figures [(5.1),(5.2)]; [(5.3),(5.4)]; and [(5.5),(5.6)]; [(5.7),(5.8)]. Also, when increases, the amplitude of the waves increases and the numbers of waves remain unchanged, for example see figures [(5.1),(5.3)]; [(5.2),(5.4)]; and [(5.5),(5.7)]; [(5.6),(5.8)].

6 Conclusion

The averaging method is applied to obtain the first order approximate periodic solutions of the problem of perturbed motions of a rotating symmetric gyrostat about a fixed point in terms of the small parameter ε. These solutions can be considered as a generalization of previously obtained ones as by Akulenko et al. [1] (in the case when $n = 0$, $k =$ constant and $\ell_3 = 0$), as by Ismail, et al [7] (in the case when $n = 0$), as Cid et al. [12] (in the case when the perturbing moments $M_i = 0$, $i = 1, 2, 3$ and $n = 0$), and as Leshchenko et al. [13] (in the case when $n = 0$, $|M_i| \ll k$, k=constant and $\ell_3 = 0$). The methodological treatment of this technique is presented in the resonant and the non-resonant cases. The axial rotational velocity r is shown decrease monotonically in an exponential fashion in accordance with (4.6). It is obvious from (4.10) that the terms representing the projections P and Q resulting from the initial values P_0 and Q_0, attenuate exponentially. On other hand, the projections P and Q contain terms of exponential forms that are proportional to the restoring moment and the third gyrostatic moment. The precession angle ψ and the self-rotations angle φ are functionally dependent on the time t and behave in logarithmic manner while the nutation angle θ remains stationary during the motion. The case of rotation with fast spin r_0 about the symmetry axis is obtained.

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Captions of Figures

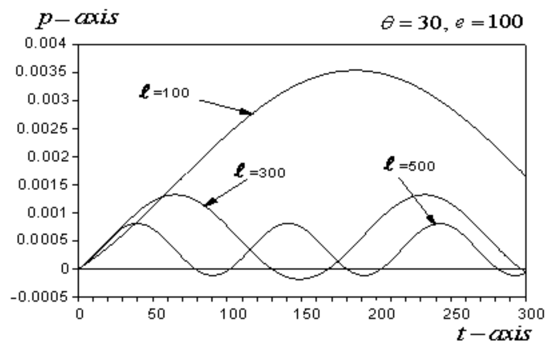

Fig. 1

The graphical representation of the solution p via t when $\theta = 30^0$, $e = 100$ *gauss* with different values of the gyrostatic moment ℓ.

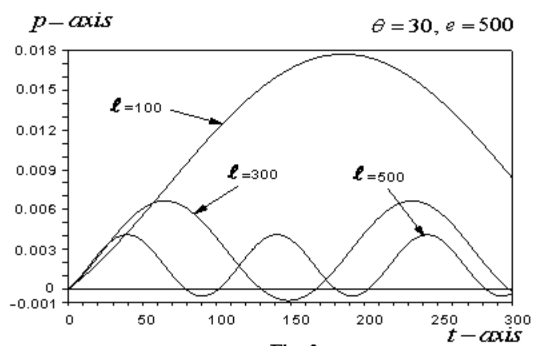

Fig. 2

The graphical representation of the solution p via t when $\theta = 30^0$, $e = 500$ *gauss* with different values of the gyrostatic moment ℓ.

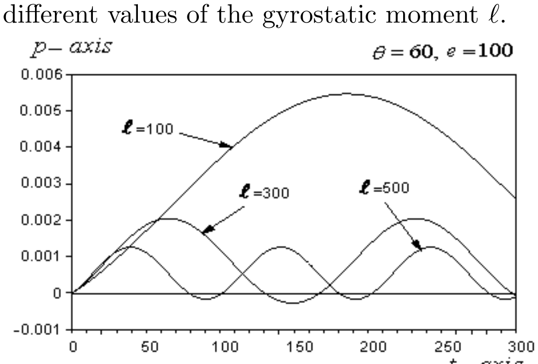

Fig. 3

The graphical representation of the solution p via t when $\theta = 60^0$, $e = 100$ *gauss* with different values of the gyrostatic moment ℓ.

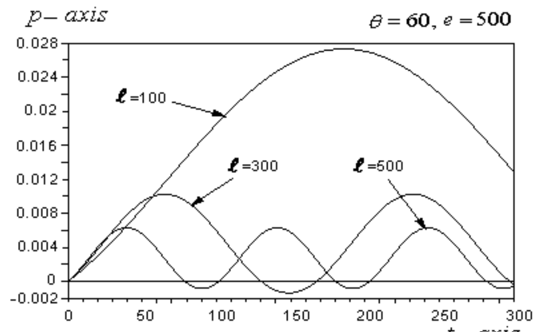

Fig. 4

The graphical representation of the solution p via t when $\theta = 60^0$, $e = 500$ *gauss* with different values of the gyrostatic moment ℓ.

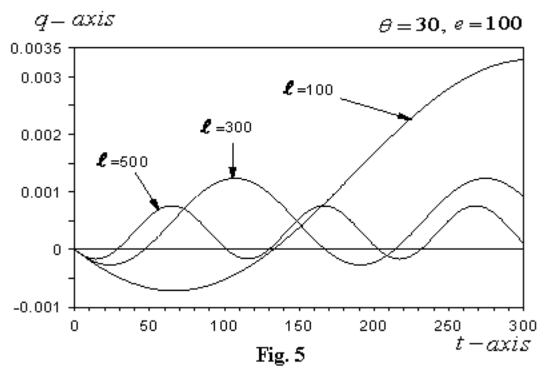

Fig. 5

The graphical representation of the solution q via t when $\theta = 30^0$, $e = 100$ *gauss* with different values of the gyrostatic moment ℓ.

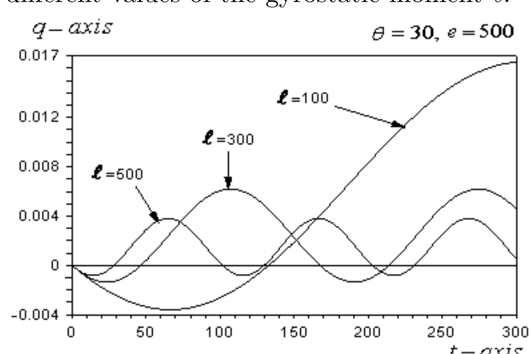

Fig. 6

The graphical representation of the solution q via t when $\theta = 30^0$, $e = 500$ *gauss* with different values of the gyrostatic moment ℓ.

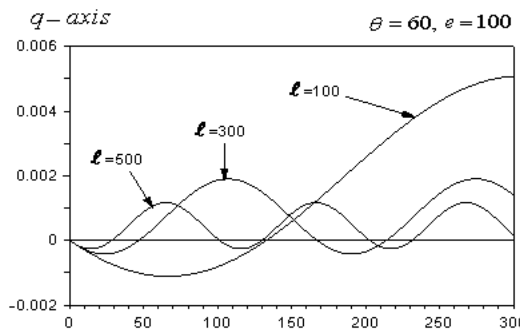

Fig. 7

The graphical representation of the solution q via t when $\theta = 60^0$, $e = 100$ *gauss* with different values of the gyrostatic moment ℓ.

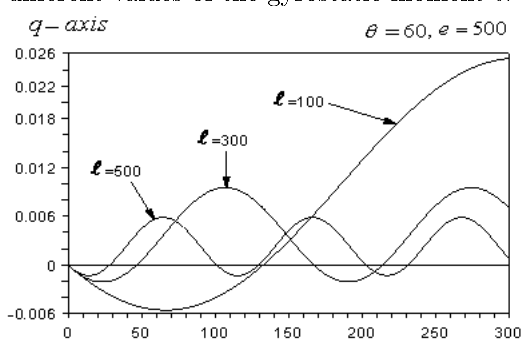

Fig. 8

The graphical representation of the solution q via t when $\theta = 60^0$, $e = 500$ *gauss* with different values of the gyrostatic moment ℓ.

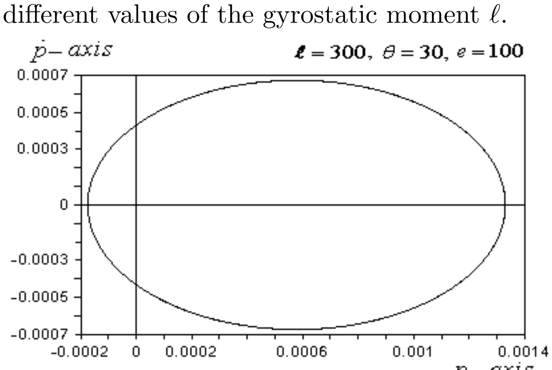

Fig. 9

The phase plane diagram of the solution p when $\ell = 300\ kg.mm^2.s^{-1}$, $\theta = 30^0$, $e = 100$ *gauss*.

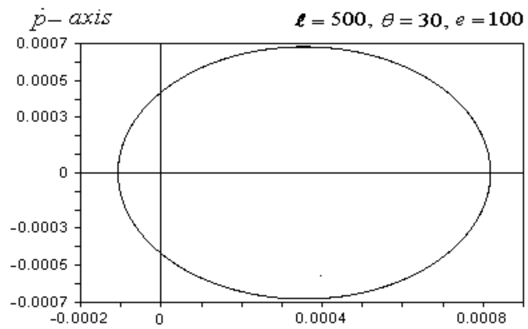

Fig. 10

The phase plane diagram of the solution p when $\ell = 500\ kg.mm^2.s^{-1}$, $\theta = 30^0$, $e = 100\ gauss$.

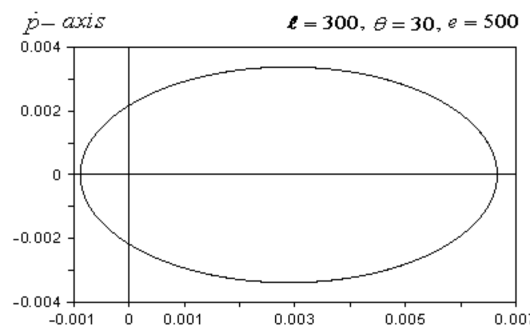

Fig. 11

The phase plane diagram of the solution p when $\ell = 300\ kg.mm^2.s^{-1}$, $\theta = 30^0$, $e = 500\ gauss$.

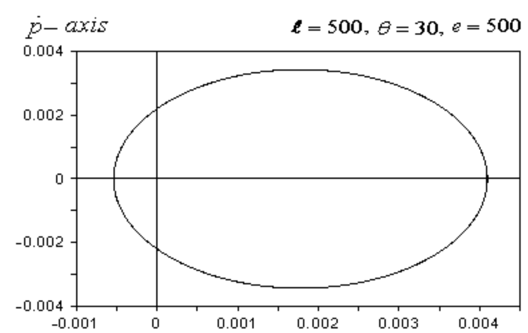

Fig. 12

The phase plane diagram of the solution p when $\ell = 500\ kg.mm^2.s^{-1}$, $\theta = 30^0$, $e = 500\ gauss$.

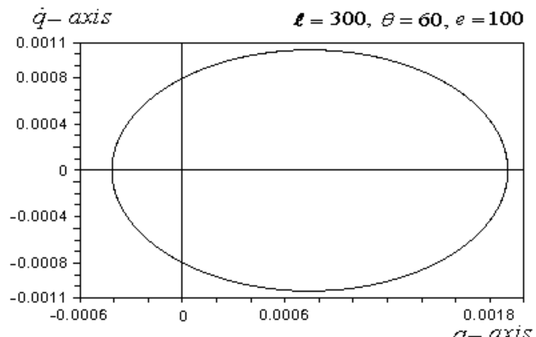

Fig. 13

The phase plane diagram of the solution q when $\ell = 300\ kg.mm^2.s^{-1}$, $\theta = 60^0$, $e = 100\ gauss$.

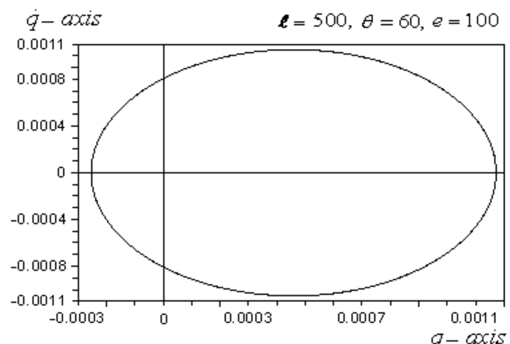

Fig. 14

The phase plane diagram of the solution q when $\ell = 500\ kg.mm^2.s^{-1}$, $\theta = 60^0$, $e = 100\ gauss$.

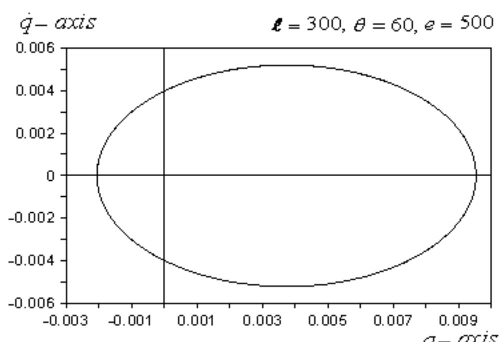

Fig. 15

The phase plane diagram of the solution q when $\ell = 300\ kg.mm^2.s^{-1}$, $\theta = 60^0$, $e = 500\ gauss$.

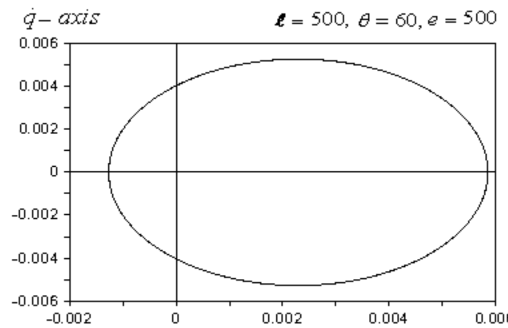

Fig. 16

The phase plane diagram of the solution q when $\ell = 500\ kg.mm^2.s^{-1}$, $\theta = 60^0$, $e = 100\ gauss$.

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T. S. Amer
Mathematics Department,
Faculty of Education for Girls
(AL-Khurmah), Taif University,
Saudi Arabia.

Mathematics Department,
Faculty of Science, Tanta University,
Tanta 31527, Egypt.
email: tarekamer30@hotmail.com