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King Type (p, q)-Bernstein Schurer Operators

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Abstract The objective of this paper is to establish the King variant of modified form of (p, q) variant of Bernstein Schurer operators and examine the estimation properties. Using King modification, we present approximation properties and estimate error of constructed operators using modulus of continuity. We also study convergence rate as well as its Voronovskaya results. Lastly, we show illustrative graphics of some numerical examples and compared the theoritical results of constructed operators graphically to various functions using MATLAB code.

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1. Introduction

Approximation theory is an explosive research area wherein different modifications have been made to Bernstein operators in order to get faster rate of convergence. Keeping this in mind, Schurer [1] proposed the speculation of foundational operators. Let $C_B[a', b']$ denotes the class of functions that are continuous on $[a', b']$. For every $\mathfrak{l} \in \mathbb{N}$, $\mathfrak{f} \in C_B[0, \mathfrak{j}+1]$ and fixed $\mathfrak{j} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the modification to Bernstein operators by Schurer is described in [1] as:

$$B_{\mathfrak{l}}^{\mathfrak{j}}(\mathfrak{f}; \varkappa) = \sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}} \left[\begin{array}{c} \mathfrak{l}+\mathfrak{j} \\ \mu \end{array} \right] \varkappa^{\mu}(1-\varkappa)^{\mathfrak{l}+\mathfrak{j}-\mu} \mathfrak{f}\left(\tfrac{\mu}{\mathfrak{l}}\right), \ \varkappa \in [0, 1].$$

Inspired by the pioneering work in modification of Bernstein operators, Faruk et al. [2] in 2020, introduces a new modification of Bernstein operators involving positive real parameter α, namely α-Bernstein-Schurer operators. In [2], Faruk et al. concluded that α-Bernstein-Schurer operators are more powerful than classical Bernstein operators as

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well as Bernstein-Schurer operators by obtaining its shape preserving and approximation properties. For $0 \leq \alpha \leq 1$, the α-Bernstein-Schurer operators are defined in [2] as:

$$B_{\mathfrak{l},\mathfrak{j}}^{\alpha}(\mathfrak{f};\varkappa)=\sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}}\left\{\begin{bmatrix}\mathfrak{l}+\mathfrak{j}-2\\ \mu\end{bmatrix}\varkappa(1-\alpha)+(1-\varkappa)(1-\alpha)\begin{bmatrix}\mathfrak{l}+\mathfrak{j}-2\\ \mu-2\end{bmatrix}\right.$$
$$\left.+\varkappa\alpha(1-\varkappa)\begin{bmatrix}\mathfrak{l}+\mathfrak{j}\\ \mu\end{bmatrix}\right\}\varkappa^{\mu-1}(1-\varkappa)^{\mathfrak{j}+\mathfrak{j}-\mu-1}\mathfrak{f}\left(\frac{\mu}{\mathfrak{l}}\right).$$

Subsequently, the generalization of Bernstein-Schurer operators in two-dimensional space is constructed and its approximation properties are examined using various mathematical tools by Mohiuddine [3]. Let $C\left(I^2\right)$ represents the class of functions that are continuous on $I^2=[0,1+\mathfrak{j}]\times[0,1+\mathfrak{j}]$. For any $\mathfrak{h}\in C\left(I^2\right),(y_1,y_2)\in[0,1]\times[0,1],v_1,v_2\in\mathbb{N}$ and $\beta_1,\beta_2\in[0,1]$, the bivariate extension of Bernstein-Schurer operators in [3] is defined as:

$$B_{v_1,v_2,\mathfrak{j}}^{\beta_1,\beta_2}(\mathfrak{h};y_1,y_2)=\sum_{k_1=0}^{v_1+\mathfrak{j}}\sum_{k_2=0}^{v_2+\mathfrak{j}}\mathfrak{h}\left(\frac{k_1}{v_1},\frac{k_2}{v_2}\right)\left\{(1-\beta_1)y_1\begin{bmatrix}v_1+\mathfrak{j}-2\\ k_1\end{bmatrix}\right.$$
$$+(1-\beta_1)(1-y_1)\begin{bmatrix}v_1+\mathfrak{j}-2\\ k_1-2\end{bmatrix}$$
$$\left.+\beta_1y_1(1-y_1)\begin{bmatrix}v_1+\mathfrak{j}\\ k_1\end{bmatrix}\right\}y_1^{k_1-1}(1-y_1)^{v_1+\mathfrak{j}-(k_1+1)}$$
$$\times\left\{(1-\beta_2)y_2\begin{bmatrix}v_2+\mathfrak{j}-2\\ k_2\end{bmatrix}+(1-\beta_2)(1-y_2)\begin{bmatrix}v_2+\mathfrak{j}-2\\ k_2-2\end{bmatrix}\right.$$
$$\left.+\beta_2y_2(1-y_2)\begin{bmatrix}v_2+\mathfrak{j}\\ k_2\end{bmatrix}\right\}y_2^{k_2-1}(1-y_2)^{v_2+\mathfrak{j}-(k_2+1)}.$$

Muraru [4] presented a speculation of Bernstein-Schurer operators based on q-calculus namely, q-Bernstein-Schurer operators, which are given as:

For fixed $\mathfrak{j}\in\in N_0$, $0<q<1$ and for all $0\leq\varkappa\leq1$,

$$B_{\mathfrak{l},q}^{\mathfrak{j}}(\mathfrak{f};\varkappa)=\sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}}\begin{bmatrix}\mathfrak{l}+\mathfrak{j}\\ \mu\end{bmatrix}_q\varkappa^{\mu}\prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1}(1-q^s\varkappa)\mathfrak{f}\left(\frac{[\mu]_q}{[\mathfrak{l}]_q}\right). \tag{1.1}$$

By putting $\mathfrak{j}=0$, (1.1) becomes classical q-Bernstein operators.

After numerous speculations of notable positive linear operators for faster rate of convergence, Mursaleen [5] was the first to construct and study the generalization of Bernstein operators in q calculus namely (p,q) variant of Bernstein operators along with its approximation properties. For $1\geq p>q>0$ and for some $\mathfrak{l}\in\mathbb{N}$, $\mathfrak{f}\in C[0,1]$, the (p,q) variant of Bernstein operators is defined by:

$$B_{\mathfrak{l}}^{p,q}(\mathfrak{f};\varkappa)=\sum_{\mu=0}^{\mathfrak{l}}\begin{bmatrix}\mathfrak{l}\\ \mu\end{bmatrix}_{p,q}\varkappa^{\mu}\prod_{s=0}^{\mathfrak{l}-\mu-1}(p^s-q^s\varkappa)\,\mathfrak{f}\left(\frac{[\mu]_{p,q}}{[\mathfrak{l}]_{p,q}}\right),\varkappa\in[0,1].$$

In addition to that, Mursaleen et.al [6] established and investigated the (p,q) variant of Bernstein-Schurer operators. In [6] Mursaleen and others generalized Bernstein-Schurer operators in q calculus using (p,q)-integers and obtained Korovkin's type approximation theorem, rate of convergence by utilizing modulus of continuity. These operators were elucidated for fixed $\mathfrak{j} \in \mathbb{N}_0$ and for every $\varkappa \in [0,1]$ as:

For $1 \geq p > q > 0$ and for some $\mathfrak{l} \in \mathbb{N}$, $\mathfrak{f} \in C_B[0, \mathfrak{j}+1]$ and $\varkappa \in [0,1]$,

$$B_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa) = \frac{1}{p^{\frac{(\mathfrak{l}+\mathfrak{j})(\mathfrak{l}+\mathfrak{j}-1)}{2}}} \sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}} \begin{bmatrix} \mathfrak{l}+\mathfrak{j} \\ \mu \end{bmatrix}_{p,q} p^{\frac{\mu(\mu-1)}{2}} \varkappa^{\mu} \prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1} (p^s - q^s \varkappa)\, \mathfrak{f}\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}} \right). \tag{1.2}$$

Before moving further, we recall some basic notations and outline of (p,q)-calculus.

For every non-negative integer r, the (p,q)-integer $[r]_{p,q}$ is given by

$[r]_{p,q} := \frac{p^r - q^r}{p-q}$, $r = 0, 1, 2, \ldots$ $1 \geq p > q > 0$.

The binomial expansion in (p,q) form is given by

$(ax+by)_{p,q}^r := \sum_{\mu=0}^{r} \begin{bmatrix} r \\ \mu \end{bmatrix}_{p,q} p^{\frac{(r-\mu)(r-\mu-1)}{2}} q^{\frac{\mu(\mu-1)}{2}} a^{r-\mu} b^{\mu} \varkappa^{r-\mu} y^{\mu}$

$(\varkappa+y)_{p,q}^r := (\varkappa+y)(px+qy)(p^2\varkappa+q^2y)\ldots(p^{r-1}\varkappa+q^{r-1}y)$.

The binomial coefficients in (p,q) form are given by

$\begin{bmatrix} r \\ \mu \end{bmatrix}_{p,q} := \frac{[r]_{p,q}!}{[\mu]_{p,q}![r-\mu]_{p,q}!}$.

Later on the idea of modification of linear positive operators using (p,q) variant was utilized by various researchers. Recently, Sharma and Abid [7] defined and obtained (p,q) variant of Szasz-beta-Stancu operators along with its approximation properties by utilizing modulus of continuity and weighted approximation. Mursaleen et.al [8] proposed GBS (generalized Boolean sum) operators of bivariate generalization of BernsteinSchurerStancu with the purpose of approximating continuous and differentiable functions in B*ȯgel* space. More details and applications on (p,q)-calculus can be obtained in ([6], [9]-[16]).

Let us assume that $e_i(t) = t^i, i = 0, 1, 2$.

Lemma 1.1. [6] *If $\varkappa \in [0,1]$ and $1 \geq p > q > 0$ then, the following results holds:*

(i) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(e_0;\varkappa) = 1$,

(ii) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(e_1;\varkappa) = \frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}\varkappa$,

(iii) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(e_2;\varkappa) = \frac{p^{\mathfrak{l}+\mathfrak{j}-1}[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}^2}\varkappa + \frac{q[\mathfrak{l}+\mathfrak{j}]_{p,q}[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}]_{p,q}^2}\varkappa^2$.

Lemma 1.2. [6] *If $\varkappa \in [0,1]$ and $1 \geq p > q > 0$ then, the following results holds:*

(i) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(e_1 - 1;\varkappa) = \frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}\varkappa - 1$,

(ii) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(e_1-\varkappa;\varkappa)=\left(\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}-1\right)\varkappa$,

(iii) $B_{\mathfrak{l},\mathfrak{j}}^{p,q}((e_1-\varkappa)^2;\varkappa)=\frac{p^{\mathfrak{l}+\mathfrak{j}-1}[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}^2}\varkappa+\left(1-2\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}+\frac{q[\mathfrak{l}+\mathfrak{j}]_{p,q}[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}]_{p,q}^2}\right)\varkappa^2$.

In order to get better error estimation, the another modification to linear positive operators was presented by King. In [17], King introduced an exotic sequence that preserves two test functions e_0 and e_2. Using this sequence of linear positive operators, King modified Bernstein operators to get better error estimation than the classical one. King's thought motivates numerous different mathematicians to construct other modifications of notable estimation measures fixing certain functions and examining their approximation and shape-preserving properties.

In [18] Duman, Özarslan and Aktuglu accompolished similar problems for Szász-Mirakyan-Beta type operators. In [19] Modified Sz'asz-Mirakyan type operators have been found to have local approximation features. Rempulska and Tomczak [20] have set another moderation to the King operators by introducing them on weighted space and by giving some important applications in relation to Baskakov, Post-Widder and Stancu operators. In [21], Mei-Ying Ren introduced another moderation to Bernstein-Schurer operators in q-calculus by introducing its King variant. Recently, Deo and Bhardwaj investigated new moderation of generalized Durrmeyer, Baskakov and Balázs type operators in [22], [23] and [24] respectively. Such type of modification is also studied by Kwun et.al. in [25] and by Radu in [26].

The objective of the present paper is to consider (p,q)-variant of Bernstein Schurer operators for the King modification and discussed its approximation properties. We derive asymptotic theorem namely Voronvskaja type theorem and using Peetre's μ-functional mathematical tool, the order of approximation for the recalled operators is established.

2. Construction of King Operators and Basic Properties

We assume that $\{u_n(p,q,\varkappa)(\varkappa)\}$ is a sequence consisting of continuous real-valued functions which are defined on $[0,\infty)$ with $0\le u_n(p,q,\varkappa)<\infty$, then , for $1\ge p>q>0$ and for some $\mathfrak{l}\in\mathbb{N}$ and fixed $\mathfrak{j}\in\mathbb{N}_0$, $\mathfrak{f}\in C_B[0,\mathfrak{j}+1]$,

$$\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa)=\frac{1}{p^{\frac{(\mathfrak{l}+\mathfrak{j})(\mathfrak{l}+\mathfrak{j}-1)}{2}}}\sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}}\left[\begin{array}{c}\mathfrak{l}+\mathfrak{j}\\ \mu\end{array}\right]_{p,q}p^{\frac{\mu(\mu-1)}{2}}(u_n(p,q,\varkappa))^{\mu}$$
$$\times\prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1}(p^s-q^s(u_n(p,q,\varkappa)))\,\mathfrak{f}\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}}\right),\tag{2.1}$$

where

$$u_n(p,q,\varkappa)=\frac{[\mathfrak{l}]_{p,q}}{[\mathfrak{j}+\mathfrak{l}]_{p,q}}\varkappa,\quad\varkappa\in[0,1]\,.$$

For $\mathfrak{f}\in C_B[0,\infty)$, the classical Peetre's K_2-functional is defined as:

$$K_2(\mathfrak{f},\delta)=\inf\left\{\|\mathfrak{f}-\mathfrak{g}\|+\delta\,\|\mathfrak{g}''\|:\mathfrak{g},\mathfrak{g}',\mathfrak{g}''\in C_B[0,\mathfrak{j}+1]\right\},\delta>0.$$

$$K_2(\mathfrak{f},\delta)\le C\omega_2\left(\mathfrak{f},\sqrt{\delta}\right),\tag{2.2}$$

where C is the non-negative constant.

The smoothness of a function $\mathfrak{f} \in C_B[0,\infty)$ (second order) is given by

$$\omega_2(\mathfrak{f}, \sqrt{\delta}) = \sup_{0<\mathfrak{h}\leq\delta} \sup_{\varkappa\in[0,\mathfrak{j}+1]} |\mathfrak{f}(\varkappa+2h) - 2f(\varkappa+\mathfrak{h}) + \mathfrak{f}(\varkappa)|.$$

Subsequently, following results are obatined.

Lemma 2.1. *Suppose $e_i(\varkappa) = \varkappa^i$ and $i = 0,1,2$, then for some $\varkappa \in [0,1]$ and $0 < q < p \leq 1$, the following results are obtained:*

(i) $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_0;\varkappa) = 1$,

(ii) $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_1;\varkappa) = \varkappa$,

(iii) $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_2;\varkappa) = \frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa + \frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}\varkappa^2$.

Lemma 2.2. *If $0 \leq \varkappa \leq 1$, $1 \geq p > q > 0$ and $\varphi_x(t) = e_1 - e_0\varkappa$, the following results are obtained:*

(i) $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\varphi_x;\varkappa) = 0$,

(ii) $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\varphi_x^2;\varkappa) = \frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa + \left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}} - 1\right)\varkappa^2$.

3. Error Estimation

In this part, for the modified operators $\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}$, a direct local approximation theorem in ordinary approximation is obtained then we build up the convergence rate and asymptotic theorem namely Voronovskaya result for these operators (2.1).

Theorem 3.1. *If $\mathfrak{f} \in C_B[0,\mathfrak{j}+1]$, then for all $\varkappa \in [0,1]$, we have*

$$\left|\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa) - \mathfrak{f}(\varkappa)\right| \leq 2\omega(\mathfrak{f},\delta^*_{\mathfrak{l}}),$$

where $\delta^*_{\mathfrak{l}} = \left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa + \left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}+\mathfrak{j}]_{p,q}} - 1\right)\varkappa^2\right)^{1/2}$.

Proof.

$$\begin{aligned}
&\left|\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa) - \mathfrak{f}(\varkappa)\right| \\
&\leq \frac{1}{p^{\frac{(\mathfrak{l}+\mathfrak{j})(\mathfrak{l}+\mathfrak{j}-1)}{2}}} \sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}} \left[\begin{array}{c}\mathfrak{l}+\mathfrak{j}\\ \mu\end{array}\right]_{p,q} p^{\frac{\mu(\mu-1)}{2}} (u_n(p,q,\varkappa))^{\mu} \\
&\times \prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1} (p^s - q^s(u_n(p,q,\varkappa))) \left|\mathfrak{f}\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}}\right) - \mathfrak{f}(\varkappa)\right| \\
&\leq \frac{1}{p^{\frac{(\mathfrak{l}+\mathfrak{j})(\mathfrak{l}+\mathfrak{j}-1)}{2}}} \sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}} \left[\begin{array}{c}\mathfrak{l}+\mathfrak{j}\\ \mu\end{array}\right]_{p,q} p^{\frac{\mu(\mu-1)}{2}} (u_n(p,q,\varkappa))^{\mu} \\
&\times \prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1} (p^s - q^s(u_n(p,q,\varkappa))) \left(\frac{\left|\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}} - \varkappa\right|}{\delta} + 1\right)\omega(\mathfrak{f},\delta).
\end{aligned}$$

Now, applying Cauchy-Schwarz inequality and Lemma 2.1,

$$\begin{aligned}
&\left|\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right| \\
&\leq\left[1+\frac{1}{\delta}\left\{\frac{1}{p^{\frac{(\mathfrak{l}+\mathfrak{j})(\mathfrak{l}+\mathfrak{j}-1)}{2}}}\sum_{\mu=0}^{\mathfrak{l}+\mathfrak{j}}\left[\begin{array}{c}\mathfrak{l}+\mathfrak{j}\\ \mu\end{array}\right]_{p,q}p^{\frac{\mu(\mu-1)}{2}}(u_n(p,q,\varkappa))^{\mu}\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}}\right)\right.\right. \\
&\times\left.\left.\prod_{s=0}^{\mathfrak{l}+\mathfrak{j}-\mu-1}(p^s-q^s(u_n(p,q,\varkappa)))\right\}^{\frac{1}{2}}\right]\left(\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_0;\varkappa)\right)^{\frac{1}{2}}\omega(\mathfrak{f},\delta) \\
&=\left\{\frac{1}{\delta}\left(\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_2;\varkappa)-2x\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_1;\varkappa)+\varkappa^2\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(e_0;\varkappa)\right)^{\frac{1}{2}}+1\right\}\omega(\mathfrak{f},\delta) \\
&=\left\{\frac{1}{\delta}\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}\varkappa^2-2\varkappa^2+\varkappa^2\right)^{\frac{1}{2}}+1\right\}\omega(\mathfrak{f},\delta) \\
&=\left\{\frac{1}{\delta}\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right)^{\frac{1}{2}}+1\right\}\omega(\mathfrak{f},\delta)
\end{aligned}$$

Taking $\delta^2=\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2$, we get the desired result by choosing $\delta^*_{\mathfrak{l}}=\delta$. ■

Remark 3.2. Under the similar states of Theorem 3.1,

$$\left|B^{p,q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right|\leq 2\omega(\mathfrak{f},\delta_{\mathfrak{l}}),$$

where $\delta_{\mathfrak{l}}=\varkappa\left|\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}-1\right|+\sqrt{\frac{[\mathfrak{j}+\mathfrak{l}]_{p,q}}{[\mathfrak{l}]_{p,q}}}\sqrt{\frac{\left(q[-1+\mathfrak{j}+\mathfrak{l}]_{p,q}-[\mathfrak{j}+\mathfrak{l}]_{p,q}\varkappa^2\right)+p^{\mathfrak{j}+\mathfrak{l}-1}\varkappa}{[\mathfrak{l}]_{p,q}}}$.

We estimate error for (2.1) through the use modulus of continuity and compared it with operators (1.2) for $p=1,\mathfrak{j}=2,q=0.6$ and different values of $\mathfrak{l}=2,3,12$ as shown in Table 1 and Table 2. Graphical illustration of error estimation $\delta_{\mathfrak{l}}$ and $\delta^*_{\mathfrak{l}}$ for both operators, (1.2) and (2.1) respectively are shown in Fig.1.

$\mathfrak{l}$	at $\varkappa=0.3$	at $\varkappa=0.6$	at $\varkappa=0.9$
2	0.5305	0.6677	0.6006
3	0.4079	0.4853	0.3911
12	0.2908	0.3112	0.1913

Table 1. Error estimation $\delta_{\mathfrak{l}}$, for the operators given by (1.2) for $\mathfrak{j}$=2, p=1, q=0.6.

$\mathfrak{l}$	at $\varkappa=0.3$	at $\varkappa=0.6$	at $\varkappa=0.9$
2	0.3823	0.4578	0.4362
3	0.3377	0.3873	0.3284
12	0.290	0.3105	0.1911

Table 2. Error estimation $\delta^*_{\mathfrak{l}}$, for operators given by (2.1) where $u_n(p,q,\varkappa)=\frac{[\mathfrak{l}]_{p,q}}{[\mathfrak{l}+\mathfrak{j}]_{p,q}}\varkappa$, $\varkappa\in[0,1]$ for $\mathfrak{j}=2$, $p=1$, $q=0.6$.

In all above cases, $\delta_{\mathfrak{l}}^{*}<\delta_{\mathfrak{l}}$. Hence, $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ has better error estimation than $B_{\mathfrak{l},\mathfrak{j}}^{p,q}$.

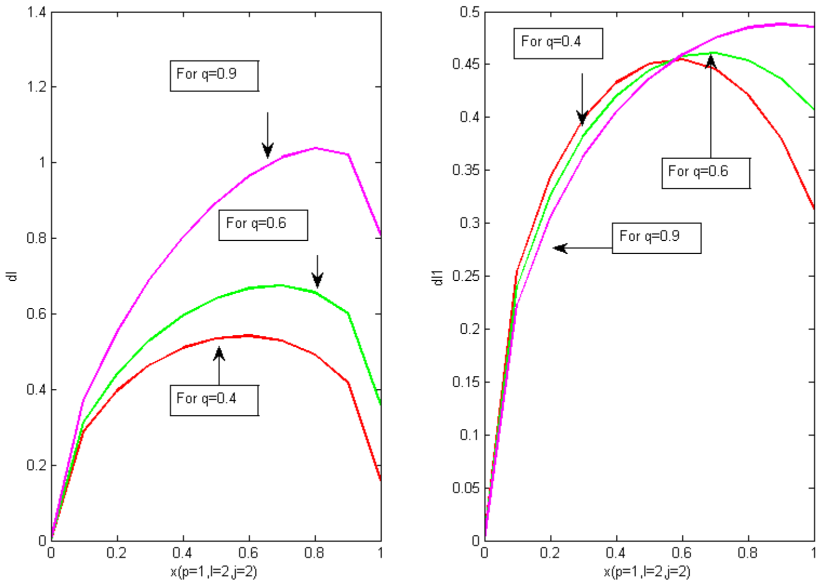

Fig. 1. Subplots of error estimation $\delta_{\mathfrak{l}}$ and $\delta_{\mathfrak{l}}^{*}$ for $B_{\mathfrak{l},\mathfrak{j}}^{p,q}$ and $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ respectively, where $d\mathfrak{l}=\delta_{\mathfrak{l}}$ and $d\mathfrak{l}1=\delta_{\mathfrak{l}}^{*}$.

4. Voronovskaya Type Results

In this segment, asymptotic results of the modified operators (2.1) are obtained, by means of Peetre's μ functional, continuity and usual Lipschitz class.

Theorem 4.1. *If $\mathfrak{f}\in C_B[0,\mathfrak{j}+1]$ satisfying $1\geq p>q>0$, then for $0\leq\varkappa\leq 1$ and constant $C>0, n\in\mathbb{N}$, we get*

$$\left|\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right|\leq C\omega_2(\mathfrak{f},\delta_{\mathfrak{l}}^{*}),$$

where $(\delta_{\mathfrak{l}}^{*})^2=\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right)$.

Proof. Let $\mathfrak{h}\in W^2$ and applying Taylor's expansion,

$$\mathfrak{h}(y)=\mathfrak{h}(\varkappa)+\mathfrak{h}'(\varkappa)(y-\varkappa)+\int_x^y(y-v)\,\mathfrak{h}''(v)\,du, y\in[0,n], n>0.$$

From Lemma 2.2, we get

$$\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{h};\varkappa)-\mathfrak{h}(\varkappa)=\left(\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\int_x^y(y-v)\,\mathfrak{h}''(v)\,du;p,q;\varkappa\right).$$

We know that

$$\left|\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{h};\varkappa)-\mathfrak{h}(\varkappa)\right|\leq\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\left(\left|\int_x^y|(y-v)|\,|\mathfrak{h}''(v)|\,du;p,q;\varkappa\right|\right)$$

and
$$\left|\int_x^y (y-v)\,\mathfrak{h}''(v)\,du\right| \le (y-\varkappa)^2 \|\mathfrak{h}''\|.$$
Therefore,
$$\left|\tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{h};\varkappa)-\mathfrak{h}(\varkappa)\right| \le \tilde{B}^{p,q}_{\mathfrak{l},\mathfrak{j}}\left((y-\varkappa)^2;p,q;\varkappa\right)\|\mathfrak{h}''\|.$$
$$\left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{h};\varkappa)-\mathfrak{h}(\varkappa)\right| \le \|\mathfrak{h}''\|\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[-1+\mathfrak{j}+\mathfrak{l}]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right).$$
By Lemma 2.1, we get
$$\left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)\right| \le \|\mathfrak{f}\|$$
Also
$$\begin{aligned}\left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right| &\le \left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}((\mathfrak{f}-\mathfrak{h});\varkappa)-(\mathfrak{f}-\mathfrak{h})(\varkappa)\right|+\left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{h};\varkappa)-\mathfrak{h}(\varkappa)\right|\\ &\le \|\mathfrak{f}-\mathfrak{h}\|+\|\mathfrak{h}''\|\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right).\end{aligned}$$
Taking infimum over $\mathfrak{h}\in W^2$ and using the definition of Peetre's K_2-functional, we get
$$\left|\tilde{B}^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right| \le CK_2\left(\mathfrak{f},\delta^{*2}_{\mathfrak{l}}\right),$$
where $\delta^{*2}_{\mathfrak{l}}=\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right)$.

Now using (2.2), we get the desired result. ■

Remark 4.2. Under the same conditions of Theorem 4.1,
$$\left|B^{p;q}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)-xh'(\varkappa)\left(\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}-1\right)\right| \le C\omega_2(\mathfrak{f},\delta_{\mathfrak{l}}),$$
where $\delta_{\mathfrak{l}}=\varkappa\left|\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}-1\right|+\sqrt{\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}}\sqrt{\frac{\left(q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}-[\mathfrak{l}+\mathfrak{j}]_{p,q}\varkappa^2\right)+p^{\mathfrak{l}+\mathfrak{j}-1}\varkappa}{[\mathfrak{l}]_{p,q}}}$.

Theorem 4.3. *If a function $\mathfrak{f}\in C_B[0,\mathfrak{j}+1]$ such that $\mathfrak{f}',\mathfrak{f}''\in C_B[0,\mathfrak{j}+1]$, satisfying $0<q_{\mathfrak{l}}<p_{\mathfrak{l}}\le 1$ where $\{p_{\mathfrak{l}}\}$ and $\{q_{\mathfrak{l}}\}$ are sequences with $\lim_{\mathfrak{l}\to\infty}p_{\mathfrak{l}}=1$ and $\lim_{\mathfrak{l}\to\infty}q_{\mathfrak{l}}=1$. Then*
$$\lim_{\mathfrak{l}\to\infty}[\mathfrak{l}]_{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left\{\tilde{B}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right\}=\frac{\mathfrak{f}''(\varkappa)}{2}\left(\lambda^*\varkappa+\alpha^*\varkappa^2\right),$$
where $\lambda^*,\alpha^*\in(0,1]$.

Proof. Applying Taylor's theorem, we get
$$\mathfrak{f}(t)-\mathfrak{f}(\varkappa)=(t-\varkappa)\mathfrak{f}'(\varkappa)+\frac{(t-\varkappa)^2}{2!}\mathfrak{f}''(\varkappa)+\frac{(t-\varkappa)^2}{2!}\xi(t,\varkappa),$$
where $\xi(t,\varkappa)$ is a remainder term and $\lim_{t\to\varkappa}\xi(t,\varkappa)=0$. Using Lemma 2.1, we have
$$\tilde{B}^{p_{\mathfrak{l}},q_m}_{\mathfrak{l},\mathfrak{j}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)=\mathfrak{f}'(\varkappa)\tilde{B}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}_{\mathfrak{l},\mathfrak{j}}(\varphi_x,\varkappa)+\frac{\mathfrak{f}''(\varkappa)}{2}\tilde{B}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}_{\mathfrak{l},\mathfrak{j}}(\varphi_x^2,\varkappa)+\frac{1}{2}\tilde{B}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}_{\mathfrak{l},\mathfrak{j}}(\varphi_x^2.\xi(t,\varkappa),\varkappa).\tag{4.1}$$

Using Cauchy-Schwarz inequality, we have

$$\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\xi(t,\varkappa).\varphi_x^2;\varkappa\right) \leq \sqrt{\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\xi^2(t,\varkappa);\varkappa\right)}\sqrt{\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\varphi_x^4(t,\varkappa);\varkappa\right)}.$$

Since $\xi^2(\varkappa,\varkappa)=0$ and $\xi^2(t,\varkappa)=0$.
Therefore $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\xi^2(t,\varkappa);\varkappa\right)=\xi^2(\varkappa,\varkappa)=0$ gives $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\xi^2(t,\varkappa)(t-\varkappa)^2;\varkappa\right)=0$.

$$\lim_{\mathfrak{l}\to\infty}[\mathfrak{l}]_{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left\{\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left((t-\varkappa)^2;\varkappa\right)\right\}=\lim_{\mathfrak{l}\to\infty}[\mathfrak{l}]_{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right). \tag{4.2}$$

Using Lemma 2.2, (4.1) and (4.2), we get

$$\lim_{\mathfrak{l}\to\infty}[\mathfrak{l}]_{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left\{\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right\}=\frac{\mathfrak{f}''(\varkappa)}{2}\left(\lambda^*\varkappa+\alpha^*\varkappa^2\right),$$

where $\lambda^*=\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}$ and $\alpha^*=\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{p[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1$ and $\lambda^*,\alpha^*\in(0,1]$ depending on the sequence $\{p_{\mathfrak{l}}\}$ and $\{q_{\mathfrak{l}}\}$. ∎

Remark 4.4. Under the conditions of Theorem 4.3,

$$\lim_{\mathfrak{l}\to\infty}[\mathfrak{l}]_{p_{\mathfrak{l}},q_{\mathfrak{l}}}\left\{B_{\mathfrak{l},\mathfrak{j}}^{p_{\mathfrak{l}},q_{\mathfrak{l}}}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right\}=\frac{\mathfrak{f}''(\varkappa)}{2}\left(\lambda\varkappa-\alpha\varkappa^2\right),$$

where $\lambda=\frac{p^{\mathfrak{l}+\mathfrak{j}-1}[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}^2}$, $\alpha=1-2\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}+\frac{q[\mathfrak{l}+\mathfrak{j}]_{p,q}[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}]_{p,q}^2}$ and $\lambda,\alpha\in(0,1]$.

Now using Lipschitz class $Lip_M(\gamma)$, $(0<\gamma\leq 1)$, convergence rate of the operators of $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ is obtained. We say that $\mathfrak{f}$ belongs to $Lip_M(\gamma)$, if following holds

$$|\mathfrak{f}(t)-\mathfrak{f}(\varkappa)|\leq\mathfrak{l}|t-\varkappa|^{\gamma}. \tag{4.3}$$

Theorem 4.5. *If $\mathfrak{f}\in Lip_M(\gamma)$ and for $1\geq p\geq q\geq 0$, we have*

$$\left|\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right|\leq\mathfrak{l}(\delta_{\mathfrak{l}}^*(\varkappa))^{\gamma}$$

where $\delta_{\mathfrak{l}}^*=\left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa+\left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}+\mathfrak{j}]_{p,q}}-1\right)\varkappa^2\right)^{1/2}$.

Proof. Since $\mathfrak{f}\in Lip_M(\gamma)$, from inequality (4.3) and Holder inequality with $p=\frac{2}{\gamma},q=\frac{2}{2-\gamma}$, we have

$$\begin{aligned}\left|\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa)-\mathfrak{f}(\varkappa)\right| &\leq \tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\left(|\mathfrak{f}(t)-\mathfrak{f}(\varkappa)|;p,q;\varkappa\right)\\ &\leq \tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\left(\left|\mathfrak{f}\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}}\right)-\mathfrak{f}(\varkappa)\right|\right)\\ &\leq \mathfrak{l}\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\left(\left|\left(\frac{[\mu]_{p,q}}{p^{\mu-\mathfrak{l}-\mathfrak{j}}[\mathfrak{l}]_{p,q}}\right)-\varkappa\right|^{\gamma}\right)\\ &\leq \mathfrak{l}\left\{\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}\left(\varphi_x^2;\varkappa\right)\right\}^{\gamma/2}\end{aligned}$$

Taking $\delta = \delta_{\mathfrak{l}}^{*}(\varkappa)$, we get

$$\begin{aligned}\left|\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{u};\varkappa) - \mathfrak{u}(\varkappa)\right| &\leq \mathfrak{l}\Big(\left(\delta_{\mathfrak{l}}^{*}(\varkappa)\right)^{2}\Big)^{\frac{\gamma}{2}} \\ &= \mathfrak{l}(\delta_{\mathfrak{l}}^{*}(\varkappa))^{\gamma},\end{aligned}$$

where $\delta_{\mathfrak{l}}^{*} = \left(\frac{p^{\mathfrak{l}+\mathfrak{j}}}{[\mathfrak{l}]_{p,q}}\varkappa + \left(\frac{q[\mathfrak{l}+\mathfrak{j}-1]_{p,q}}{[\mathfrak{l}+\mathfrak{j}]_{p,q}} - 1\right)\varkappa^{2}\right)^{1/2}$.
This completes the proof. ∎

Remark 4.6. Using Lemma 2.1, for the operator $B_{\mathfrak{l},\mathfrak{j}}^{p,q}$ under same conditions, then

$$\left|B_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{u};\varkappa) - \mathfrak{u}(\varkappa)\right| \leq \mathfrak{l}\delta_{\mathfrak{l}}^{\gamma}(\varkappa),$$

where $\delta_{\mathfrak{l}} = \varkappa\left|\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}} - 1\right| + \sqrt{\frac{[\mathfrak{l}+\mathfrak{j}]_{p,q}}{[\mathfrak{l}]_{p,q}}}\sqrt{\frac{\left(q[\mathfrak{l}+\mathfrak{j}-1]_{p,q} - [\mathfrak{l}+\mathfrak{j}]_{p,q}\varkappa^{2}\right) + p^{\mathfrak{l}+\mathfrak{j}-1}\varkappa}{[\mathfrak{l}]_{p,q}}}$.

5. Graphical Analysis

For $f_1(\varkappa) = \left(\varkappa - \frac{1}{2}\right)\left(\varkappa - \frac{1}{3}\right)\left(\varkappa - \frac{3}{4}\right)$ and $f_2(\varkappa) = 1 + \sin\left(-4\varkappa^{2}\right)$, and $\mathfrak{l} = 2, 3, 12$, the convergence of the operators $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ given by (2.1) for functions f_1 and f_2 are illustrated in Fig 2 and Fig 3., using MATLAB code.

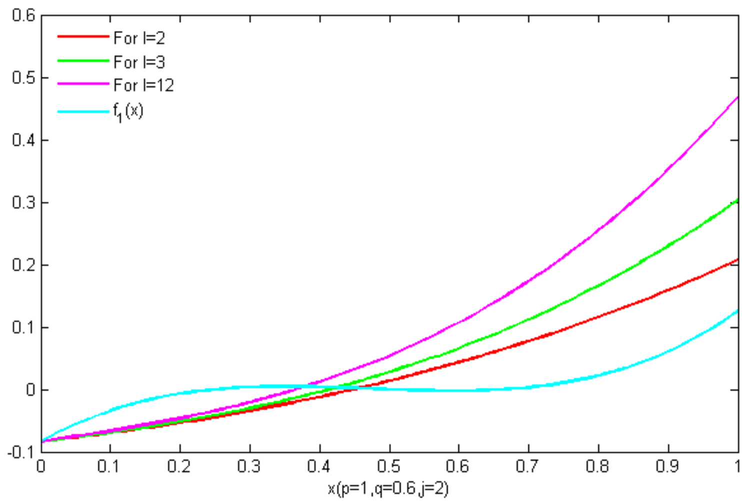

Fig. 2. For $\mathfrak{l} = 2, 3, 12$; Convergence of $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ to function $f_1(\varkappa) = \left(\varkappa - \frac{1}{2}\right)\left(\varkappa - \frac{2}{3}\right)\left(\varkappa - \frac{1}{4}\right)$.

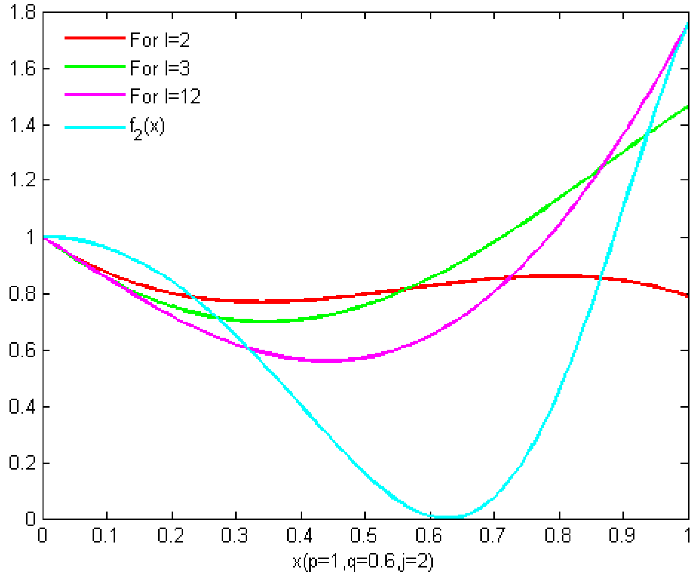

FIG. 3. For $\mathfrak{l}=2,3,12$; Convergence of $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ to function $f_2(\varkappa)=1+\sin\left(-4\varkappa^2\right)$.

6. Conclusion

The purpose of this paper is to construct the King modification of (p,q)-variant of Bernstein Schurer operators and discuss its approximation properties. The motivation behind introducing this new type of operators is to have better error estimation. In order to demonstrate that these newly defined operators $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ furnished better error approximation than (p,q)-variant of Bernstein Schurer operators $B_{\mathfrak{l},\mathfrak{j}}^{p,q}$, graphical illustration of error estimation for both operators are shown in Fig 1. We observe that the error in approximation by $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ is much smaller than the error in approximation by $B_{\mathfrak{l},\mathfrak{j}}^{p,q}(\mathfrak{f};\varkappa)$. Thus we conclude that King modification enables better error estimation in comparison to the (p,q)-variant of Bernstein Schurer operators. Also from the above illustrative graphics, we conclude that the rate of convergence of $\tilde{B}_{\mathfrak{l},\mathfrak{j}}^{p,q}$ to the function becomes faster as the parameter $\mathfrak{l}$ decreases i.e the error decreases with the decrease in the parameter $\mathfrak{l}$.

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