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On the Diophantine Equation $x^2 - kxy + ky^2 + 3^n y = 0,\ n = 1, 2, 3$

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Abstract In this article, we determine all values of k when the equation $x^2 - kxy + ky^2 + 3^n y = 0$ where $n = 1, 2, 3$ has infinitely many positive solutions.

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1. Introduction

There have been many works on the Diophantine equation of the form

$$x^2 - kxy + ly^2 + mx + ny = 0$$

for different values of the integers k, l, m and n. In 2004, Marlewski and Zarzycki [1] proved that the equation $x^2 - kxy + y^2 + x = 0$ has infinitely many positive integer solutions if and only if $k = 3$. Their results are based on data obtained by computer experiments. The results suggest that for many k there are infinitely many integer solutions. Thus, Yuan and Hu [2] demonstrated in 2011 that $x^2 - kxy + y^2 + 2x = 0$ has infinitely many positive integer solutions if and only if $k = 3, 4$ and the equation $x^2 - kxy + y^2 + 4x = 0$ has infinitely many positive integer solutions if and only if $k = 3, 4, 6$. Motivated by Yuan and Hu [2], Keskin, Karaatli and Siar [3] showed that the Diophantine equation $x^2 - kxy + y^2 + 2^n = 0$ for $0 \leq n \leq 10$ has infinitely many solutions in positive integers and provided all positive integer solutions. Moreover, in the same year, Karaatli and Siar [4] determined the values of k when the Diophantine equation $x^2 - kxy + ky^2 + ly = 0$ for $l \in \{1, 2, 4, 8\}$ has infinitely many positive integers x and y. The values of such (l, k) are $(1, 5), (2, 5), (2, 6), (4, 5), (4, 6), (4, 8), (8, 5), (8, 6), (8, 8)$, and $(8, 12)$.

Expanding on the work of Yuan and Hu [2] and Karaatli and Siar [4], Keskin, Karaatli and Siar [5] showed that the Diophantine equation $x^2 - kxy + y^2 - 2^n = 0$ for $0 \leq n \leq 10$ has infinitely many solutions in positive integers and also gave all positive integer solutions in terms of generalized Fibonacci sequence. Moreover they formulated a conjecture when

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the Diophantine equation of the form $x^2-kxy+y^2-2^n=0$ has positive integer solutions which was proved later by Boumahdi, Kihel, Mavecha [6] in 2018.

In 2017, Mavecha [7] studied the equation of the form the equation $x^2-kxy+ky^2+ly=0$ for $l=2^n$ and showed that $k=5$ is the only odd integer k such that the equation has an infinitely many solutions in positive integers. The same equations were studied again in 2021 by Alkabouss, Benseba, Berbara, Earp-Lynch, and Luca [8]. They determined conditions when the equation

$$x^2-kxy+ky^2+ly=0 \tag{1.1}$$

has infinitely many positive integer solutions. They showed that if $l^2<k$ and equation (1.1) has infinitely many positive integer solutions, then $(l,k)\in\{(1,5),(2,5),(2,6)\}$. Moreover, they also determined the values of k when $l=2^s$, $l=p^n$ for $p\equiv 3 \mod 4$, $k\equiv 2 \mod p$, and $l=2^a3^b$, $k=2k'+1$ where $k'\equiv 2 \mod 3$.

Inspired by Mavecha [7] as well as Alkabouss, Benseba, Berbara, Earp-Lynch, and Luca [8], we will use the techniques that we develop to determine when the equation $x^2-kxy+ky^2+3^ny=0$ where $n=1,2,3$ has infinitely many positive solutions.

2. Preliminaries

In this section, we will review some results and provide a definition that is necessary for the proof of our main theorems.

Lemma 2.1. [9] *Let N,D be odd positive integers with D non-square. Suppose that the equation*

$$x^2-Dy^2=4, gcd(x,y)=1$$

is solvable and let $x_0+y_0\sqrt{D}$ be the least solution. If the equation

$$u^2-Dv^2=-4N, u,v\in \mathrm{Z},$$

where $gcd(u,v)\mid 2$, is solvable, then $u^2-Dv^2=-4N$ has a solution $u_0+v_0\sqrt{D}$ with the following property:

$$0<v_0\le \frac{y_0\sqrt{N}}{\sqrt{(x_0-2)}},\quad 0\le u_0\le\sqrt{(x_0-2)N}.$$

Lemma 2.2. [9] *Let N,D be positive integers with D non-square. Suppose that $x_0+y_0\sqrt{D}$ is the fundamental solution of the Pell equation $x^2-Dy^2=1$ and the equation*

$$u^2-Dv^2=-N, gcd(u,v)=1$$

is solvable. Then $u^2-Dv^2=-N$ has a solution $u_0+v_0\sqrt{D}$ with the following property:

$$0<v_0\le \frac{y_0\sqrt{N}}{\sqrt{2(x_0-1)}},\quad 0\le u_0\le\sqrt{\frac{1}{2}(x_0-2)N}.$$

In the next section, we will consider when the Diophantine equation of the form $x^2-kxy+ky^2+ly=0$ has an infinitely many positive integer solutions. So, for the sake of convenience, we define the following.

Definition 2.3. For a positive integer l, let $T(l)$ be the set of integers k for which the equation

$$x^2 - kxy + ky^2 + ly = 0 \tag{2.1}$$

has infinitely many positive integer solutions and let $T'(l)$ be the set of integers k for which the equation

$$x^2 - kxy + ky^2 + l = 0 \tag{2.2}$$

has infinitely many positive integer solutions (x, y) where $\gcd(x, y) = 1$.

Some results concerning the set $T(1), T(2), T(4)$ and $T(8)$ are provided in [3].

Theorem 2.4. [3] *$T(1) = \{5\}$, $T(2) = \{5, 6\}$, $T(4) = \{5, 6, 8\}$, and $T(8) = \{5, 6, 8, 12\}$.*

3. Main Results

In this section, we will show that $T(3) = \{5, 7\}$, $T(9) = \{5, 7, 9, 13\}$ and $T(27) = \{5, 6, 7, 9, 13, 18, 31\}$. First, we give a lemma which is applied in the following theorem.

Lemma 3.1. *Let k and l be positive integers. If $x^2 - kxy + ky^2 + ly = 0$ then there exist positive integers d, a, b such that $\gcd(a, b) = 1$, $d \mid l$, $x = dab$ and $y = da^2$.*

Proof. Assume that $x^2 - kxy + ky^2 + ly = 0$ for some positive integers x and y. Then it follows that $y \mid x^2$ and thus $x^2 = yz$ for some positive integer z. Let $d = gcd(y, z)$. Then $y = dm$ and $z = dn$ for some positive integers m, n and $gcd(m, n) = 1$. Since $x^2 = yz = d^2mn$, we obtain $m = a^2$ and $n = b^2$ for some positive integers a, b and $\gcd(a, b) = 1$. Hence, $x = dab$ and $y = da^2$. Substituting the values of x and y into equation $x^2 - kxy + ky^2 + ly = 0$, we have $(dab)^2 - k(dab)(da^2) + k(da^2)^2 + lda^2 = 0$. Thus, we have $db^2 - kdab + kda^2 + l = 0$. It follows that $d \mid l$. ∎

Our next theorem shows that the set of integers k for which the equation (2.1) has infinitely many positive solutions can be obtained from the set of integers k for which the equation (2.2) has infinitely many positive solutions.

Theorem 3.2. *Let p be a prime and n be a positive integer. Then*

$$T(p^n) = \bigcup_{k=0}^{n} T'(p^k).$$

Moreover, $T(p^n) = T(p^{n-1}) \cup T'(p^n)$.

Proof. Assume that $x^2 - lxy + ly^2 + p^n y = 0$ has infinitely many solutions. By Lemma 3.1, there exist positive integers d, a, b such that $\gcd(a, b) = 1$, $x = dab, y = da^2$ and $d \mid p^n$. Thus $d = p^k$ for some $0 \le k \le n$. Suppose $d = p^k$. Then $x = p^k ab$, $y = p^k a^2$, and $gcd(a, b) = 1$. Substituting the values of x and y into the equation $x^2 - lxy + ly^2 + p^n y = 0$, we obtain $b^2 - lab + la^2 + p^{n-k} = 0$. If $l \in T(p^n)$ then $l \in T'(p^k)$ for some $k \le n$. Conversely, suppose $k \in T'(p^l)$ for some $l \le n$. Then the equation $x^2 - kxy + ky^2 + p^l = 0$ has infinitely many positive solutions (x, y) where $\gcd(x, y) = 1$. Multiplying by $p^{2n-2l}y^2$, we obtain

$$(p^{n-l}xy)^2 - k(p^{n-l}xy)(p^{n-l}y^2) + k(p^{n-l}y^2)^2 + p^n(p^{n-l}y^2) = 0.$$

Thus $k \in T(p^n)$ as desired. ∎

In the next lemma, we provide the bound for a solution a of the equation $b^2 - kab + ka^2 + N = 0$ for given integers k and N. Later we give a lower bound for k.

Lemma 3.3. *Let $k > 4$ be a positive integer and N be an odd integer. If $b^2 - kab + ka^2 + N = 0$ and $2b - ka \neq 0$ then $0 < a \leq \frac{\sqrt{N}}{\sqrt{k-4}}$ and there are infinitely many positive integers a, b satisfying $b^2 - kab + ka^2 + N = 0$.*

Proof. Suppose $b^2 - kab + ka^2 + N = 0$ where $\gcd(a, b) = 1$. Then

$$\begin{aligned} 4b^2 - 4kab + 4ka^2 &= -4N \\ (2b - ka)^2 - k^2a^2 + 4ka^2 &= -4N \\ (2b - ka)^2 - ((k-2)^2 - 4)a^2 &= -4N. \end{aligned}$$

Let $u = 2b - ka$,$v = a$. Since $\gcd(a, b) = 1$, $\gcd(u, v) = 1$ or 2. We have

$$u^2 - ((k-2)^2 - 4)v^2 = -4N.$$

Note that $(x_0, y_0) = (k - 2, 1)$ is a fundamental solution of the equation

$$x^2 - ((k-2)^2 - 4)y^2 = 4.$$

For $k > 4$, we have $(k-3)^2 < (k-2)^2 - 4 < (k-2)^2$. Thus $(k-2)^2 - 4$ is not a perfect square. By Lemma 2.1, we have a solution (u_0, v_0) of equation $u^2 - ((k-2)^2 - 4)v^2 = -4N$ with the following property: $0 < v_0 \leq \frac{\sqrt{N}}{\sqrt{k-4}}$. ∎

Lemma 3.4. *For positive integers l and k, if an equation $x^2 - kxy + ky^2 + l = 0$ is solvable then $k > 4$.*

Proof. Suppose $x^2 - kxy + ky^2 + l = 0$. Then $(2x - ky)^2 + y^2(4k - k^2) = -4l$. This implies that $4k - k^2 < 0$. Since $k > 0$, we have $k > 4$. ∎

We are now ready to prove our main theorems.

Theorem 3.5. *The equation $x^2 - kxy + ky^2 + 3y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 7$.*

Proof. By Theorem 3.2, we have that $T(3) = T(1) \cup T'(3)$. By Theorem 2.4, we have $T(1) = \{5\}$. We next find the set $T'(3)$. Suppose $b^2 - kab + ka^2 + 3 = 0$ where $\gcd(a, b) = 1$. By Lemma 3.4, we consider the case $k > 4$. If $2b - ka = 0$, then there are finitely positive solutions (a, b). We now suppose that $2b - ka \neq 0$. Then by Lemma 3.3, we have $0 < a \leq \frac{\sqrt{3}}{\sqrt{k-4}} < 2$. Thus $a = 1$. This implies that $b^2 - kb + k + 3 = 0$. Hence $(k - 1 - b)(-1 + b) = 4$. Since b is positive, we obtain that $b - 1 = 1$ and $k - 1 - b = 4$ or $b - 1 = 2$ and $k - 1 - b = 2$ or $b - 1 = 4$ and $k - 1 - b = 1$. We have $(b, k) = (2, 7)$, $(3, 6)$ or $(5, 7)$. Since $2b - k \neq 0$, only the case $k = 7$ holds. ∎

Theorem 3.6. *The equation $x^2 - kxy + ky^2 + 9y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 7, 9, 13$.*

Proof. By Theorem 3.2, we have that $T(9) = T(3) \cup T'(9)$. By Theorem 3.5, we have $T(3) = \{5, 7\}$. We next find the set $T'(9)$. By Lemma 3.4, we consider the case $k > 4$. If $2b - ka = 0$, then there are finitely positive solutions (a, b). We now suppose that $2b - ka \neq 0$ and $b^2 - kab + ka^2 + 9 = 0$ where $\gcd(a, b) = 1$. By Lemma 3.3, we have $0 < a \leq \frac{\sqrt{9}}{\sqrt{k-4}} \leq 3$. Thus $a = 1, 2$ or 3.

Case 1 $a = 1$. We have $b^2 - kb + k + 9 = 0$. Hence $(k - b - 1)(b - 1) = 10$. Then $k = 10/(b-1) + b + 1$. Since b is positive and $(b-1) \mid 10$, we have $(b, k) \in \{(2, 13), (3, 9), (6, 9), (11, 13)\}$.

Case 2 $a = 2$. Then $b^2 - 2kb + 4k + 9 = 0$. Hence $(2k - b - 2)(b - 2) = 13$. Then $k = 13/2(b-2) + b/2 + 1$. Since k is an integer and $b > 0$, $b = 3$ or $b = 15$. Thus $k = 9$.

Case 3 $a = 3$. Then $b^2 - 3kb + 9k + 9 = 0$. Hence $(3k - b - 3)(b - 3) = 18$. Then $k = 6/(b-3) + b/3 + 1$. Since k is an integer and $b > 0$, we have $b = 6$ or $b = 9$. Thus $k = 5$. This completes the proof. ∎

Theorem 3.7. *The equation $x^2 - kxy + ky^2 + 27y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 6, 7, 9, 13, 18, 31$.*

Proof. By Theorem 3.2, we have that $T(27) = T(9) \cup T'(27)$. By Theorem 3.6, we have $T(9) = \{5, 7, 9, 13\}$. We next find the set $T'(27)$. Suppose $b^2 - kab + ka^2 + 27 = 0$ where $\gcd(a, b) = 1$. By Lemma 3.3, we have $0 < a \le \frac{\sqrt{27}}{\sqrt{k-4}} < 6$. Thus $a = 1, 2, 3, 4$ or 5.

Case 1 $a = 1$. We have $b^2 - kb + k + 27 = 0$. Hence $(k - b - 1)(b - 1) = 28$. Then $k = 28/(b-1) + b + 1$. Since k is an integer and $b > 0$, we have $b = 2, 3, 5, 8, 29$. Thus $k = 13, 18, 31$.

Case 2 $a = 2$. Then $b^2 - 2kb + 4k + 27 = 0$. Hence $(2k - b - 2)(b - 2) = 31$. Then $k = 31/2(b-2) + b/2 + 1$. Since k is an integer and $b > 0$, we have $b = 3, 33$. Thus $k = 18$.

Case 3 $a = 3$. Then $b^2 - 3kb + 9k + 27 = 0$. Hence $(3k - b - 3)(b - 3) = 36$. $k = 12/(b-3) + b/3 + 1$. Since k is an integer and $b > 0$, we have $b = 6, 9, 15$. Thus $k = 6, 7$.

Case 4 $a = 4$. Then $b^2 - 4kb + 16k + 27 = 0$. Hence $(4k - b - 4)(b - 4) = 43$. $k = 43/4(b-4) + b/4 + 1$. Since k is an integer and $b > 0$, we have $b = 5, 47$. Thus $k = 13$.

Case 5 $a = 5$. Then $b^2 - 5kb + 25k + 27 = 0$. Hence $(5k - b - 5)(b - 5) = 52$. $k = 52/5(b-5) + b/5 + 1$. Since k is an integer, no such integer b satisfies. ∎

In conclusion, we present helpful results that enable us to determine for which values of k there exist infinitely many positive solutions to the Diophantine equation (2.1) and we then use our tools to derive all possible values of k for $l = 3, 9$ and 27.

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