Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 137–149

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Common Fixed Point Theorems

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Abstract : In this note we establish a common fixed point theorem for a quadruple of self-mappings satisfying a generalized contractive condition in a normed space which extends the result of Rashwan [2]. We also prove some fixed point theorems with asymptotic regularity condition for a quadruple of mappings. These theorems generalize and extend results of Sastry et al. [3] and Zeqing Liu et al. [4].

Keywords : Common fixed points; Generalized contractive; Mann iteration; Normed spaces; Orbit; Orbitally continuous; Orbitally complete; Asymptotically regular.
2000 Mathematics Subject Classification : 54H25, 47H10.

1 Introduction

The following definitions were used in [1] and [2] respectively.

Definition 1.1 Let $(N, ||.||)$ be a normed space. Then T_1 and T_2 be two self-mappings of N called a *generalized contractive pair of mappings* if

$$||T_1x - T_2y|| \leq \max\left\{||x-y||, \frac{||x-T_1x||[1-||x-T_2y||]}{1+||x-T_1x||}, \frac{||x-T_2y||[1-||x-T_1x||]}{1+||x-T_2y||}, \frac{||T_1x-y||[1-||y-T_2y||]}{1+||T_1x-y||}, \frac{||y-T_2y||[1-||T_1x-y||]}{1+||y-T_2y||}\right\},$$

for all x, y in X, where $0 < q < 1$.

Definition 1.2 Let T_1 and T_2 be two self-mappings of a Banach space B. The *Mann iterative process* associated with T_1 and T_2 is defined in the following manner. Let x_0 be in N and let

$$x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1x_{2n},$$
$$x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2x_{2n+1},$$

for $n = 0, 1, 2, \ldots$, where c_n satisfies (i) $c_0 = 1$, (ii) $0 < c_n < 1$, $n = 1, 2, \ldots$ and (iii) $\lim_{n\to\infty} c_n = h > 0$.

In [1], Pathak proved the following common fixed point theorem:

Theorem 1.3 *Let X be a closed convex subset of a normed space N and let T_1 and T_2 be two continuous self mappings satisfying Definition 1.1 on X. Let x_0 be an arbitrary point in X. Then sequence of Mann iterates $\{x_n\}$ associated with T_1 and T_2 is defined by*

$$x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1x_{2n},$$
$$x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2x_{2n+1},$$

for $n = 0, 1, 2, \ldots$, where $\{c_n\}$ satisfies conditions (i), (ii) and (iii) of Definition 1.2. If $\{x_n\}$ converges to u in X and if u is fixed point of of either T_1 or T_2, then u is the common fixed point of T_1 and T_2.

In [2], Rashwan extended Theorem 1.3 for three mappings as follows:

Theorem 1.4 *Let X be a closed convex subset of a normed space N. Let T_1 and T_2 be mappings of X into X and f a continuous mapping of X into X such that*

$$\begin{aligned}
||T_1x - T_2y|| \leq q\max\Bigg\{&||fx - fy||, \frac{||fx - T_1x||[1 - ||fx - T_2y||]}{1 + ||fx - T_1x||}, \\
&\frac{||fx - T_2y||[1 - ||fx - T_1x||]}{1 + ||fx - T_2y||}, \frac{||T_1x - fy||[1 - ||fy - T_2y||]}{1 + ||T_1x - fy||}, \\
&\frac{||fy - T_2y||[1 - ||T_1x - fy||]}{1 + ||fy - T_2y||}\Bigg\}, \\
||fx - fy|| \leq ||T_1x - fx|| &+ ||T_1x - T_2y|| + ||T_2y - fy||,
\end{aligned}$$

for all x, y in X, where $0 < q < 1$, and the sequence $\{fx_n\}$ associated with T_1 and T_2 is given by

$$fx_{2n+1} = (1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n},$$
$$fx_{2n+2} = (1 - c_{2n+1})fx_{2n+1} + c_{2n+1}T_2x_{2n+1},$$

for $n = 0, 1, 2, \ldots$, where $\{c_n\}$ satisfies conditions (i), (ii) and (iii) as given above and x_0 is an arbitrary point in X. If $\{fx_n\}$ converges to a point u in X, then u is a common fixed point of T_1, T_2 and f.

2 Main Results

We extend Theorem 1.4 for a quadruple of self-mappings as follows:

Theorem 2.1 *Let X be a closed convex subset of a normed space N. Let T_1, T_2 be mappings of X into X and let f and g be injective and continuous mappings of X into X satisfying*

$$\begin{aligned} ||T_1x - T_2y|| \leq q \max\Bigg\{ & ||fx - gy||, \frac{||fx - T_1x||[1 - ||fx - T_2y||]}{1 + ||fx - T_1x||}, \\ & \frac{||fx - T_2y||[1 - ||fx - T_1x||]}{1 + ||fx - T_2y||}, \frac{||T_1x - gy||[1 - ||gy - T_2y||]}{1 + ||T_1x - gy||}, \\ & \frac{||gy - T_2y||[1 - ||T_1x - gy||]}{1 + ||gy - T_2y||}\Bigg\} \end{aligned} \tag{2.1}$$

$$\begin{aligned} ||fx - fgy|| \leq & ||T_1x - fx|| + ||T_1x - T_2y|| \\ & + ||T_2y - gy|| + ||gy - fx|| \end{aligned} \tag{2.2}$$

$$\begin{aligned} ||gy - gfx|| \leq & ||T_1x - gy|| + ||T_1x - T_2y|| \\ & + ||T_2y - fx|| + ||gy - fx|| \end{aligned} \tag{2.3}$$

for all x, y in X, where $0 < q < 1$,

$$(1 - \lambda)f(X) + \lambda T_1(X) \subseteq g(X), \tag{2.4}$$

$$(1 - \mu)g(X) + \mu T_2(X) \subseteq f(X) \tag{2.5}$$

for all $\lambda, \mu \in (0, 1]$, the sequence $\{x_n\}$ associated with the mappings T_1, T_2, f and g is defined by

$$x_{2n+1} \in g^{-1}[(1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n}], \tag{2.6}$$

$$x_{2n+2} \in f^{-1}[(1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1}] \tag{2.7}$$

$n = 0, 1, 2, \ldots$, where x_0 is an arbitrary point in X and $\{y_n\}$ is the sequence defined by $y_{2n-1} = fx_{2n-1}$ and $y_{2n} = gx_{2n}$ for $n = 1, 2, \ldots$ and $\{c_n\}$ satisfies conditions (i), (ii) and (iii) given above. If $\{y_n\}$ converges to a point u in X, then u is the unique common fixed point of T_1, T_2, f and g.

Proof. Since f and g are injective and satisfy conditions (2.4) and (2.5), the sequence $\{x_n\}$ defined by equations (2.6) and (2.7) is unique. Also from equation (2.6), we have

$$T_1x_{2n} = \frac{gx_{2n+1} - (1 - c_{2n})fx_{2n}}{c_{2n}},$$

and so

$$\begin{aligned} \lim_{n\to\infty} T_1x_{2n} &= \lim_{n\to\infty} \frac{gx_{2n+1} - (1 - c_{2n})fx_{2n}}{c_{2n}} \\ &= \frac{u - (1 - h)u}{h} = u. \end{aligned}$$

Similarly

$$\lim_{n\to\infty} T_2x_{2n+1} = u.$$

From equation (2.2), we have

$$\begin{aligned}||fx_{2n} - fgx_{2n+1}|| \le\ & ||T_1x_{2n} - fx_{2n}|| + ||T_1x_{2n} - T_2x_{2n+1}|| \\ & + ||T_2x_{2n+1} - gx_{2n+1}|| + ||gx_{2n+1} - fx_{2n}||,\end{aligned}$$

and so

$$\lim_{n\to\infty} ||fx_{2n} - fgx_{2n+1}|| = \lim_{n\to\infty} ||y_{2n} - fy_{2n+1}|| = ||u - fu|| \le 0.$$

It follows that $u = fu$.

Also from (2.3), we have

$$\begin{aligned}||gx_{2n+1} - gfx_{2n}|| \le\ & ||T_1x_{2n} - gx_{2n+1}|| + ||T_1x_{2n} - T_2x_{2n+1}|| \\ & + ||T_2x_{2n+1} - fx_{2n}|| + ||gx_{2n+1} - fx_{2n+1}||,\end{aligned}$$

and so

$$\lim_{n\to\infty} ||gx_{2n+1} - gfx_{2n}|| = \lim_{n\to\infty} ||y_{2n+1} - gy_{2n}|| = ||u - gu|| \le 0.$$

It follows that $u = gu$.

Further, using inequality (2.1), we have

$$\begin{aligned}||u - T_2u|| &\le ||u - gx_{2n+1}|| + ||gx_{2n+1} - T_2u|| \\ &\le ||u - gx_{2n+1}|| + ||(1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n} - T_2u|| \\ &\le ||u - gx_{2n+1}|| + (1 - c_{2n})||fx_{2n} - T_2u|| + c_{2n}||T_1x_{2n} - T_2u|| \\ &\le ||u - gx_{2n+1}|| + (1 - c_{2n})||fx_{2n} - T_2u|| \\ &\quad + c_{2n}q \max\left\{||fx_{2n} - gu||, \frac{||fx_{2n} - T_1x_{2n}||[1 - ||fx_{2n} - T_2u||]}{1 + ||fx_{2n} - T_1x_{2n}||}\right., \\ &\quad \frac{||fx_{2n} - T_2u||[1 - ||fx_{2n} - T_1x_{2n}||]}{1 + ||fx_{2n} - T_2u||}, \\ &\quad \frac{||T_1x_{2n} - gu||[1 - ||gu - T_2u||]}{1 + ||T_1x_{2n} - gu||}, \\ &\quad \left.\frac{||gu - T_2u||[1 - ||T_1x_{2n} - gu||]}{1 + ||gu - T_2u||}\right\}.\end{aligned}$$

Assuming that $T_2u \neq u$, we have on letting n tends to infinity

$$\begin{aligned}||u - T_2u|| &\le 0 + (1-h)||u - T_2u|| \\ &\quad + hq \max\left\{0, 0, \frac{||u - T_2u||}{1 + ||u - T_2u||}, 0, \frac{||u - T_2u||}{1 + ||u - T_2u||}\right\} \\ &\le (1-h)||u - T_2u|| + hq\frac{||u - T_2u||}{1 + ||u - T_2u||} \\ &< (1 - h + hq)||u - T_2u|| \\ &< ||u - T_2u||,\end{aligned}$$

a contradiction, and so $u = T_2u$.

Similarly

$$\begin{aligned}\|u - T_1u\| &\le \|u - fx_{2n+2}\| + \|fx_{2n+2} - T_1u\| \\ &\le \|u - fx_{2n+2}\| + \|(1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1} - T_1u\| \\ &\le \|u - fx_{2n+2}\| + (1 - c_{2n+1})\|gx_{2n+1} - T_1u\| \\ &\quad + c_{2n+1}\|T_1u - T_2x_{2n+1}\| \\ &\le \|u - fx_{2n+2}\| + (1 - c_{2n+1})\|gx_{2n+1} - T_1u\| \\ &\quad + c_{2n+1}q \max\Big\{||fu - gx_{2n+1}||, \\ &\quad \frac{\|fu - T_1u\|[1 - \|fu - T_2x_{2n+1}\|]}{1 + \|fu - T_1u\|}, \\ &\quad \frac{\|fu - T_2x_{2n+1}\|[1 - \|fu - T_1u\|]}{1 + \|fu - T_2x_{2n+1}\|}, \\ &\quad \frac{\|T_1u - gx_{2n+1}\|[1 - \|gx_{2n+1} - T_2x_{2n+1}\|]}{1 + \|T_1u - gx_{2n+1}\|}, \\ &\quad \frac{\|gx_{2n+1} - T_2x_{2n+1}\|[1 - \|T_1u - gx_{2n+1}\|]}{1 + \|gx_{2n+1} - T_2x_{2n+1}\|}\Big\}.\end{aligned}$$

Assuming that $T_1u \neq u$, we have on letting n tend to infinity

$$\begin{aligned}||u - T_1u|| &\le 0 + (1-h)||u - T_1u|| + \\ &\quad h \max\left\{0, \frac{||u - T_1u||}{1 + ||u - T_1u||}, 0, \frac{||u - T_1u||}{1 + ||u - T_1u||}\right\} \\ &\le (1-h)||u - T_1u|| + \frac{hq||u - T_1u||}{1 + ||u - T_1u||} \\ &< (1 - h + hq)||u - T_1u|| \\ &< ||u - T_1u||,\end{aligned}$$

a contradiction, so that $u = T_1u$. We have therefore proved that u is a common fixed point of T_1, T_2, f and g.

To prove the uniqueness of u, suppose that v is a second common fixed point of T_1, T_2, f and g. Then

$$\begin{aligned}
||u-v|| &= ||T_1u - T_2v|| \\
&\le q\max\left\{||fu-gv||, \frac{||fu-T_1u||[1-||fu-T_2v||]}{1+||fu-T_1u||},\right. \\
&\quad \frac{||fu-T_2v||[1-||fu-T_1u||]}{1+||fu-T_2v||}, \frac{||T_1u-gv||[1-||gv-T_2v||]}{1+||T_1u-gv||}, \\
&\quad \left.\frac{||gv-T_2v||[1-||T_1u-gv||]}{1+||gv-T_2v||}\right\} \\
&= q\max\left\{||u-v||, \frac{||u-u||[1-||u-v||]}{1+||u-u||},\right. \\
&\quad \frac{||u-v||[1-||u-u||]}{1+||u-v||}, \frac{||u-v||[1-||v-v||]}{1+||u-v||}, \\
&\quad \left.\frac{||v-v||[1-||u-v||]}{1+||v-v||}\right\} \\
&= q\max\left\{||u-v||, 0, \frac{||u-v||}{1+||u-v||}, \frac{||u-v||}{1+||u-v||}, 0\right\} \\
&= q||u-v||,
\end{aligned}$$

a contradiction and so $u = v$. This proves the uniqueness of u. □

When $f = g = I_X$ the identity mapping on X, conditions (2.2) and (2.3) are trivial and we have the following corollary:

Corollary 2.2 *Let X be a closed convex subset of a normed vector space N. Let T_1 and T_2 be mappings of X into X satisfying*

$$\begin{aligned}
||T_1x - T_2y|| \le q\max&\left\{||x-y||, \frac{||x-T_1x||[1-||x-T_2y||]}{1+||x-T_1x||},\right. \\
&\frac{||x-T_2y||[1-||x-T_1x||]}{1+||x-T_2y||}, \frac{||T_1x-y||[1-||y-T_2y||]}{1+||T_1x-y||}, \\
&\left.\frac{||y-T_2y||[1-||T_1x-y||]}{1+||y-T_2y||}\right\},
\end{aligned}$$

for all x, y in X, where $0 < q < 1$,

$$(1-\lambda)X + \lambda T_1(X) \subseteq X,$$
$$(1-\mu)X + \mu T_2(X) \subseteq X,$$

for all $\lambda, \mu \in (0,1]$, the sequence $\{x_n\}$ is defined as in Theorem 1.3 and $\{c_n\}$ satisfies conditions (i), (ii) and (iii), given above. If $\{x_n\}$ converges to a point u in X, then u is the unique common fixed point of T_1 and T_2.

Example 2.3 Let $X = [0,1] \subset \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers with the usual norm and $T_1, T_2, f, g : X \to X$

$$T_1x = \frac{x}{4}, \quad T_2x = \frac{x^{2/3}}{4},$$
$$fx = x^{1/2}, \quad gx = x^{1/3}.$$

Clearly the mappings g^{-1} and f^{-1} defined by

$$g^{-1}x = x^3 \text{ and } f^{-1}x = x^2$$

exist.

Suppose that $\{y_n\}$ is a sequence of elements of X such that

$$y_{2n+1} = gx_{2n+1} = (1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n},$$
$$y_{2n+2} = fx_{2n+2} = (1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1},$$

and

$$c_n = \frac{n+1}{2n+1}.$$

If $x_0 = \frac{1}{2}$, then with the help of equations (2.6) and (2.7), we obtain the sequence $\{x_n\}$, where

$$\begin{aligned}
x_1 &= g^{-1}[(1-c_0)fx_0 + c_0T_1x_0] \\
&= g^{-1}\Big[(1-1)f\Big(\frac{1}{2}\Big) + \frac{1}{2}\Big] = \Big(\frac{1}{8}\Big)^3, \\
x_2 &= f^{-1}[(1-c_1)gx_1 + c_1T_2x_1] \\
&= f^{-1}\Big[\Big(1-\frac{2}{3}\Big)\frac{1}{8} + \frac{2}{3}\Big(\frac{1}{8}\Big)^2\frac{1}{4}\Big] \\
&= \Big(\frac{17}{3.128}\Big)^2, \\
x_3 &= g^{-1}[(1-c_2)fx_2 + c_2T_1x_2] \\
&= \Big[\frac{17}{3.5.128}\Big(2 + \frac{17}{512}\Big)\Big]^3, \\
x_4 &= f^{-1}[(1-c_3)gx_3 + c_3T_2x_3] \\
&= \Big[\frac{17}{3.5.7.128}\Big(2 + \frac{17}{512}\Big)\Big(3 + \frac{17}{3.5.128}\Big(2 + \frac{17}{128}\Big)\Big)\Big]^2,
\end{aligned}$$

and so on. Then

$$\begin{aligned}
y_1 &= gx_1 = \frac{1}{3}, \\
y_2 &= fx_2 = \frac{17}{3.144}, \\
y_3 &= gx_3 = (\frac{17}{3.5.128})\Big(2 + \frac{17}{512}\Big), \\
y_4 &= fx_4 = \frac{17}{3.5.7.128}\Big(2 + \frac{17}{512}\Big)\Big(3 + \frac{17}{3.5.128}\Big(2 + \frac{17}{128}\Big)\Big),
\end{aligned}$$

and so on. It is evident that $y_n \to 0$ as $n \to \infty$.

We note that T_1, T_2, f and g are continuous and satisfy all the conditions of Theorem 2.1 with $0 < q = \frac{1}{2} < 1$. Indeed we have

$$\begin{aligned}||T_1x - T_2y|| &= \frac{1}{4}||x - y^{\frac{2}{3}}||\\ &\leq \frac{(||x^{1/2}|| + ||y^{1/3}||)(||x^{1/2} - y^{1/3}||)}{4}\\ &\leq \frac{||x^{1/2} - y^{1/3}||}{2}\\ &\leq \frac{||fx - gy||}{2}.\end{aligned}$$

Further, 0 is the common fixed point of T_1, T_2, f and g.

3 Fixed Point Theorems with Asymptotic Regularity Condition

Let $\mathbb{R}^+$ denote the set of nonnegative real numbers, $W : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $0 < W(t) < t$ for all $t \in \mathbb{R}^+$ and let T_1, T_2, f and g be selfmaps on a metric space (X, d). For a point $x_0 \in X$, if there exists a sequence $\{y_n\}$ in X such that

$$\begin{aligned}y_{2n} &= T_1x_{2n} = gx_{2n+1},\\ y_{2n+1} &= T_2x_{2n+1} = fx_{2n+2},\end{aligned}$$

for $n = 0, 1, 2, \ldots$, then $O(T_1, T_2, f, g, x_0) = \{y_n : n = 1, 2, \ldots\}$ is called the *orbit* of (T_1, T_2, f, g) at x_0. T_1 and T_2 are said to be *orbitally continuous* at x_0 if and only if they are continuous on $O(T_1, T_2, f, g, x_0)$. X is said to be *orbitally complete* at x_0 if and only if every Cauchy sequence in $O(T_1, T_2, f, g, x_0)$ converges in X. The pair (T_1, T_2) is said to be *asymptotically regular* (a.r.) with respect to (g, f) at x_0 if there exists a sequence $\{y_n\}$ in X such that

$$\begin{aligned}y_{2n} &= T_1x_{2n} = gx_{2n+1},\\ y_{2n+1} &= T_2x_{2n+1} = fx_{2n+2},\end{aligned}$$

for $n = 0, 1, 2, \ldots$ and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Zeqing Liu et al. [4] proved the following theorem :

Theorem 3.1 *Let f, g and h be selfmaps on a metric space (X, d) and let $fh = hf$ or $gh = hg$. Suppose that there exists a point $x_0 \in X$ such that (f, g) is a.r. with* *respect to h at x_0, X is orbitally complete at x_0, and h is orbitally continuous at x_0. If*

$$d(fx, gy) \leq M(x, y) - W(M(x, y)) \tag{3.1}$$

holds for all $x, y \in X$, then f, g and h have a unique common fixed point in X, where

$$M(x, y) = \max\{d(hx, hy), d(hx, fx), d(hy, gy), d(hx, gy), d(hy, fx)\}.$$

4 Main Result

Now we present our second main theorem :

Theorem 4.1 *Let T_1, T_2, f and g be selfmaps on a metric space (X, d) and let $T_1 f = fT_1$ and $T_2 g = gT_2$. Suppose that there exists a point $x_0 \in X$ such that (T_1, T_2) is a.r. with respect to (g, f) at x_0, X is orbitally complete at x_0, and g, f are orbitally continuous at x_0. If*

$$d(T_1 x, T_2 y) \leq M^{'}(x, y) - W(M^{'}(x, y)) \tag{4.1}$$

holds for all $x, y \in X$, then T_1, T_2, f and g have a unique common fixed point in X, where

$$M^{'}(x, y) = \max\{d(fx, gy), d(fx, T_1 x), d(fx, T_2 y), d(T_1 x, gy), d(gy, T_2 y)\}.$$

Proof. Since (T_1, T_2) is a.r. with respect to (g, f) at x_0, there exists a sequence $\{y_n\}$ in X such that

$$y_{2n} = T_1 x_{2n} = g x_{2n+1},$$
$$y_{2n+1} = T_2 x_{2n+1} = f x_{2n+2},$$

for $n = 0, 1, 2, \ldots$ and

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{4.2}$$

In order to show that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that the result is not true. Then there will be a positive number ϵ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > 2k$ and

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{4.3}$$

For each integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (4.3) so that

$$d(y_{2m(k)-2}, y_{2n(k)}) \leq \epsilon. \tag{4.4}$$

Then for each even integer $2k$,

$$\begin{aligned}d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)-2}, y_{2n(k)}) + d(y_{2m(k)-2}, y_{2m(k)-1}) \\ + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

From (4.2), (4.3), (4.4) and the above inequality we have,

$$\lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)}) \leq \epsilon. \tag{4.5}$$

Using the triangular inequality and putting $d(y_n, y_{n+1}) = d_n$, we obtain

$$\begin{aligned}|d(y_{2m(k)+1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)})| &\leq d_{2m(k)}, \\ |d(y_{2m(k)+1}, y_{2n(k)+1}) - d(y_{2m(k)+1}, y_{2n(k)})| &\leq d_{2n(k)}, \\ |d(y_{2m(k)+2}, y_{2n(k)+1}) - d(y_{2m(k)+1}, y_{2n(k)+1})| &\leq d_{2m(k)+1}, \\ |d(y_{2m(k)+2}, y_{2n(k)}) - d(y_{2m(k)+1}, y_{2n(k)})| &\leq d_{2m(k)+1},\end{aligned}$$

and from (4.2), (4.5) and the above inequalities, we have

$$\begin{aligned}\lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) &\leq \epsilon, \\ \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)+1}) &\leq \epsilon, \\ \lim_{k\to\infty} d(y_{2m(k)+2}, y_{2n(k)+1}) &\leq \epsilon, \\ \lim_{k\to\infty} d(y_{2m(k)+2}, y_{2n(k)}) &\leq \epsilon.\end{aligned}$$

It follows from (4.1) that

$$\begin{aligned} d(y_{2n(k)+1}, y_{2m(k)+2}) &= d(T_2x_{2n(k)+1}, T_1x_{2m(k)+2}) \\ &= d(T_2x_{2n(k)+1}, T_1x_{2m(k)+2}) \\ &\leq \max\Big\{d(fx_{2m(k)+2}, gx_{2n(k)+1}), d(fx_{2m(k)+2}, T_1x_{2m(k)+2}), \\ &\quad d(fx_{2m(k)+2}, T_2x_{2n(k)+1}), d(T_1x_{2m(k)+2}, gx_{2n(k)+1}), \\ &\quad d(gx_{2n(k)+1}, T_2x_{2n(k)+1})\Big\} \\ &\quad - W\Big(d(fx_{2m(k)+2}, gx_{2n(k)+1}), d(fx_{2m(k)+2}, T_1x_{2m(k)+2}), \\ &\quad d(fx_{2m(k)+2}, T_2x_{2n(k)+1}), d(T_1x_{2m(k)+2}, gx_{2n(k)+1}), \\ &\quad d(gx_{2n(k)+1}, T_2x_{2n(k)+1})\}\Big) \\ &\leq \max\Big\{d(y_{2m(k)+1}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2m(k)+2}), \\ &\quad d(y_{2m(k)+1}, y_{2n(k)+1}), d(y_{2m(k)+2}, y_{2n(k)}), \\ &\quad d(y_{2n(k)}, y_{2n(k)+1})\Big\} \\ &\quad - W\Big(d(y_{2m(k)+1}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2m(k)+2}), \\ &\quad d(y_{2m(k)+1}, y_{2n(k)+1}), d(y_{2m(k)+2}, y_{2n(k)}), \\ &\quad d(y_{2n(k)}, y_{2n(k)+1})\}\Big). \end{aligned}$$

As k tends to infinity, we have

$$\epsilon \leq \max\{\epsilon, 0, \epsilon, \epsilon, 0\} - W(\max\{\epsilon, 0, \epsilon, \epsilon, 0\}),$$

or

$$\epsilon \leq \epsilon - W(\epsilon).$$

That is, $W(\epsilon) \leq 0$, which implies $\epsilon = 0$, a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence.

Since X is (T_1, T_2, f, g) orbitally complete at x_0, there exists a point z such that $y_n \to z$ as n tends to ∞.

Now applying (4.1) to $d(gx_{2n+1}, T_2z)$ and $d(T_1z, fx_{2n+2})$ and letting n tend to infinity, we have

$$\begin{aligned} d(gx_{2n+1}, T_2z) &= d(T_1x_{2n}, T_2z) \\ &\leq \max\{d(fx_{2n}, gz), d(fx_{2n}, T_1x_{2n}), d(fx_{2n}, T_2z), \\ &\quad d(T_1x_{2n}, gz), d(gz, T_2z)\} \\ &\quad - W(\max\{d(fx_{2n}, gz), d(fx_{2n}, T_1x_{2n}), d(fx_{2n}, T_2z), \\ &\quad d(T_1x_{2n}, gz), d(gz, T_2z)\}) \end{aligned}$$

or

$$d(z, T_2z) \le \max\{d(z, gz), d(z, z), d(z, T_2z), d(z, gz), d(gz, T_2z)\} \\ - W(\max\{d(z, gz), d(z, z), d(z, T_2z), d(z, gz), d(gz, T_2z)\})$$

or

$$d(z, T_2z) \le \max\{d(z, gz), d(z, z), d(z, T_2z), d(z, gz), d(gz, T_2z)\} \\ - W(\max\{d(z, gz), d(z, z), d(z, T_2z), d(z, gz), d(gz, T_2z)\}) \qquad (4.6)$$

and

$$\begin{aligned} d(T_1z, fx_{2n+2}) &= d(T_1z, T_2x_{2n+1}) \\ &\le \max\{d(fz, gx_{2n+1}), d(fz, T_1z), d(fz, T_2x_{2n+1}), \\ &\quad d(T_1z, gx_{2n+1}), d(gx_{2n+1}, T_2x_{2n+1})\} \\ &\quad - W(\max\{d(fz, gx_{2n+1}), d(fz, T_1z), d(fz, T_2x_{2n+1}), \\ &\quad d(T_1z, gx_{2n+1}), d(gx_{2n+1}, T_2x_{2n+1})\}) \end{aligned}$$

or

$$d(T_1z, z) \le \max\{d(fz, z), d(fz, T_1z), d(fz, z), d(T_1z, z), d(z, z)\} \\ - W(\max\{d(fz, z), d(fz, T_1z), d(fz, z), d(T_1z, z), d(z, z)\})$$

or

$$d(T_1z, z) \le \max\{d(fz, z), d(fz, T_1z), d(fz, z), d(T_1z, z), 0\} \\ - W(\max\{d(fz, z), d(fz, T_1z), d(fz, z), d(T_1z, z), 0\}). \qquad (4.7)$$

Since $T_1f = fT_1$, we have $T_1fx_{2n+2} = fT_1x_{2n+2} \to fz$.
Again, since T_1 is orbitally continuous at x_0 we have by (4.1)

$$\begin{aligned} d(T_1fx_{2n+2}, fx_{2n+2}) &= d(T_1fx_{2n+2}, T_2x_{2n+1}) \\ &\le \max\{d(ffx_{2n+2}, gx_{2n+1}), d(ffx_{2n+2}, T_1fx_{2n+2}), d(ffx_{2n+2}, T_2x_{2n+1}), \\ &\quad d(T_1fx_{2n+2}, gx_{2n+1}), d(gx_{2n+1}, T_2x_{2n+1})\} \\ &\quad - W(\max\{d(ffx_{2n+2}, gx_{2n+1}), d(ffx_{2n+2}, T_1fx_{2n+2}), \\ &\quad d(ffx_{2n+2}, T_2x_{2n+1}), d(T_1fx_{2n+2}, gx_{2n+1}), d(gx_{2n+1}, T_2x_{2n+1})\}) \\ &\le \max\{d(fz, z), d(fz, fz), d(fz, z), d(fz, z), d(z, z)\} \\ &\quad - W(\max\{d(fz, z), d(fz, fz), d(fz, z), d(fz, z), d(z, z)\}). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(fz, z) \le d(fz, z) - W(d(fz, z),$$

which implies $fz = z$.

Similarly if $T_2g = gT_2$, then $T_2gx_{2n+1} = gT_2x_{2n+1} \to gz$.

Since T_2 is orbitally continuous on x_0, then we see on applying (4.1) to $d(gx_{2n+1}, T_2gx_{2n+1})$ and letting $n \to \infty$, we obtain $gz = z$.

In equation (4.6), if we put $z = gz$, then we get $T_2z = z$. Again in (4.7), if we put $z = fz$, then we get $T_1z = z$. Thus z is a common fixed point of T_1, T_2, f and g. Uniqueness of z is obvious. This completes the proof of the theorem. □

Remark 4.2 When $f = g$, Theorem 3.1 strictly extends Theorem 2.1 of Liu Zeqing et al. [4]. Furthermore, taking $W(t) = (1-r)t : \ r \in (0,1)$, we obtain Theorem 1 of Sastry et al. [3].

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(Received 4 March 2007)

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